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FINITE-SCALE LAURENT NORMING FOR REPRESENTATIONS OF GRAPH C^*-ALGEBRAS

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Abstract. Let a representation of the graph C^*-algebra of a finite directed graph be nonzero on every vertex projection. We study its lower norm on canonical path-length subspaces. The spectra of represented vertex-simple cycles without exits determine compact subsets of the circle and associated Laurent norming constants. We prove an exact finite-scale formula that expresses the lower norm as the minimum of these constants. The same value persists at every matrix level. We also derive asymptotic consequences in terms of logarithmic capacity and Green functions.

Keywords. graph C^*-algebra, Cuntz–Krieger uniqueness, cycle without exits, Laurent polynomial norming, logarithmic capacity
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1. Introduction

Let E be a finite directed graph with at least one vertex and let $C^*(E)$ be its graph C^*-algebra. The general Cuntz–Krieger uniqueness theorem says that a $*$-homomorphism out of $C^*(E)$ is injective exactly when every vertex projection has nonzero image and the image of every vertex-simple cycle without exits has spectrum in the codomain containing $\mathbb{T}$ [7, Theorem 1.2]. The classical uniqueness framework for graph C^*-algebras is developed systematically in [4, Chapters 2–3]. Once every vertex projection has nonzero image, noninjectivity is therefore detected on the no-exit cycles. This criterion does not quantify the norm lost on finite spans of the canonical graph monomials.

Given a C^*-algebra B and a $*$-homomorphism $\pi : C^*(E) \to B$ that is nonzero on every vertex projection, let $\mathcal{V}_N(E)$ be the path-length subspace spanned by $s_\mu s_\nu^*$ with $r(\mu) = r(\nu)$ and $|\mu|, |\nu| \leq N$. We study the lower norm

$$\kappa_N(\pi) = \inf_{0 \neq a \in \mathcal{V}_N(E)} \frac{\|\pi(a)\|}{\|a\|}$$

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and its matrix analogues. Faithfulness forces $\kappa_N(\pi) = 1$ at every scale because a faithful $*$-homomorphism is isometric. The problem is to compute these lower norms without assuming faithfulness.

Let $\mathscr{C}_{\mathrm{ne}}(E)$ contain one representative of each vertex-simple cycle without exits, modulo cyclic permutation. For $c \in \mathscr{C}_{\mathrm{ne}}(E)$ of length ℓ_c, write $v_c = s(c)$, and let $X_c(\pi) \subseteq \mathbb{T}$ be the spectrum of the represented cycle unitary in the corner $\pi(p_{v_c})B\pi(p_{v_c})$. Let $Y_c(\pi) = \{z \in \mathbb{T} : z^{\ell_c} \in X_c(\pi)\}$. With this corner-spectrum convention, the uniqueness condition above is $X_c(\pi) = \mathbb{T}$ for every $c \in \mathscr{C}_{\mathrm{ne}}(E)$. Write $\kappa_N^{(r)}(\pi)$ for the corresponding lower norm at matrix level r. The main theorem identifies the graph-algebra lower norm with a Laurent norming constant:

$$\kappa_N(\pi) = \min_{c \in \mathscr{C}_{\mathrm{ne}}(E)} \lambda_N^{\mathrm{L}}\big(Y_c(\pi)\big), \qquad \kappa_N^{(r)}(\pi) = \kappa_N(\pi) \quad (r \geq 1).$$

Here $\lambda_N^{\mathrm{L}}(Y)$ is the Laurent norming constant, namely the optimal lower norm for restricting Laurent polynomials with exponents in $[-N, N]$ from $\mathbb{T}$ to Y, and the empty minimum is 1.

The non-gauge-invariant primitive ideals associated with no-exit cycles are parametrized by circle families in [2, Theorem 2.10 and Corollary 2.12]. The row-finite ideal description [1, Theorem 5.5.3] decomposes a kernel containing no vertex projection into orthogonal ideals supported on cycles without exits. On the ideal associated with c, the gauge action changes the circle parameter t to $z^{\ell_c}t$; the parameters for which the rotated representation annihilates the kernel consequently form a rotation of $Y_c(\pi)$. Laurent norming gives the lower estimate, while a unitary assembled from the edges of c gives the reverse estimate. Orthogonality keeps the cycle contributions independent even in the presence of common upstream entrances, and the same argument at every matrix level gives no additional amplification loss.

Multiplication by z^N turns the Laurent degree window $[-N, N]$ into algebraic degree at most $2N$. The Bernstein–Walsh inequality [6, Lemma 3.7] and the Siciak extremal characterization [3, Theorem 5.1.7] yield the asymptotic formula in Corollary 2.4. Under the hypotheses of that corollary, let $G(X)$ denote the maximum on $\mathbb{T}$ of the Green function $g_{\widehat{\mathbb{C}}\setminus X}(\,\cdot\,, \infty)$, with $G(\mathbb{T}) = 0$. Then

$$\lim_{N\to\infty} -\frac{1}{N} \log \kappa_N(\pi) = \max_{c \in \mathscr{C}_{\mathrm{ne}}(E)} \frac{2}{\ell_c} G\big(X_c(\pi)\big).$$

The capacity formula is an asymptotic consequence of the finite-scale identity. If all no-exit cycle spectra are $\mathbb{T}$, every Laurent norming constant equals 1, in agreement with the qualitative uniqueness theorem.

2. Main results

Let $E = (E^0, E^1, r, s)$ be a finite directed graph with $E^0 \neq \varnothing$, with sinks and sources allowed, and let $A = C^*(E)$ be generated by its canonical Cuntz–Krieger family $\{p_v, s_e : v \in E^0,\ e \in E^1\}$. The gauge action is denoted by γ, so that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$ for $z \in \mathbb{T}$. Let $\mathscr{C}_{\mathrm{ne}}(E)$ contain one representative of each vertex-simple cycle without exits, modulo cyclic permutation. For $c \in \mathscr{C}_{\mathrm{ne}}(E)$, put $\ell_c = |c|$, choose $v_c = s(c)$, and write s_c for the corresponding cycle

partial isometry. The finiteness assumption ensures that $\mathscr{C}_{\mathrm{ne}}(E)$ is finite and is also used in the saturation argument in Lemma 2.1.

Let B be a C^*-algebra and fix a $*$-homomorphism $\pi : A \to B$ such that $\pi(p_v) \neq 0$ for every $v \in E^0$. Since s_c is a unitary in $p_{v_c} A p_{v_c}$, the sets

$$X_c(\pi) = \sigma_{\pi(p_{v_c})B\pi(p_{v_c})}\big(\pi(s_c)\big) \subseteq \mathbb{T}, \qquad Y_c(\pi) = \{z \in \mathbb{T} : z^{\ell_c} \in X_c(\pi)\}$$

are nonempty and compact. Changing the starting vertex of c gives corner-conjugate cycle unitaries, so these sets depend only on the cyclic class of c.

For $N \geq 0$, set

$$\mathcal{V}_N(E) = \operatorname{span}\{s_\mu s_\nu^* : r(\mu) = r(\nu),\ |\mu|, |\nu| \leq N\}.$$

Define

$$\kappa_N(\pi) = \inf_{0 \neq a \in \mathcal{V}_N(E)} \frac{\|\pi(a)\|}{\|a\|}, \qquad \kappa_N^{(r)}(\pi) = \inf_{0 \neq A_0 \in M_r(\mathcal{V}_N(E))} \frac{\|(\mathrm{id}_{M_r} \otimes \pi)(A_0)\|}{\|A_0\|}$$

for $r \geq 1$. If Y is a nonempty compact subset of $\mathbb{T}$, its Laurent norming constant is

$$\lambda_N^{\mathrm{L}}(Y) = \inf_{\substack{0 \neq q(z) = \sum_{k=-N}^{N} a_k z^k \\ a_k \in \mathbb{C}}} \frac{\|q\|_Y}{\|q\|_{\mathbb{T}}}, \qquad \|q\|_Y = \sup_{z \in Y} |q(z)|.$$

For later use, define the algebraic polynomial norming constant by

$$\lambda_m^{\mathrm{alg}}(Y) = \inf_{\substack{0 \neq p \in \mathbb{C}[z] \\ \deg p \leq m}} \frac{\|p\|_Y}{\|p\|_{\mathbb{T}}}.$$

Multiplication by z^N is a bijection from the Laurent polynomials with exponents in $[-N, N]$ onto the algebraic polynomials of degree at most $2N$. Since $|z| = 1$ on both norming sets,

$$\lambda_N^{\mathrm{L}}(Y) = \lambda_{2N}^{\mathrm{alg}}(Y). \tag{2.1}$$

Lemma 2.1. *Let $I = \ker \pi$, let $Q : A \to A/I$ be the quotient map, and, for $c \in \mathscr{C}_{\mathrm{ne}}(E)$, let $J_c = \overline{A p_{v_c} A}$. Then*

$$J_c J_d = \{0\} \quad (c \neq d), \qquad I = \bigoplus_{c \in \mathscr{C}_{\mathrm{ne}}(E)} (I \cap J_c).$$

The projection p_{v_c} is full in J_c, and

$$p_{v_c} J_c p_{v_c} = C^*(s_c) \cong C(\mathbb{T}), \qquad p_{v_c} I p_{v_c} = \{f(s_c) : f \in C(\mathbb{T}),\ f|_{X_c(\pi)} = 0\}.$$

If ρ is an irreducible representation of A such that $\rho(I) \neq 0$, there exist unique $c \in \mathscr{C}_{\mathrm{ne}}(E)$ and a unique $t \in \mathbb{T}$ such that $\rho|_{J_c} \neq 0$ and its kernel corresponds to t under $\mathrm{Prim}(J_c) \cong \mathbb{T}$. *For every $z \in \mathbb{T}$,*

$$(\rho \circ \gamma_z)(I) = \{0\} \quad \Longleftrightarrow \quad z^{\ell_c} t \in X_c(\pi).$$

Proof. For the ideal I, the hereditary saturated set $H_I = \{v \in E^0 : p_v \in I\}$ is empty. The graph is row-finite, so its ideal data have no breaking vertices. The description of closed ideals of row-finite graph C^*-algebras in [1, Theorem 5.5.3 and (5.6)], specialized to $H_I = \varnothing$, writes I as the sum of its parts supported on cycles without exits in E. The supporting ideals J_c are pairwise orthogonal, as verified below. For each such cycle, the no-exit condition gives $p_{v_c} J_c p_{v_c} = C^*(s_c)$,

and p_{v_c} is full in J_c by the definition of J_c. Since automorphisms preserve spectra and $\gamma_z(s_c) = z^{\ell_c} s_c$, the spectrum of s_c in $p_{v_c} A p_{v_c}$ is invariant under multiplication by z^{ℓ_c} for every $z \in \mathbb{T}$. As z^{ℓ_c} ranges over $\mathbb{T}$, this nonempty spectrum is all of $\mathbb{T}$. Hence $C^*(s_c) \cong C(\mathbb{T})$.

The full-corner correspondence [4, Lemma 5.10 and Example 5.16] identifies the ideals and primitive ideals of J_c with those of $C(\mathbb{T})$; in particular, $\mathrm{Prim}(J_c) \cong \mathbb{T}$. This agrees with the circle parametrization in [2, Theorem 2.10 and Corollary 2.12]. The row-finite ideal theorem further identifies $I \cap J_c$ with the ideal generated by the corresponding closed ideal of this full corner [1, Theorem 5.5.3 and (5.6)].

For $c \in \mathscr{C}_{\mathrm{ne}}(E)$, let H_c denote the saturated hereditary closure of the vertex set c^0. Since c has no exits, c^0 is hereditary. Define

$$\Sigma_0^{(c)} = c^0, \qquad \Sigma_{n+1}^{(c)} = \Sigma_n^{(c)} \cup \left\{ v \in E^0 : 0 < |s^{-1}(v)| < \infty,\ r(s^{-1}(v)) \subseteq \Sigma_n^{(c)} \right\}.$$

Because E is finite, $H_c = \bigcup_{n \geq 0} \Sigma_n^{(c)}$. If $v \in H_c \setminus c^0$ and n is minimal with $v \in \Sigma_n^{(c)}$, then every edge emitted by v has range in $\Sigma_{n-1}^{(c)}$. Hence along any infinite path starting at v, the minimal saturation level decreases until the path reaches c^0; from that point onward it remains on c, since c has no exits. Consequently, if $c \neq d$, no vertex can belong to both H_c and H_d, and therefore $H_c \cap H_d = \varnothing$. Since the lattice correspondence for gauge-invariant ideals preserves intersections [1, Theorem 5.5.3], it follows that $J_c \cap J_d = \{0\}$, and therefore $J_c J_d = \{0\}$.

Under the corner identification, the closed ideal has the form $\{f \in C(\mathbb{T}) : f|_X = 0\}$ for a unique closed set $X \subseteq \mathbb{T}$. The quotient corner is generated by $Q(s_c)$, whose spectrum is X. The injective factorization of π through A/I, followed by spectral permanence in $\pi(p_{v_c}) B \pi(p_{v_c})$, identifies this set with $X_c(\pi)$.

If $\rho(I) \neq 0$, orthogonality gives a unique c such that $\rho(I \cap J_c) \neq 0$. Then $\rho|_{J_c}$ is nonzero and nondegenerate, and its primitive kernel corresponds through the full corner to evaluation at a unique $t \in \mathbb{T}$. Since $J_c J_d = \{0\}$ for $d \neq c$, irreducibility of ρ implies $\rho(J_d) = 0$ for every $d \neq c$.

Since $\gamma_z(s_c) = z^{\ell_c} s_c$, the corner parameter of $\rho \circ \gamma_z$ is $z^{\ell_c} t$. By the full-corner ideal correspondence, $(\rho \circ \gamma_z)(I \cap J_c) = \{0\}$ if and only if evaluation at $z^{\ell_c} t$ annihilates $p_{v_c} I p_{v_c}$, which is equivalent to $z^{\ell_c} t \in X_c(\pi)$. Because $\gamma_z(J_d) = J_d$ and $\rho(J_d) = 0$ for $d \neq c$, this is equivalent to $(\rho \circ \gamma_z)(I) = \{0\}$. □

Lemma 2.2. *Let D be a complex Banach space, let $Y \subseteq \mathbb{T}$ be nonempty and compact, and let*

$$F(z) = \sum_{k=-N}^{N} z^k x_k, \qquad x_k \in D.$$

Then

$$\sup_{z \in Y} \|F(z)\| \geq \lambda_N^{\mathrm{L}}(Y) \sup_{z \in \mathbb{T}} \|F(z)\|.$$

Moreover, $\lambda_N^{\mathrm{L}}(\alpha Y) = \lambda_N^{\mathrm{L}}(Y)$ for every $\alpha \in \mathbb{T}$.

Proof. The inequality is trivial when $F = 0$. Otherwise, choose $z_0 \in \mathbb{T}$ such that $\|F(z_0)\| = \sup_{z \in \mathbb{T}} \|F(z)\|$. There is $\varphi \in D^*$ with $\|\varphi\| = 1$ and $\varphi(F(z_0)) =$

$\|F(z_0)\|$. The scalar Laurent polynomial $q = \varphi \circ F$ satisfies

$$\|q\|_{\mathbb{T}} = \sup_{z\in\mathbb{T}} \|F(z)\|, \qquad \|q\|_Y \le \sup_{z\in Y} \|F(z)\|.$$

The defining inequality for $\lambda_N^{\mathrm{L}}(Y)$ gives the desired estimate. The substitution $z \mapsto \alpha z$ is a bijection of the admissible Laurent polynomials and proves rotation invariance. □

Theorem 2.3. *For every $N \ge 0$,*

$$\kappa_N(\pi) = \min_{c\in\mathscr{C}_{\mathrm{ne}}(E)} \lambda_N^{\mathrm{L}}\big(Y_c(\pi)\big),$$

where the minimum over the empty set is 1. For every $r \ge 1$, $\kappa_N^{(r)}(\pi) = \kappa_N(\pi)$.

Proof. Let $I = \ker \pi$, and let $Q : A \to A/I$ be the quotient map. Put $\lambda = \min_{c\in\mathscr{C}_{\mathrm{ne}}(E)} \lambda_N^{\mathrm{L}}\big(Y_c(\pi)\big)$, with $\lambda = 1$ when $\mathscr{C}_{\mathrm{ne}}(E) = \varnothing$. The map induced by π from A/I into B is injective and hence completely isometric. It is therefore enough to prove, for $A_0 \in M_r(\mathcal{V}_N(E))$, that

$$\|(\mathrm{id}_{M_r} \otimes Q)(A_0)\| \ge \lambda \|A_0\|.$$

Let ρ be an irreducible representation of A. If $\rho(I) = 0$, then $\mathrm{id}_{M_r} \otimes \rho$ factors through $M_r(A/I)$, and consequently

$$\|(\mathrm{id}_{M_r} \otimes Q)(A_0)\| \ge \|(\mathrm{id}_{M_r} \otimes \rho)(A_0)\|.$$

Suppose that $\rho(I) \ne 0$, and take c, t from Lemma 2.1. Choose $w \in \mathbb{T}$ such that $w^{\ell_c} = t^{-1}$, and set $Z = \overline{w}\, Y_c(\pi)$. Every entry of A_0 has gauge degree between $-N$ and N. Thus

$$F(z) = (\mathrm{id}_{M_r} \otimes \rho)\big((\mathrm{id}_{M_r} \otimes \gamma_z)(A_0)\big)$$

is an operator-valued Laurent polynomial with exponents in $[-N, N]$. For $z \in Z$, one has $z^{\ell_c} t = (zw)^{\ell_c} \in X_c(\pi)$. Lemma 2.1 therefore implies that $\rho \circ \gamma_z$ factors through A/I, so

$$\|(\mathrm{id}_{M_r} \otimes Q)(A_0)\| \ge \sup_{z\in Z} \|F(z)\|.$$

Lemma 2.2 and its rotation invariance give

$$\begin{aligned} \|(\mathrm{id}_{M_r} \otimes Q)(A_0)\| &\ge \lambda_N^{\mathrm{L}}\big(Y_c(\pi)\big) \sup_{z\in\mathbb{T}} \|F(z)\| \\ &\ge \lambda \|(\mathrm{id}_{M_r} \otimes \rho)(A_0)\|. \end{aligned}$$

Here we used $\lambda \le \lambda_N^{\mathrm{L}}(Y_c(\pi))$ and $\sup_{z\in\mathbb{T}} \|F(z)\| \ge \|F(1)\| = \|(\mathrm{id}_{M_r} \otimes \rho)(A_0)\|$. Since $\lambda \le 1$, the same estimate also holds in the case $\rho(I) = 0$. Finally, the C^*-norm on $M_r(A)$ is the supremum of $\|(\mathrm{id}_{M_r} \otimes \rho)(A_0)\|$ over the irreducible representations ρ of A, so taking this supremum yields the required lower bound.

Fix $c \in \mathscr{C}_{\mathrm{ne}}(E)$. Enumerate its vertices and edges cyclically as

$$c = e_0 e_1 \cdots e_{\ell_c - 1}, \qquad s(e_j) = v_j, \quad r(e_j) = v_{j+1},$$

where indices are read modulo ℓ_c, and put

$$P_c = \sum_{j=0}^{\ell_c - 1} p_{v_j}, \qquad U_c = \sum_{j=0}^{\ell_c - 1} s_{e_j}.$$

The absence of exits gives $U_c^*U_c = U_cU_c^* = P_c$; hence U_c is a unitary in the corner P_cAP_c. Its spectrum in that corner is $\mathbb{T}$: it is nonempty, lies in $\mathbb{T}$, and is invariant under every rotation because $\gamma_z(U_c) = zU_c$.

Let $V_c = \pi(U_c)$. If c_j denotes the cyclic rotation of c based at v_j, then

$$V_c^{\ell_c} = \sum_{j=0}^{\ell_c-1} \pi(s_{c_j}).$$

Each summand belongs to the corner $\pi(p_{v_j})B\pi(p_{v_j})$. Conjugation by the path partial isometry from v_0 to v_j identifies its spectrum with $X_c(\pi)$. The displayed block-diagonal decomposition and spectral mapping therefore give

$$\{z^{\ell_c} : z \in \sigma_{\pi(P_c)B\pi(P_c)}(V_c)\} = X_c(\pi).$$

For $\omega^{\ell_c} = 1$, the unitary

$$D_\omega = \sum_{j=0}^{\ell_c-1} \overline{\omega}^j \pi(p_{v_j})$$

satisfies $D_\omega V_c D_\omega^* = \omega^{-1}V_c$. The spectrum of V_c is therefore invariant under all ℓ_c-th roots of unity, and the preceding power identity implies $\sigma_{\pi(P_c)B\pi(P_c)}(V_c) = Y_c(\pi)$.

For a nonzero Laurent polynomial $q(z) = \sum_{k=-N}^{N} a_k z^k$, define

$$q(U_c) = a_0P_c + \sum_{k=1}^{N} a_kU_c^k + \sum_{k=1}^{N} a_{-k}(U_c^*)^k.$$

Every summand is a linear combination of $s_\mu s_\nu^*$ with $|\mu|, |\nu| \le N$, so $q(U_c) \in \mathcal{V}_N(E)$. Continuous functional calculus gives $\|q(U_c)\| = \|q\|_{\mathbb{T}}$ and $\|\pi(q(U_c))\| = \|q\|_{Y_c(\pi)}$. Placing $q(U_c)$ in one diagonal entry of $M_r(A)$ and then taking the infimum over q shows that $\kappa_N^{(r)}(\pi) \le \lambda_N^{\mathrm{L}}\big(Y_c(\pi)\big)$ for every c and every $r \ge 1$. Together with the lower bound, this identifies every $\kappa_N^{(r)}(\pi)$, including the scalar case, with λ. If there is no cycle without exits, Lemma 2.1 gives $I = 0$, and the empty-minimum value is 1. □

Corollary 2.4. *Assume that each $X_c(\pi)$ is either $\mathbb{T}$ or a proper regular compact subset of $\mathbb{T}$ with positive logarithmic capacity. Here a proper compact set $X \subseteq \mathbb{T}$ is called regular if $g_{\widehat{\mathbb{C}}\setminus X}(\,\cdot\,,\infty)$ extends continuously to X with boundary value 0; see [5]. For a proper set X of the latter type, put $G(X) = \max_{\zeta\in\mathbb{T}} g_{\widehat{\mathbb{C}}\setminus X}(\zeta,\infty)$, and set $G(\mathbb{T}) = 0$. Then*

$$\lim_{N\to\infty} -\frac{1}{N}\log\kappa_N(\pi) = \max_{c\in\mathscr{C}_{\mathrm{ne}}(E)} \frac{2}{\ell_c} G\big(X_c(\pi)\big),$$

where the maximum over the empty set is 0.

Proof. By (2.1), it suffices to determine the asymptotic behavior of $\lambda_m^{\mathrm{alg}}(Y)$. Suppose that Y is proper, regular, and of positive logarithmic capacity. Its complement in $\widehat{\mathbb{C}}$ is connected. Since Y is regular, the Green function $g_{\widehat{\mathbb{C}}\setminus Y}(\,\cdot\,,\infty)$

extends continuously by 0 to Y. The Bernstein–Walsh inequality [6, Lemma 3.7] gives

$$|p(z)| \le \|p\|_Y \exp\big(m g_{\widehat{\mathbb{C}}\setminus Y}(z,\infty)\big), \qquad z \in \mathbb{C}\setminus Y, \quad \deg p \le m.$$

By continuity, the same estimate holds on Y, where the extended Green function vanishes, and hence throughout $\mathbb{T}$. The Siciak extremal characterization [3, Theorem 5.1.7] gives

$$g_{\widehat{\mathbb{C}}\setminus Y}(z,\infty) = \sup_{\substack{d\ge 1,\ 0\ne p\in\mathbb{C}[z]\\ \deg p\le d}} \frac{1}{d}\log\frac{|p(z)|}{\|p\|_Y}.$$

Let

$$L_m(Y) = \frac{1}{\lambda_m^{\mathrm{alg}}(Y)} = \sup_{\substack{0\ne p\in\mathbb{C}[z]\\ \deg p\le m}} \frac{\|p\|_{\mathbb{T}}}{\|p\|_Y}.$$

The first inequality gives

$$\limsup_{m\to\infty} \frac{1}{m}\log L_m(Y) \le \max_{\zeta\in\mathbb{T}} g_{\widehat{\mathbb{C}}\setminus Y}(\zeta,\infty).$$

Choose a maximizing point ζ_0. The extremal characterization provides, for every $\varepsilon > 0$, a polynomial p of positive degree d such that

$$\frac{1}{d}\log\frac{|p(\zeta_0)|}{\|p\|_Y} > \max_{\zeta\in\mathbb{T}} g_{\widehat{\mathbb{C}}\setminus Y}(\zeta,\infty) - \varepsilon.$$

For $k = \lfloor m/d\rfloor$, p^k is admissible in $L_m(Y)$, and $L_m(Y) \ge (|p(\zeta_0)|/\|p\|_Y)^{\lfloor m/d\rfloor}$. Letting $m\to\infty$ and then $\varepsilon\downarrow 0$ gives the reverse bound. Hence

$$\lim_{N\to\infty} -\frac{1}{N}\log\lambda_N^{\mathrm{L}}(Y) = 2\max_{\zeta\in\mathbb{T}} g_{\widehat{\mathbb{C}}\setminus Y}(\zeta,\infty).$$

Let $X \subsetneq \mathbb{T}$ be regular with positive capacity, and let $Y = \{z\in\mathbb{T} : z^\ell \in X\}$. Since $z\mapsto z^\ell$ maps $\mathbb{C}\setminus Y$ into $\mathbb{C}\setminus X$, harmonicity is preserved under holomorphic composition. Thus the function $h(z) = \frac{1}{\ell} g_{\widehat{\mathbb{C}}\setminus X}(z^\ell,\infty)$ is harmonic on $\mathbb{C}\setminus Y$, vanishes continuously on Y, and has the Green-function normalization at infinity. Uniqueness of the Green function yields

$$g_{\widehat{\mathbb{C}}\setminus Y}(z,\infty) = \frac{1}{\ell} g_{\widehat{\mathbb{C}}\setminus X}(z^\ell,\infty), \qquad G(Y) = \frac{1}{\ell}G(X).$$

The same identity shows that Y is regular and that $\mathrm{cap}(Y) = \mathrm{cap}(X)^{1/\ell}$. If $X = \mathbb{T}$, then $Y = \mathbb{T}$ and $\lambda_N^{\mathrm{L}}(Y) = 1$.

Applying these facts to $Y_c(\pi)$ gives

$$\lim_{N\to\infty} -\frac{1}{N}\log\lambda_N^{\mathrm{L}}\big(Y_c(\pi)\big) = \frac{2}{\ell_c}G\big(X_c(\pi)\big).$$

If $\mathscr{C}_{\mathrm{ne}}(E) \ne \varnothing$, then Theorem 2.3 gives

$$-\frac{1}{N}\log\kappa_N(\pi) = \max_{c\in\mathscr{C}_{\mathrm{ne}}(E)} \left\{-\frac{1}{N}\log\lambda_N^{\mathrm{L}}\big(Y_c(\pi)\big)\right\}.$$

The index set is finite, so the limit of the right-hand side is the maximum of the individual limits. If $\mathscr{C}_{\mathrm{ne}}(E) = \varnothing$, then $\kappa_N(\pi) = 1$. □

Corollary 2.5. *Recall that a compact set is polar if its logarithmic capacity is zero; see* [5]. *For each* $c \in \mathscr{C}_{\mathrm{ne}}(E)$, *the following hold.*

(1) *If* $X_c(\pi) = \mathbb{T}$, *then* $\lambda_N^{\mathrm{L}}(Y_c(\pi)) = 1$ *for every* N.
(2) *If* $X_c(\pi)$ *is finite, then*

$$\lambda_N^{\mathrm{L}}\big(Y_c(\pi)\big) = 0 \quad \Longleftrightarrow \quad N \geq \left\lceil \frac{\ell_c |X_c(\pi)|}{2} \right\rceil.$$

(3) *If* $X_c(\pi)$ *is infinite and polar, then* $\lambda_N^{\mathrm{L}}(Y_c(\pi)) > 0$ *for every* N, *while*

$$\lim_{N\to\infty} -\frac{1}{N} \log \lambda_N^{\mathrm{L}}\big(Y_c(\pi)\big) = +\infty.$$

Consequently, if at least one cycle has finite spectrum, the first index at which $\kappa_N(\pi) = 0$ *is*

$$\min_{\substack{c\in\mathscr{C}_{\mathrm{ne}}(E)\\ |X_c(\pi)|<\infty}} \left\lceil \frac{\ell_c |X_c(\pi)|}{2} \right\rceil.$$

If no cycle has a finite spectrum but some $X_c(\pi)$ *is infinite and polar, then every* $\kappa_N(\pi)$ *is positive and*

$$\lim_{N\to\infty} -\frac{1}{N} \log \kappa_N(\pi) = +\infty.$$

Proof. If $X_c(\pi) = \mathbb{T}$, then $Y_c(\pi) = \mathbb{T}$, and the defining infimum is 1. If $X_c(\pi)$ is finite, then $Y_c(\pi)$ has exactly $\ell_c|X_c(\pi)|$ points. Finite dimensionality shows that the Laurent norming constant vanishes exactly when the restriction to $Y_c(\pi)$ has a nonzero kernel. By the bijection underlying (2.1), a nonzero admissible Laurent polynomial vanishes on $Y_c(\pi)$ exactly when a nonzero algebraic polynomial of degree at most $2N$ does. This polynomial exists exactly when $2N \geq \ell_c|X_c(\pi)|$, proving the second assertion.

If $X_c(\pi)$ is infinite and polar, the polynomial-preimage formula for logarithmic capacity [5, Theorem 5.2.5] gives $\operatorname{cap}\big(Y_c(\pi)\big) = \operatorname{cap}\big(X_c(\pi)\big)^{1/\ell_c} = 0$. The set $Y_c(\pi)$ is infinite. The supremum norm in $Y_c(\pi)$ is a norm in each finite-dimensional polynomial space, because a nonzero polynomial cannot vanish in an infinite set. Thus, $\lambda_N^{\mathrm{L}}(Y_c(\pi)) > 0$ for every N. Let $t_m(Y_c(\pi)) = \inf\{\|p\|_{Y_c(\pi)} : p \text{ is monic and } \deg p = m\}$. The Chebyshev characterization of logarithmic capacity [6, Theorem 1.18(a)] gives $t_m(Y_c(\pi))^{1/m} \to 0$. By Cauchy's coefficient formula, a monic polynomial of degree m has norm at least 1 in $\mathbb{T}$. Therefore

$$\frac{1}{m} \log \frac{1}{\lambda_m^{\mathrm{alg}}(Y_c(\pi))} \geq -\frac{1}{m} \log t_m(Y_c(\pi)) \longrightarrow +\infty.$$

Equation (2.1) proves the third assertion. By Theorem 2.3, $\kappa_N(\pi) = 0$ exactly when one of the cycle Laurent norming constants is zero, which gives the stated first index. When every cycle spectrum is infinite, all of the corresponding Laurent norming constants are positive. If $X_c(\pi)$ is infinite and polar, then $\kappa_N(\pi) \leq \lambda_N^{\mathrm{L}}(Y_c(\pi))$; the third assertion forces the last limit. □

Example 2.6. *Let* E *consist of one vertex and one loop. Then* $C^*(E) \cong C(\mathbb{T})$ *and* $\mathcal{V}_N(E) = \operatorname{span}\{z^k : -N \leq k \leq N\}$. *For any nonempty compact set* $X \subseteq \mathbb{T}$,

take $B = C(X)$ and let $\pi_X : C(\mathbb{T}) \to C(X)$ be the restriction map. The unique vertex projection is sent to 1*, while for the unique cycle c,*

$$X_c(\pi_X) = Y_c(\pi_X) = X, \qquad \kappa_N(\pi_X) = \lambda_N^{\mathrm{L}}(X).$$

Thus, every nonempty compact subset of $\mathbb{T}$ occurs as a cycle spectrum.

3. Conclusion

The finite-scale lower norm of π is therefore determined by the Laurent norming constants associated with the spectra of the cycles without exits, and its value is unchanged under matrix amplification. Finite cycle spectra lead to vanishing at a finite scale, proper regular spectra of positive logarithmic capacity give exponential decay determined by the corresponding Green functions, and infinite polar spectra give superexponential decay.

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