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NUMERICAL-ANALYTIC APPROACH TO THREE-POINT PROBLEMS FOR VOLTERRA INTEGRO-DIFFERENTIAL SYSTEMS

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Abstract. This article investigates Volterra integro-differential systems with linear three-point boundary conditions. Because exact analytical solutions are rare, we apply a numerical-analytic method of successive approximations. We construct a recurrent sequence of functions satisfying the boundary conditions and prove its uniform convergence using majorizing matrices. An algorithm based on an approximate determining equation is proposed to find the required initial values. A numerical example validates the theoretical results by demonstrating the method's effectiveness.

Keywords. Integro-differential equation, three-point boundary value problem, numerical-analytic method of successive approximations, determining equation.
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1. Introduction

The analysis of fundamental problems in modern mechanics, physics, control theory, engineering sciences, and mathematical biology inevitably leads to the study of integro-differential equations. The foundations of the general theory of such equations, including integral and Volterra integro-differential equations, were laid in the classical work of V. Volterra [7]. Since then, these models have become indispensable for describing processes possessing "memory" or nonlocal properties. Unlike ordinary differential equations, integro-differential models take into account the cumulative effect of past states of the system, making them more accurate for describing complex real processes.

Of particular mathematical complexity and practical significance are boundary value problems for such systems. In real applications, boundary conditions often do not reduce to classical Cauchy conditions or standard two-point conditions.

Nonlocal and multipoint boundary conditions are widely encountered in practice. In particular, three-point boundary value problems arise in modeling processes with intermediate control, vibrations of elastic structures with additional

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supports, and various economic models. Recently, various analytical and topological approaches have been applied to such problems; for instance, in [3] the existence and uniqueness of solutions to nonlinear systems of integro-differential equations with three-point and nonlinear integral boundary conditions are investigated. However, the mathematical tools for investigating such problems is significantly more complicated due to the nonlocality of conditions connecting solution values at various points of the interval.

Since obtaining exact analytical solutions for nonlinear integro-differential equations with multipoint boundary conditions is, in most cases, impossible, approximate methods of investigation are widely used. Among them, the numerical-analytic method of successive approximations, originally developed by A. M. Samoilenko and N. I. Ronto for periodic and general two-point boundary value problems, occupies a special place [5]. The main advantage of this method over purely numerical approaches is that it not only constructively builds a sequence of functions converging to the exact solution, but also allows one to obtain rigorous a priori and a posteriori error estimates, as well as to prove existence and uniqueness theorems within a unified computational process.

The theoretical foundations of this approach, including the treatment of degenerate and Fredholm-type boundary conditions, were further developed in [1]. Over the years, the numerical-analytic method has been successfully extended to various classes of boundary-value problems. In particular, a modification of the successive approximation scheme for nonlinear systems of ordinary differential equations with linear three-point boundary conditions of a general form was proposed in [6]. Furthermore, the applicability of a modification of this method to systems of integro-differential equations was demonstrated in [2, 4], where sufficient conditions for the solvability of periodic and multipoint boundary value problems were established, respectively.

Despite the extensive literature devoted to two-point and three-point problems for ordinary differential equations, the application of the numerical-analytic method to Volterra integro-differential systems with linear three-point boundary conditions remains insufficiently studied. The presence of Volterra integral operators introduces fundamentally new technical difficulties related to the estimation of double integrals and the construction of corresponding majorizing matrices, which prevents a direct combination of the results of [2, 6].

The main goal of this paper is to obtain more precise majorizing estimates specifically adapted to the three-point structure, as well as explicit sufficient conditions for existence that are weaker than the general conditions for multipoint boundary problems [4].

The paper is organized as follows. In Section 2, the main assumptions are formulated, a recurrent sequence of functions satisfying the three-point conditions is constructed, and a theorem on its uniform convergence is proved. In Section 3, sufficient conditions for the existence of a solution to the original boundary value problem are derived based on the analysis of the determining equation. Finally, Section 4 presents an illustrative numerical example demonstrating the effectiveness of the proposed algorithm and confirming the theoretical convergence estimates.

2. Form of successive approximations and their convergence

Consider a system of integro-differential equations of the form

$$\frac{dx(t)}{dt} = f\left(t, x(t), \int_0^t \varphi(t, s, x(s))\, ds\right), \qquad t \in [0, T], \tag{2.1}$$

subject to the linear non-separated three-point boundary conditions

$$Ax(0) + Bx(t_1) + Cx(T) = d, \tag{2.2}$$

where $x \in \mathbb{R}^n$, $f : [0,T] \times D \times D_1 \to \mathbb{R}^n$, and $\varphi : [0,T] \times [0,T] \times D \to \mathbb{R}^n$; D is a closed, bounded domain in $\mathbb{R}^n$, and D_1 is a ball in $\mathbb{R}^n$, $D_1 : \ \|\varphi_h\| \le Th$, $h = \max\limits_{\substack{[0,T]\times[0,T],\\ x\in D}} \|\varphi(t,s,x)\|$.

Let $t_1 \in [0,T]$ and let A, B, C be constant $n \times n$ matrices such that $\det(t_1 B + TC) \neq 0$, and let $d \in \mathbb{R}^n$ be a given constant vector.

Assume that the functions f and φ appearing in system (2.1) and condition (2.2) satisfy the following assumptions:

1) The functions $f(t,x,z)$ and $\varphi(t,s,x)$ are defined and continuous in the domain

$$(t, s, x, z) \in [0,T] \times [0,T] \times D \times D_1. \tag{2.3}$$

2) The function $f(t,x,z)$ is bounded by a constant vector M and satisfies a Lipschitz condition with respect to x and z with matrices K_1 and K_2, respectively. Similarly, the function $\varphi(t,s,x)$ satisfies a Lipschitz condition with respect to x with matrix K_3. That is, for all (t,x,z), $(t,\bar{x},\bar{z})$, (t,s,x), and $(t,s,\bar{x})$ from the domain (3.1), the inequalities

$$\begin{gathered} |f(t,x,z)| \le M, \\ |f(t,x,z) - f(t,\bar{x},\bar{z})| \le K_1|x-\bar{x}| + K_2|z-\bar{z}|, \\ |\varphi(t,s,x) - \varphi(t,s,\bar{x})| \le K_3|x-\bar{x}| \end{gathered} \tag{2.4}$$

hold. Here $M = (M_1, M_2, \dots, M_n)$ with $M_i > 0$ for $i = 1, 2, \dots, n$, and $K_l = \{k_{ij}^{(l)}\}$ are nonnegative matrices ($k_{ij}^{(l)} \ge 0$) for $l = 1, 2, 3$ and $i, j = 1, \dots, n$. The notation $|f| = (|f_1|, |f_2|, \dots, |f_n|)$ denotes the componentwise absolute value, and inequalities between vectors are understood componentwise.

3) The set D_β of points $x_0 \in \mathbb{R}^n$ whose β-neighborhoods are entirely contained within the region D is nonempty:

$$D_\beta \neq \varnothing,$$

where

$$\beta(x_0) = \frac{T}{2}M + \beta_1(x_0), \qquad \beta_1(x_0) = |H[d - (A+B+C)x_0]| + G\alpha_1(t_1)M,$$

with

$$\alpha_1(t_1) = 2t_1\left(1 - \frac{t_1}{T}\right), \qquad H = \left(\frac{t_1}{T}B + C\right)^{-1}, \qquad G = |HB|.$$

4) The largest eigenvalue $\lambda(Q)$ of the matrix $Q = \dfrac{T}{2}(Q_1 + Q_2)$ is less than one:

$$\lambda(Q) < 1, \tag{2.5}$$

where

$$Q_1 = K_1 + \frac{3T}{4}K_2K_3, \qquad Q_2 = GQ_1. \tag{2.6}$$

Under assumptions 1) – 4), the numerical-analytic method of successive approximations [5] can be applied to investigate the solutions of the boundary value problem (2.1), (2.2). We first construct a sequence of functions $x_m(t, x_0)$ that satisfy the boundary conditions (2.2). To this end, we construct a sequence of functions of the form

$$\begin{aligned} x_m(t, x_0) = x_0 + \int_0^t \Bigg[& f\left(\tau, x_{m-1}(\tau, x_0), \int_0^\tau \varphi(\tau, s, x_{m-1}(s, x_0))\, ds\right) \\ & - \frac{1}{T}\int_0^T f\left(\tau, x_{m-1}(\tau, x_0), \int_0^\tau \varphi(\tau, s, x_{m-1}(s, x_0))\, ds\right) d\tau \Bigg] d\tau + \alpha t, \end{aligned} \tag{2.7}$$

where $m = 1, 2, \ldots$, $x_0(t, x_0) = x_0 = (x_{01}, x_{02}, \ldots, x_{0n})$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. The vector parameter α is selected such that the functions defined by (2.7) satisfy the boundary conditions (2.2) for every $m = 1, 2, \ldots$ and for any initial vector $x_0 \in D_\beta$. Substituting expression (2.7) into (2.2), we obtain the following expression for α:

$$\begin{aligned} \alpha = \frac{1}{T}H\Bigg\{ & d - (A + B + C)x_0 \\ & - B\int_0^{t_1} \Bigg[f\left(\tau, x_{m-1}(\tau, x_0), \int_0^\tau \varphi(\tau, s, x_{m-1}(s, x_0))\, ds\right) \\ & - \frac{1}{T}\int_0^T f\left(\tau, x_{m-1}(\tau, x_0), \int_0^\tau \varphi(\tau, s, x_{m-1}(s, x_0))\, ds\right) d\tau \Bigg] d\tau \Bigg\}. \end{aligned}$$

Substituting this value of α into (2.7), we obtain

$$\begin{aligned} x_m(t, x_0) = x_0 + \int_0^t \Bigg[& f\left(\tau, x_{m-1}(\tau, x_0), \int_0^\tau \varphi(\tau, s, x_{m-1}(s, x_0))\, ds\right) \\ & - \frac{1}{T}\int_0^T f\left(\tau, x_{m-1}(\tau, x_0), \int_0^\tau \varphi(\tau, s, x_{m-1}(s, x_0))\, ds\right) d\tau \Bigg] d\tau \\ & + \frac{t}{T}H\Bigg\{ d - (A + B + C)x_0 - B\int_0^{t_1} \Bigg[f\left(\tau, x_{m-1}(\tau, x_0), \int_0^\tau \varphi(\tau, s, x_{m-1}(s, x_0))\, ds\right) \\ & - \frac{1}{T}\int_0^T f\left(\tau, x_{m-1}(\tau, x_0), \int_0^\tau \varphi(\tau, s, x_{m-1}(s, x_0))\, ds\right) d\tau \Bigg] d\tau \Bigg\}, \end{aligned} \tag{2.8}$$

where $m = 1, 2, \ldots$, $x_0(t, x_0) = x_0$.

Every function of the sequence (2.8) satisfies the boundary conditions (2.2).

The sequence of functions $x_m(t,x_0)$ defined by (2.8) converges, as stated in the following theorem.

Theorem 2.1. *Suppose that conditions 1) – 4) hold for the boundary value problem (2.1), (2.2). Then the sequence of functions $x_m(t,x_0)$ of the form (2.8), satisfying the boundary conditions (2.2), converges uniformly as $m\to\infty$ on the domain $(t,x_0)\in[0,T]\times D_\beta$ to a limit function $x^*(t,x_0)$. Moreover, the function $x^*(t,x_0)$ passing through the point $x^*(0,x_0)=x_0$ at $t=0$ is a solution of the integral equation*

$$\begin{aligned}x(t,x_0)=x_0+\int_0^t\Bigg[f\left(\tau,x(\tau,x_0),\int_0^\tau\varphi(\tau,s,x(s,x_0))\,ds\right)\\ -\frac{1}{T}\int_0^T f\left(\tau,x(\tau,x_0),\int_0^\tau\varphi(\tau,s,x(s,x_0))\,ds\right)d\tau\Bigg]d\tau\\ +\frac{t}{T}H\Bigg\{d-(A+B+C)x_0-B\int_0^{t_1}\Bigg[f\left(\tau,x(\tau,x_0),\int_0^\tau\varphi(\tau,s,x(s,x_0))\,ds\right)\\ -\frac{1}{T}\int_0^T f\left(\tau,x(\tau,x_0),\int_0^\tau\varphi(\tau,s,x(s,x_0))\,ds\right)d\tau\Bigg]d\tau\Bigg\}\end{aligned}\tag{2.9}$$

and, in addition, $x^*(t,x_0)$ satisfies the boundary conditions (2.2), that is, it is a solution of the boundary value problem

$$\frac{dx}{dt}=f\left(t,x,\int_0^t\varphi(t,s,x(s))\,ds\right)+\Delta(x_0),$$

$$Ax(0)+Bx(t_1)+Cx(T)=d,$$

where

$$\begin{aligned}\Delta(x_0)=\frac{1}{T}H\Bigg\{d-(A+B+C)x_0-B\int_0^{t_1}\Bigg[f\left(t,x(t,x_0),\int_0^t\varphi(t,s,x(s,x_0))\,ds\right)\\ -\frac{1}{T}\int_0^T f\left(\tau,x(\tau,x_0),\int_0^\tau\varphi(\tau,s,x(s,x_0))\,ds\right)d\tau\Bigg]dt\Bigg\}\\ -\frac{1}{T}\int_0^T f\left(t,x(t,x_0),\int_0^t\varphi(t,s,x(s,x_0))\,ds\right)dt.\end{aligned}$$

For the deviation of $x^*(t,x_0)$ from $x_m(t,x_0)$, the following estimate is true for all $m=1,2,\dots$

$$|x^*(t,x_0)-x_m(t,x_0)|\le Q^m(E-Q)^{-1}\beta(x_0),\tag{2.10}$$

where E is the identity matrix, $Q=\dfrac{T}{2}(Q_1+Q_2)$, and Q_1,Q_2 are defined in (2.6).

Proof. We first show that if $x_0 \in D_\beta$, then all functions $x_m(t, x_0)$ belong to D. Indeed, using Lemma 2.1 in [5, p. 29], relation (2.8) implies

$$
\begin{aligned}
|x_1(t,x_0)-x_0| \le & \left| \int_0^t \left[f\left(\tau, x_0, \int_0^\tau \varphi(\tau,s,x_0)\,ds\right)\right.\right. \\
& \left.\left. -\frac{1}{T}\int_0^T f\left(\tau, x_0, \int_0^\tau \varphi(\tau,s,x_0)\,ds\right) d\tau \right] d\tau \right| \\
& + |H[d-(A+B+C)x_0]| + |HB| \left| \int_0^{t_1} \left[f\left(\tau, x_0, \int_0^\tau \varphi(\tau,s,x_0)\,ds\right)\right.\right. \\
& \left.\left. -\frac{1}{T}\int_0^T f\left(\tau, x_0, \int_0^\tau \varphi(\tau,s,x_0)\,ds\right) d\tau \right] d\tau \right| \\
& \le \left(1-\frac{t}{T}\right)\int_0^t \left| f\left(\tau, x_0, \int_0^\tau \varphi(\tau,s,x_0)\,ds\right)\right| d\tau \\
& + \frac{t}{T}\int_t^T \left| f\left(\tau, x_0, \int_0^\tau \varphi(\tau,s,x_0)\,ds\right)\right| d\tau \\
& + |H[d-(A+B+C)x_0]| \\
& + |HB| \left\{ \left(1-\frac{t_1}{T}\right)\int_0^{t_1} \left| f\left(\tau, x_0, \int_0^\tau \varphi(\tau,s,x_0)\,ds\right)\right| d\tau \right. \\
& \left. + \frac{t_1}{T}\int_{t_1}^T \left| f\left(\tau, x_0, \int_0^\tau \varphi(\tau,s,x_0)\,ds\right)\right| d\tau \right\} \\
& \le \alpha_1(t)M + |H[d-(A+B+C)x_0]| \\
& + |HB|\alpha_1(t_1)M = \alpha_1(t)M + \beta_1(x_0), \qquad (2.11)
\end{aligned}
$$

where $\alpha_1(t) = 2t\left(1-\dfrac{t}{T}\right) \le \dfrac{T}{2}$, $t \in [0,T]$.

It follows that if $x_0 \in D_\beta$, then $x_1(t,x_0) \in D$. By mathematical induction, it can be shown that, for all $m = 1,2,\ldots$, $t \in [0,T]$, and every $x_0 \in D_\beta$, all functions $x_m(t,x_0)$ belong to D.

To establish convergence of the sequence $x_m(t,x_0)$ as $m \to \infty$, we use the Cauchy criterion. Therefore, it is necessary to estimate the difference $x_{m+k}(t,x_0) - x_m(t,x_0)$ for an arbitrary natural number $k \ge 1$.

By (2.4), (2.8), and Lemma 2.1 in [5, p. 29], for all $m \geq 1$ and $t \in [0,T]$ we obtain the inequality

$$\begin{aligned}
|x_{m+1}(t,x_0) - x_m(t,x_0)| \leq K_1 &\left[\left(1 - \frac{t}{T}\right) \int_0^t |x_m(\tau,x_0) - x_{m-1}(\tau,x_0)|d\tau \right. \\
&\left. + \frac{t}{T} \int_t^T |x_m(\tau,x_0) - x_{m-1}(\tau,x_0)|d\tau \right] \\
&+ K_2K_3 \left[\left(1 - \frac{t}{T}\right) \int_0^t \int_0^\tau |x_m(s,x_0) - x_{m-1}(s,x_0)|dsd\tau \right. \\
&\left. + \frac{t}{T} \int_t^T \int_0^\tau |x_m(s,x_0) - x_{m-1}(s,x_0)|dsd\tau \right] \\
&+ GK_1 \left[\left(1 - \frac{t_1}{T}\right) \int_0^{t_1} |x_m(\tau,x_0) - x_{m-1}(\tau,x_0)|d\tau \right. \\
&\left. + \frac{t_1}{T} \int_{t_1}^T |x_m(\tau,x_0) - x_{m-1}(\tau,x_0)|d\tau \right] \\
&+ GK_2K_3 \left[\left(1 - \frac{t_1}{T}\right) \int_0^{t_1} \int_0^\tau |x_m(s,x_0) - x_{m-1}(s,x_0)|dsd\tau \right. \\
&\left. + \frac{t_1}{T} \int_{t_1}^T \int_0^\tau |x_m(s,x_0) - x_{m-1}(s,x_0)|dsd\tau \right]. \qquad (2.12)
\end{aligned}$$

Using the inequality $|x_1(t,x_0) - x_0| \leq \alpha_1(t)M + \beta_1(x_0)$ from (2.11) and the fact that $\alpha_1(t) \leq \dfrac{T}{2}$ for all $t \in [0,T]$, we can uniformly bound the difference by:

$$|x_1(t,x_0) - x_0| \leq \frac{T}{2}M + \beta_1(x_0) = \beta(x_0).$$

Substituting this bound into the right-hand side of (2.12) for $m = 1$, we obtain

$$\begin{aligned}
|x_2(t,x_0) - x_1(t,x_0)| &\leq K_1\beta(x_0)\alpha_1(t) + \frac{3T}{4}K_2K_3\beta(x_0)\alpha_1(t) \\
&\quad + GK_1\beta(x_0)\alpha_1(t_1) + \frac{3T}{4}GK_2K_3\beta(x_0)\alpha_1(t_1) \\
&= \left(K_1 + \frac{3T}{4}K_2K_3\right)\beta(x_0)\alpha_1(t) + G\left(K_1 + \frac{3T}{4}K_2K_3\right)\beta(x_0)\alpha_1(t_1) \\
&\leq \frac{T}{2}\left(K_1 + \frac{3T}{4}K_2K_3\right)\beta(x_0) + \frac{T}{2}G\left(K_1 + \frac{3T}{4}K_2K_3\right)\beta(x_0) \\
&= \frac{T}{2}(Q_1 + GQ_1)\beta(x_0) = Q\beta(x_0),
\end{aligned}$$

where $Q = \dfrac{T}{2}(Q_1 + Q_2)$, $Q_1 = K_1 + \dfrac{3T}{4}K_2K_3$, and $Q_2 = GQ_1$.

Proceeding by mathematical induction, we establish the estimate

$$|x_{m+1}(t,x_0) - x_m(t,x_0)| \leq Q^m\beta(x_0).$$

Then, from

$$x_{m+k}(t,x_0)-x_m(t,x_0)=\sum_{i=0}^{k-1}\big(x_{m+k-i}(t,x_0)-x_{m+k-i-1}(t,x_0)\big),$$

we obtain

$$|x_{m+k}(t,x_0)-x_m(t,x_0)|\le\sum_{i=0}^{k-1}Q^{m+i}\beta(x_0). \tag{2.13}$$

By condition (2.5), the corresponding matrix series converges, allowing us to majorize the finite sum by the infinite geometric series:

$$\sum_{i=0}^{k-1}Q^{m+i}=Q^m\sum_{i=0}^{k-1}Q^i\le Q^m\sum_{i=0}^{\infty}Q^i=Q^m(E-Q)^{-1}.$$

Furthermore, condition (2.5) directly implies that

$$\lim_{m\to\infty}Q^m=0. \tag{2.14}$$

Relations (2.13) and (2.14) imply that, as $m\to\infty$, the sequence $x_m(t,x_0)$ converges uniformly in the domain $(t,x_0)\in[0,T]\times D_\beta$ to a limit function $x^*(t,x_0)$:

$$\lim_{m\to\infty}x_m(t,x_0)=x^*(t,x_0). \tag{2.15}$$

Since every function $x_m(t,x_0)$ of the sequence (2.8) satisfies the boundary conditions (2.2), the limit function $x^*(t,x_0)$ also satisfies these boundary conditions.

Taking the limit as $k\to\infty$ in (2.13) and using the fact that non-strict inequalities are preserved in the limit, we arrive at the estimate (2.10) for the deviation of the limit function $x^*(t,x_0)$ from $x_m(t,x_0)$, valid for all $m=1,2,\ldots$.

Next, we pass to the limit as $m\to\infty$ in (2.8). By virtue of the uniform convergence established in (2.15) and the continuity of the functions f and φ, the limit can be taken under the integral signs. This shows that the limit function $x^*(t,x_0)$ is a solution of the integral equation (2.9). Moreover, this solution passes through the point $x^*(0,x_0)=x_0$ at $t=0$ and satisfies the boundary conditions (2.2). □

The following theorem establishes the necessary and sufficient conditions under which the limit function $x^*(t,x_0)$ of the successive approximation sequence (2.8) constitutes a solution of the original boundary value problem (2.1), (2.2).

Theorem 2.2. *Assume that all conditions of Theorem 2.1 are satisfied. Then the function $x=x^*(t)$, which is a solution of system (2.1) with initial value $x^*(0)=x_0$ at $t=0$, is also a solution of the boundary value problem (2.1), (2.2)*

if and only if the determining function $\Delta(x_0)$ *vanishes at* x_0*:*

$$\Delta(x_0) = \frac{1}{T}H\Bigg\{d - (A+B+C)x_0 - B\int_0^{t_1}\Bigg[f\left(t, x^*(t,x_0), \int_0^t \varphi(t,s,x^*(s,x_0))\,ds\right) - \frac{1}{T}\int_0^T f\left(\tau, x^*(\tau,x_0), \int_0^\tau \varphi(\tau,s,x^*(s,x_0))\,ds\right)d\tau\Bigg]dt\Bigg\} - \frac{1}{T}\int_0^T f\left(t, x^*(t,x_0), \int_0^t \varphi(t,s,x^*(s,x_0))\,ds\right)dt = 0, \tag{2.16}$$

where $x^*(t,x_0)$ *is the limit function of the sequence* (2.8)*. In this case* $x^*(t) = x^*(t,x_0)$*, and the deviation of* $x^*(t)$ *from the approximate solution* $x_m(t,x_0)$ *defined by* (2.8) *is bounded by estimate* (2.10)*.*

Proof. By Theorem 2.1, the limit function $x^*(t,x_0)$ of the successive approximation sequence (2.8) satisfies the integral equation (2.9). However, the original integro-differential equation (2.1) is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t f\left(\tau, x(\tau), \int_0^\tau \varphi(\tau,s,x(s))\,ds\right)d\tau.$$

Comparing these two representations, we observe that $x^*(t,x_0)$ is a solution of the original boundary value problem (2.1), (2.2) if and only if the determining function $\Delta(x_0)$, defined by (2.16), vanishes. Thus, solving the boundary value problem (2.1), (2.2) reduces to finding an initial value x_0 such that $\Delta(x_0) = 0$. □

3. Sufficient conditions for the existence of a solution of the boundary value problem

To investigate the existence of a solution to the boundary value problem (2.1), (2.2), we introduce, alongside the exact determining equation (2.16), the approximate determining equation

$$\Delta_m(x_0) = \frac{1}{T}H\Bigg\{d - (A+B+C)x_0 - B\int_0^{t_1}\Bigg[f\left(t, x_m(t,x_0), \int_0^t \varphi(t,s,x_m(s,x_0))\,ds\right) - \frac{1}{T}\int_0^T f\left(\tau, x_m(\tau,x_0), \int_0^\tau \varphi(\tau,s,x_m(s,x_0))\,ds\right)d\tau\Bigg]dt\Bigg\} - \frac{1}{T}\int_0^T f\left(t, x_m(t,x_0), \int_0^t \varphi(t,s,x_m(s,x_0))\,ds\right)dt = 0, \tag{3.1}$$

where $x_m(t,x_0)$ denotes the m-th approximate solution defined by (2.8).

This leads to the natural question: how can one use the approximate determining equation (3.1) to locate the zeros of the exact determining function $\Delta(x_0)$ and thereby establish the existence of solutions to the boundary value problem (2.1), (2.2)?

To address this question, we first estimate the difference between $\Delta(x_0)$ and $\Delta_m(x_0)$. For $m \geq 1$, by applying equations (2.16) and (3.1), the Lipschitz condition (2.4), and the error estimate (2.10), we obtain

$$\begin{aligned}
|\Delta(x_0) - \Delta_m(x_0)| \leq \frac{G}{T}\Bigg\{ & \left(1 - \frac{t_1}{T}\right) \int_0^{t_1} \Bigg[K_1 |x^*(t,x_0) - x_m(t,x_0)| \\
& + K_2 K_3 \int_0^t |x^*(s,x_0) - x_m(s,x_0)| ds \Bigg] dt \\
& + \frac{t_1}{T} \int_{t_1}^{T} \Bigg[K_1 |x^*(t,x_0) - x_m(t,x_0)| \\
& + K_2 K_3 \int_0^t |x^*(s,x_0) - x_m(s,x_0)| ds \Bigg] dt \Bigg\} \\
& + \frac{1}{T} \int_0^T \Bigg[K_1 |x^*(t,x_0) - x_m(t,x_0)| \\
& + K_2 K_3 \int_0^t |x^*(s,x_0) - x_m(s,x_0)| ds \Bigg] dt \\
& \leq \frac{G}{T} \left[\left(K_1 + \frac{3T}{4} K_2 K_3 \right) \alpha_1(t_1) Q^m (E - Q)^{-1} \beta(x_0) \right] \\
& + \frac{1}{T} \left(K_1 + \frac{T}{2} K_2 K_3 \right) Q^m (E - Q)^{-1} \beta(x_0) \\
& = \left(\frac{1}{2} Q_2 + Q_3 \right) Q^m (E - Q)^{-1} \beta(x_0), \qquad (3.2)
\end{aligned}$$

where $Q_3 = K_1 + \dfrac{T}{2} K_2 K_3$.

It follows from inequality (3.2) that the sequence $\Delta_m(x_0)$ converges to $\Delta(x_0)$ as $m \to \infty$.

Based on this result and applying the approach developed in [5] to the present problem, we arrive at the following theorem, which establishes the conditions for the existence of solutions to the boundary value problem (2.1), (2.2).

Theorem 3.1. *Assume that all conditions of Theorem 2.1 are satisfied. Assume additionally that:*

1. For some fixed $m \geq 1$, there exists a closed convex domain $D_2 \subset D_\beta$ in which the approximate determining equation (3.1) has a unique solution $x_0 = x_{0m}$ of nonzero index;

2. On the boundary Γ_{D_2} of D_2, the following estimate holds:

$$\inf_{x_0 \in \Gamma_{D_2}} |\Delta_m(x_0)| \geq \left(\frac{1}{2} Q_2 + Q_3 \right) Q^m (E - Q)^{-1} \beta(x_0). \qquad (3.3)$$

Then the boundary value problem (2.1), (2.2) has a solution $x = x^*(t)$ such that $x^*(0) = x_0^* \in D_2$.

Proof. The proof consists in showing that $\Delta(x_0)$ has a zero in D_2. Using estimates (3.2) and (3.3), the argument follows the same lines as the proof of [5, Theorem 3.1]. □

4. Example

To illustrate the proposed method, consider the system of integro-differential equations

$$\begin{cases} \dot{x}_1 = 0.05x_2 - 0.2\displaystyle\int_0^t x_1(s)\,ds + 0.1, \\ \dot{x}_2 = 0.05x_1 + 0.5x_2^2 - 0.02t^4 + 0.395t, \end{cases} \tag{4.1}$$

defined on the interval $[0, 1]$ within the domain

$$D: \ |x_1| \le 0.5, \qquad |x_2| \le 0.5, \tag{4.2}$$

subject to the boundary conditions

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(0.5) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(1) = \begin{bmatrix} 0.2 \\ 0.05 \end{bmatrix}. \tag{4.3}$$

The structural matrices and parameters are identified as

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad d = \begin{bmatrix} 0.2 \\ 0.05 \end{bmatrix}, \quad t_1 = 0.5,$$

$$\det(t_1 B + TC) \neq 0, \qquad H = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

We can readily verify that the Lipschitz and bounding constants are given by

$$M = \begin{bmatrix} 0.225 \\ 0.565 \end{bmatrix}, \qquad K_1 = \begin{bmatrix} 0 & 0.05 \\ 0.05 & 0.5 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad K_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.15 & 0.05 \\ 0.05 & 0.5 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0.225 & 0.075 \\ 0.025 & 0.25 \end{bmatrix},$$

$$\beta(x_0) = \begin{bmatrix} 0.3375 + |0.1 - 2x_{01} - 2x_{02}| \\ 0.2825 + |0.2 - x_{01} - x_{02}| \end{bmatrix} = \begin{bmatrix} \beta^{(1)}(x_0) \\ \beta^{(2)}(x_0) \end{bmatrix}.$$

With these parameters, the domain D_β is defined by

$$D_\beta: \ |x_1| \le 0.5 - \beta^{(1)}(x_0), \qquad |x_2| \le 0.5 - \beta^{(2)}(x_0), \tag{4.4}$$

and the convergence condition $\lambda(Q) < 1$ is strictly satisfied.

From (4.4) it follows that D_β is nonempty; specifically, it contains all points x_0 satisfying

$$\beta^{(1)}(x_0) \le 0.5, \qquad \beta^{(2)}(x_0) \le 0.5,$$

and $\lambda(Q) < 1$ implies that condition (2.5) is satisfied.

Consequently, the proposed numerical-analytic scheme is applicable to the boundary value problem (4.1), (4.3) within the domain (4.2).

To determine the approximate initial vector $x_0 = (x_{01}, x_{02})$, we solve the zeroth-order approximate determining equation, $\Delta_0(x_0) = 0$, which corresponds to (3.1) with $m = 0$. This yields the following values for the components:

$$x_{01} = -0.0164, \qquad x_{02} = 0.0156.$$

Substituting this vector into the recurrence relation (2.8) yields the first approximation

$$x_1(t, x_0) = \begin{bmatrix} x_{01}(1 - 1.95t - 0.1t^2) - 2x_{02}t + 0.1t \\ -x_{01}t + x_{02}(1 - t) + 0.004t^5 + 0.1975t^2 - 0.0015t \end{bmatrix}.$$

It can be directly verified that the boundary value problem (4.1), (4.3) admits the exact solution

$$x^*(t) = \begin{bmatrix} 0.1t \\ 0.2t^2 \end{bmatrix}. \tag{4.5}$$

Further numerical computations show that the sequence of approximate solutions generated by (2.8) converges to the exact solution (4.5).

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