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WEIGHTED STEIN–WEISS INEQUALITIES ON GRAND HERZ–MORREY–HARDY SPACES

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Abstract. Motivated by recent Stein–Weiss inequalities on variable Lebesgue spaces and on Herz spaces with variable exponent, we utilize a scale of function spaces, the weighted grand variable Herz–Morrey–Hardy spaces, obtained by equipping the underlying space with a Muckenhoupt-type weight $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$ instead of a pure power weight. On this scale we establish a ball-membership lemma and the corresponding Stein–Weiss inequalities for the fractional integral operator. The hypotheses are explicit and include a sharp threshold for the Morrey parameter. The results extend known estimates on Herz spaces, variable Lebesgue spaces, and classical Lebesgue spaces.

Keywords. Stein–Weiss inequality, Herz space, variable exponent, Morrey space, grand maximal function, Hardy space, Muckenhoupt weight, fractional integral operator
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1. Introduction and preliminaries

The classical Stein–Weiss inequality, established by Stein and Weiss [1], gives the boundedness of the fractional integral operator between power-weighted Lebesgue spaces and has served for decades as a model result at the interface of harmonic analysis and weighted-norm theory. For $\alpha \in (0, n)$ and locally integrable f, the fractional integral operator I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \tag{1.1}$$

and the original inequality reads as follows.

Theorem 1.1 ([1]). *Let $n \geq 1$, $\alpha \in (0, n)$, $1 < p < \infty$, $a < n/p'$, $b < n/q$, $a + b \geq 0$, and let $1/q = 1/p + (a + b - \alpha)/n$. If $p \leq q < \infty$, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^{q,-b}} \leq C \|f\|_{L^{p,a}} \qquad \forall f \in L^{p,a}.$$

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Muckenhoupt and Wheeden [2] extended such weighted estimates to general A_p-type weights for fractional integrals, and Pérez [3] obtained sharp two-weight inequalities for potential and fractional maximal operators, laying the groundwork for the systematic weighted theory used throughout this article; weighted Stein–Weiss inequalities for fractional singular integral operators have recently been obtained on Fofana spaces as well [4]. Herz introduced the spaces that bear his name in [5] to study Lipschitz classes and absolutely convergent Fourier series; Lu, Yang, and collaborators subsequently developed a full Herz-type theory, including Hardy–Littlewood–Sobolev-type boundedness of fractional integration on Herz-type spaces [17], boundedness of sublinear operators on weighted Herz-type spaces [18], and a systematic account of Herz-type spaces and their applications in [20]. Bennett and Sharpley's monograph [6] supplies the Banach function space framework in which such scales are naturally studied.

A second line of generalization replaces the fixed exponent p by a measurable exponent function $p(\cdot)$. The foundational theory of variable exponent Lebesgue spaces is due to Cruz-Uribe, Fiorenza, and Neugebauer [7], who established boundedness of the Hardy–Littlewood maximal operator under a log-Hölder continuity condition; Diening's boundedness criterion [9] and the comprehensive monograph of Diening, Harjulehto, Hästö, and Růžička [10] organize this theory, while the book of Cruz-Uribe and Fiorenza [11] gives a complete, self-contained treatment. Weighted norm inequalities on variable Lebesgue spaces, with the Muckenhoupt-type class $\mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$, were developed by Cruz-Uribe, Fiorenza, and Neugebauer [8]. Almeida and Drihem [13] generalized Herz spaces themselves to variable exponents, giving the characterization on which the ball-testing arguments of the present article rely. Izuki [14] and Izuki and Noi [15] established boundedness of sublinear operators and fractional integrals on (weighted) Herz spaces with variable exponent, and Sawano, Ho, Yang, and Yang [21] placed such scales inside the unifying framework of ball quasi-Banach function spaces and their associated Hardy spaces. Campanato-type methods have likewise been applied to the multidimensional Katugampola operator [16].

These two threads meet on the Herz scale in the extrapolation theory of Ho for Herz spaces with variable exponents [24], in the boundedness of sublinear operators on grand Herz–Hardy spaces with variable exponent obtained by Shabbir and Zaighum [25], and, most directly relevant here, in the Stein–Weiss inequalities on Herz spaces with variable exponent due to Ho [27], which we recall below. In its statement, $\beta(\cdot)$ is assumed to be log-Hölder continuous at infinity, that is, $|\beta(x) - \beta_\infty| \le C/\log(e + |x|)$ for some $\beta_\infty \in \mathbb{R}$ (condition (2.3) of [27]), and a, b are required to satisfy the compatibility conditions (2.6)–(2.7) of [27] linking $a-b$, α, $p(\cdot)$ and $q(\cdot)$. We refer to these as conditions (2.3) and (2.6)–(2.7) throughout.

Theorem 1.2 ([27, Theorem 3.1])**.** *Let $\alpha \in (0,n)$, $\theta \in (0,\infty)$, $\beta(\cdot) \in L^\infty(\mathbb{R}^n)$, $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p(x) \le q(x)$ $\forall x \in \mathbb{R}^n$, $1 < p_- \le p_+ < \infty$, and $1 < q_- \le q_+ < \infty$. If $a, b, p(\cdot), q(\cdot)$ satisfy (2.6)–(2.7), β satisfies (2.3), and*

$$-\frac{n}{p_+} + \alpha - b < \beta_\infty < n - \frac{n}{p_-} - b,$$

then there is a constant $C > 0$ *such that, for any* $f \in K^{\beta(\cdot),b}_{p(\cdot),\theta}$,

$$\|I_\alpha f\|_{K^{\beta(\cdot),a}_{q(\cdot),\theta}} \le C\|f\|_{K^{\beta(\cdot),b}_{p(\cdot),\theta}}.$$

Cruz-Uribe and Suragan [28] established Hardy–Leray-type Stein–Weiss inequalities directly on variable Lebesgue spaces, which is the Lebesgue-level ingredient (Theorem 2.6 of [27]) on which Theorem 1.2 itself is built. A third, independent line of generalization introduces Morrey-type growth control together with a grand, Hardy-space layer built from the Fefferman–Stein grand maximal function [29]; Sultan, Hussain, and Sultan [30] carry out this construction for the Herz scale with variable exponent, introducing the *grand variable Herz–Morrey–Hardy spaces* and establishing the boundedness of a range of integral operators on them. Potential-type operators on grand variable Herz spaces were studied in [26].

We mainly take the idea of the underlying space from Sultan, Hussain, and Sultan [30] and make it new by incorporating a genuine Muckenhoupt-type weight $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$, so that our Definition 1.12 below, although built on the scale of [30], is itself new. Neither Theorem 1.2 nor the spaces of [30] carry a genuine weight w; conversely, the weighted theory of Muckenhoupt and Wheeden [2] and Pérez [3] carries no Herz, Morrey, or grand/Hardy structure.

The combination of these structures is not merely formal. Muckenhoupt-type weights describe degenerate or singular coefficients and arise in duality and extrapolation arguments; Morrey-type growth control measures local regularity, which is exactly what fractional integrals improve (see [19] for a discrete counterpart on weighted Morrey sequence spaces); the grand (Iwaniec–Sbordone-type) layer [12] captures the limiting situations in which the exponent cannot be kept fixed; and the Hardy layer, defined through the grand maximal function, is the natural setting for exponents below the Banach range and for cancellation phenomena, while smoothness of potentials can also be measured on Besov scales [22]. Each of these structures has been treated separately for the Stein–Weiss problem, and the purpose of this article is to identify the conditions under which they can be combined. Admitting a genuine weight w forces an additional compatibility hypothesis ($wv_b \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$) absent from the power-weight-only theory, and admitting the Morrey parameter $\Gamma > 0$ forces a genuinely new threshold condition, $0 \le \Gamma < a + \beta + n/\gamma_2$, governing exactly how much local growth the fractional integral operator can tolerate before boundedness fails. Both phenomena are invisible at $w \equiv 1$, $\Gamma = 0$.

Throughout, C is a positive constant independent of the main parameters, not necessarily the same at each occurrence; $\mathcal{B}(x,R) = \{y \in \mathbb{R}^n : |x-y| < R\}$; $|S|$ is Lebesgue measure; $\psi \sim \mu$ means $C^{-1}\mu \le \psi \le C\mu$.

1.1. **Variable exponent Lebesgue spaces and weights.** For measurable $\mathrm{p}(\cdot) : \mathbb{R}^n \to [1,\infty)$, the variable exponent Lebesgue space $L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ consists of all measurable ψ with $\int_{\mathbb{R}^n} |\psi(x)/s|^{\mathrm{p}(x)}dx < \infty$ for some $s > 0$, equipped with

the Luxemburg norm

$$\|\psi\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)} = \inf\Big\{s > 0 : \int_{\mathbb{R}^n} \Big|\frac{\psi(x)}{s}\Big|^{\mathrm{p}(x)} dx \le 1\Big\};$$

it is a Banach function space [11]. We write $\mathrm{p}^- = \operatorname{ess\,inf} \mathrm{p}$, $\mathrm{p}^+ = \operatorname{ess\,sup} \mathrm{p}$, and $P(\mathbb{R}^n) := \{\mathrm{p}(\cdot) : 1 < \mathrm{p}^- \le \mathrm{p}^+ < \infty\}$. An exponent $\mathrm{p}(\cdot)$ is locally log-Hölder continuous if $|\mathrm{p}(x) - \mathrm{p}(y)| \le C/\log(e + 1/|x-y|)$, and log-Hölder continuous at infinity if $\mathrm{p}(\infty) := \lim_{|x|\to\infty} \mathrm{p}(x)$ exists and $|\mathrm{p}(x) - \mathrm{p}(\infty)| \le C/\log(e+|x|)$. The class $P^{\log}(\mathbb{R}^n) = C^{\log}(\mathbb{R}^n) \cap P(\mathbb{R}^n)$ consists of the exponents in $P(\mathbb{R}^n)$ satisfying both conditions; it is stable under $\mathrm{p} \mapsto \mathrm{p}' = \mathrm{p}/(\mathrm{p}-1)$, and, by Diening's criterion [9, 10], the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ for every $\mathrm{p}(\cdot) \in P^{\log}(\mathbb{R}^n)$.

Definition 1.3 ([8])**.** Let $\mathrm{p}(\cdot) \in P(\mathbb{R}^n)$. An a.e. positive, locally integrable function w belongs to $\mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ if

$$[w]_{\mathcal{A}_{\mathrm{p}(\cdot)}} := \sup_{\mathcal{B}} \frac{1}{|\mathcal{B}|} \|w\chi_{\mathcal{B}}\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)} \|w^{-1}\chi_{\mathcal{B}}\|_{L^{\mathrm{p}'(\cdot)}(\mathbb{R}^n)} < \infty,$$

the supremum being taken over all balls $\mathcal{B}$. We set $\|\psi\|_{L^{\mathrm{p}(\cdot)}(w)} := \|\psi w\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)}$. Since this is a Luxemburg norm, $0 \le \psi \le \mu$ a.e. implies $\|\psi\|_{L^{\mathrm{p}(\cdot)}(w)} \le \|\mu\|_{L^{\mathrm{p}(\cdot)}(w)}$.

The following standard facts are used below; see [10, Section 4.5], [11, Chapter 2] and [8].

Lemma 1.4. *Let $\mathrm{p}(\cdot) \in P^{\log}(\mathbb{R}^n)$ and $w \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$.*

(a) For $\psi \in L^{\mathrm{p}(\cdot)}(w)$ and $\mu w^{-1} \in L^{\mathrm{p}'(\cdot)}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} |\psi\mu|\,dx \le (1 + 1/\mathrm{p}^- - 1/\mathrm{p}^+)\|\psi\|_{L^{\mathrm{p}(\cdot)}(w)}\|\mu\|_{L^{\mathrm{p}'(\cdot)}(w^{-1})}$; with $w \equiv 1$ this is the generalized Hölder inequality.

(b) $\|\chi_{\mathcal{B}}\|_{L^{\mathrm{p}(\cdot)}} \sim |\mathcal{B}|^{1/\mathrm{p}(x)}$ $(x \in \mathcal{B})$ if $|\mathcal{B}| \le 2^n$, and $\|\chi_{\mathcal{B}}\|_{L^{\mathrm{p}(\cdot)}} \sim |\mathcal{B}|^{1/\mathrm{p}(\infty)}$ if $|\mathcal{B}| \ge 1$.

(c) $\mathcal{M}$ is bounded on $L^{\mathrm{p}(\cdot)}(w)$, and $w^{-1} \in \mathcal{A}_{\mathrm{p}'(\cdot)}(\mathbb{R}^n)$.

(d) There are $C \ge 1$ and $\delta_1, \delta_2 \in (0,1)$ such that, for all balls $\mathcal{B}$ and measurable $S \subset \mathcal{B}$,

$$\frac{\|\chi_{\mathcal{B}}\|_{L^{\mathrm{p}(\cdot)}(w)}}{\|\chi_S\|_{L^{\mathrm{p}(\cdot)}(w)}} \le C\frac{|\mathcal{B}|}{|S|}, \quad \frac{\|\chi_S\|_{L^{\mathrm{p}(\cdot)}(w)}}{\|\chi_{\mathcal{B}}\|_{L^{\mathrm{p}(\cdot)}(w)}} \le C\Big(\frac{|S|}{|\mathcal{B}|}\Big)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{\mathrm{p}'(\cdot)}(w^{-1})}}{\|\chi_{\mathcal{B}}\|_{L^{\mathrm{p}'(\cdot)}(w^{-1})}} \le C\Big(\frac{|S|}{|\mathcal{B}|}\Big)^{\delta_2},$$

and $|\mathcal{B}|^{-1}\|\chi_{\mathcal{B}}\|_{L^{\mathrm{p}(\cdot)}(w)}\|\chi_{\mathcal{B}}\|_{L^{\mathrm{p}'(\cdot)}(w^{-1})} \le C$.

1.2. **Grand variable Herz–Morrey–Hardy spaces.** Let $\mathcal{B}_\kappa = \{x : |x| \le 2^\kappa\}$, $\mathcal{A}_\kappa = \mathcal{B}_\kappa \setminus \mathcal{B}_{\kappa-1}$, $\chi_\kappa = \chi_{\mathcal{A}_\kappa}$, $\kappa \in \mathbb{Z}$. For $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ log-Hölder continuous at $0, \infty$, with $\alpha(0) = \lim_{x\to 0} \alpha(x)$, $\alpha_\infty = \lim_{|x|\to\infty} \alpha(x)$, set

$$\phi(\kappa) := \begin{cases} \alpha(0), & \kappa \le 0, \\ \alpha_\infty, & \kappa > 0, \end{cases}$$

the standard case [14, 15] replacing $2^{\kappa\alpha(x)}$, $x \in \mathcal{A}_\kappa$, by $2^{\kappa\phi(\kappa)}$ up to a fixed constant.

Definition 1.5. For $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $0 < \upsilon < \infty$, $\mathrm{p}(\cdot) \in P(\mathbb{R}^n)$, $0 \le \Gamma < \infty$, $w \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$,

$$\|\psi\|_{M\dot{\mathcal{K}}^{\alpha(\cdot),\upsilon}_{\Gamma,\mathrm{p}(\cdot)}(w)} := \sup_{\vartheta>0}\sup_{m_0\in\mathbb{Z}} 2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0} 2^{\kappa\phi(\kappa)\upsilon(1+\vartheta)}\|\psi\chi_\kappa\|^{\upsilon(1+\vartheta)}_{L^{\mathrm{p}(\cdot)}(w)}\right)^{1/(\upsilon(1+\vartheta))}.$$

Definition 1.6 ([29])**.** For $N > n+1$, set $\varphi_\tau(x) = \tau^{-n}\varphi(x/\tau)$ and

$$\mathfrak{A}_N = \Big\{\varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\gamma|,|\beta|\le N} |x^\gamma D^\beta \varphi(x)| \le 1\Big\};$$

then

$$\mathcal{G}_N\psi(x) := \sup_{\varphi\in\mathfrak{A}_N}\sup_{|y-x|<\tau} |\varphi_\tau * \psi(y)|, \qquad \psi \in \mathcal{S}'(\mathbb{R}^n).$$

Definition 1.7.

$$HM\dot{\mathcal{K}}^{\alpha(\cdot),\upsilon)}_{\Gamma,\mathrm{p}(\cdot)}(w) := \Big\{\mu \in \mathcal{S}'(\mathbb{R}^n) : \|\mu\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),\upsilon)}_{\Gamma,\mathrm{p}(\cdot)}(w)} := \|\mathcal{G}_N\mu\|_{M\dot{\mathcal{K}}^{\alpha(\cdot),\upsilon)}_{\Gamma,\mathrm{p}(\cdot)}(w)} < \infty\Big\}.$$

Remark 1.8. For $w \equiv 1$, Definition 1.7 reduces to the grand variable Herz–Morrey–Hardy space of [30]; if moreover $\Gamma = 0$ and the grand parameter is frozen, one obtains the Herz-type Hardy spaces with variable exponent of [30], and, for constant α and p, the classical Herz-type Hardy spaces of [20]. For $\Gamma = 0$ and general w one obtains a weighted Herz-type Hardy space within the ball quasi-Banach framework of [21].

1.3. **Weighted grand variable Herz–Morrey–Hardy spaces.** We now equip the grand variable Herz–Morrey–Hardy scale of §1.2 with a Muckenhoupt-type weight $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$ and a power weight $v_b(x) = |x|^b$, following Ho's passage from $K^{\beta(\cdot)}_{p(\cdot),\theta}$ to $K^{\beta(\cdot),b}_{p(\cdot),\theta}$ in [27], but with the plain norm $\|\cdot\|_{L^{p(\cdot)}}$ replaced throughout by the weighted norm $\|\cdot\|_{L^{p(\cdot)}(w)}$ of Definition 1.3.

Definition 1.9. Let $b \in \mathbb{R}$, $\theta \in (0,\infty)$, $p(\cdot) \in P(\mathbb{R}^n)$, $\beta(\cdot) \in L^\infty(\mathbb{R}^n)$ satisfying (2.3), and $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$. The power-weighted Herz space with variable exponent relative to w is

$$K^{\beta(\cdot),b}_{p(\cdot),\theta}(w) := \Big\{\psi \in M(\mathbb{R}^n) : \|\psi\|_{K^{\beta(\cdot),b}_{p(\cdot),\theta}(w)} < \infty\Big\},$$

where

$$\|\psi\|_{K^{\beta(\cdot),b}_{p(\cdot),\theta}(w)} := \Big(\sum_{\kappa=0}^{\infty} 2^{\kappa\phi(\kappa)\theta}\big\|\psi\,\chi_\kappa\,v_b\big\|^\theta_{L^{p(\cdot)}(w)}\Big)^{1/\theta}.$$

When $w \equiv 1$ this is exactly the space $K^{\beta(\cdot),b}_{p(\cdot),\theta}$ of [27].

Assumption 1.10. Let $p(\cdot) \in P^{\log}(\mathbb{R}^n)$, $1 < p_- \le p_+ < \infty$, $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$. We say b is *admissible* for w if $-n/p_+ < b$ and $wv_b \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$. For $w \equiv 1$ this reduces to $v_b \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$, the variable-exponent analogue of the classical range $-n < b(p-1) < n$ for $|x|^b \in A_p$ studied by Muckenhoupt and Wheeden [2].

Lemma 1.11. *Let $p(\cdot) \in P^{\log}(\mathbb{R}^n)$, $1 < p_- \le p_+ < \infty$, $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$, and let b be admissible for w. Then there exist $C_0, C_1 > 0$ such that, for every $R > 0$,*

$$C_0R^b\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)} \le \|\chi_{B(0,R)}v_b\|_{L^{p(\cdot)}(w)} \le C_1R^b\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}.$$

Proof. Fix $R > 0$, $\kappa_R \in \mathbb{Z}$ with $2^{\kappa_R - 1} < R \le 2^{\kappa_R}$; split $B(0,R) = \bigcup_{i \le \kappa_R} (\mathcal{A}_i \cap B(0,R))$, on which $v_b \sim 2^{ib}$ pointwise. For the upper bound, subadditivity of $\|\cdot\|_{L^{p(\cdot)}(w)}^{\min(1,p_-)}$ together with the doubling/reverse-doubling estimates of Lemma 1.4(d) (applied to $\mathcal{A}_i \subset \mathcal{B}_i \subset \mathcal{B}_{\kappa_R}$) sums a geometric series in $i \to -\infty$, convergent precisely because $b > -n/p_+$ matches the sharp decay exponent $\delta_1 > 1/p_+$ of Lemma 1.4(d). For the lower bound, restrict to the top shell $i = \kappa_R$, of measure comparable to $|B(0,R)|$, and apply Lemma 1.4(d) once more. Admissibility of b for w is used so that $w v_b$ itself carries the same doubling package as w alone.

Definition 1.12. For $b \in \mathbb{R}$, $\upsilon \in (0,\infty)$, $\Gamma \ge 0$, $p(\cdot) \in P(\mathbb{R}^n)$, $\beta(\cdot) \in L^\infty(\mathbb{R}^n)$ satisfying (2.3), $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$, write β for β_∞ (by [27, Proposition 2.8], only β_∞ matters on the annuli $\kappa \ge 0$) and set

$$\|\psi\|_{K^{\beta,b,\upsilon)}_{\Gamma,p(\cdot)}(w)} := \sup_{\vartheta>0} \sup_{m_0 \ge 0} 2^{-m_0\Gamma} \Big(\sum_{\kappa=0}^{m_0} 2^{\kappa\beta\upsilon(1+\vartheta)} \big\| \psi\, \chi_\kappa v_b \big\|_{L^{p(\cdot)}(w)}^{\upsilon(1+\vartheta)} \Big)^{1/(\upsilon(1+\vartheta))},$$

$$\|\mu\|_{HK^{\beta,b,\upsilon)}_{\Gamma,p(\cdot)}(w)} := \|\mathcal{G}_N \mu\|_{K^{\beta,b,\upsilon)}_{\Gamma,p(\cdot)}(w)}.$$

At $\Gamma = 0$ this coincides with Definition 1.9 (Lemma 1.13 below); the family generalizes simultaneously the grand variable Herz–Morrey–Hardy spaces of Sultan, Hussain, and Sultan [30] (no weight w), the power-weighted Herz spaces of Ho [27] (no Morrey or grand/Hardy structure), and the classical Herz-type Hardy spaces of Lu, Yang, and Hu [20] (neither weight, variable exponent, Morrey, nor grand structure).

Lemma 1.13. *For $(c_\kappa)_{\kappa \ge 0} \subset [0,\infty)$ with $\sum_{\kappa=0}^\infty c_\kappa^\upsilon < \infty$,*

$$\sup_{\vartheta>0} \sup_{m_0 \ge 0} \Big(\sum_{\kappa=0}^{m_0} c_\kappa^{\upsilon(1+\vartheta)} \Big)^{1/(\upsilon(1+\vartheta))} = \Big(\sum_{\kappa=0}^{\infty} c_\kappa^{\upsilon} \Big)^{1/\upsilon}.$$

Proof. "$\ge$": monotone convergence as $\vartheta \to 0^+$, $m_0 \to \infty$. "$\le$": by $\sum c_\kappa^{\upsilon(1+\vartheta)} = \sum c_\kappa^{\upsilon\vartheta} c_\kappa^\upsilon \le (\sup_\kappa c_\kappa)^{\upsilon\vartheta} \sum c_\kappa^\upsilon$ and $a^p b^{1-p} \le \max(a,b)$ $(p \in (0,1))$ with $a = \sup_\kappa c_\kappa \le (\sum c_\kappa^\upsilon)^{1/\upsilon} = b$, the left side is $\le b$ for every ϑ, m_0.

The following statements collect the estimates on the weighted Lebesgue and ball norms that are used repeatedly in §2.

Assumption 1.14. There exist $\sigma_w \in \mathbb{R}$ and $C_2 > 0$ such that $\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)} \le C_2 R^{\sigma_w}$ for every $R \ge 1$; this is the weighted analogue of Lemma 1.4(b) (see [10, Corollary 4.5.9]), and $\sigma_w = n/p(\infty)$ when $w \equiv 1$.

Proposition 1.15. *Let $B = B(0,R)$ and $N > n+1$. Then $\mathcal{G}_N \chi_B \le C_N$ on $\mathbb{R}^n$; and, for $|x| \ge 2R$, $\mathcal{G}_N \chi_B(x) \le C_{n,N} R^n |x|^{-n}$.*

Proof. $\varphi \in \mathfrak{A}_N \Rightarrow \|\varphi\|_{L^1} \le C_N$, hence $|\varphi_\tau * \chi_B(y)| \le \|\varphi_\tau\|_{L^1} \|\chi_B\|_{L^\infty} \le C_N$: first bound. $\varphi \in \mathfrak{A}_N \Rightarrow |\varphi(x)| \le C_{n,N}(1+|x|)^{-N}$ (Schwartz seminorm equivalence), so $|\varphi_\tau(x)| \le C_{n,N} \tau^{N-n} (\tau + |x|)^{-N}$; for fixed $d > 0$, $\sup_{\tau>0} \tau^{N-n}(\tau+d)^{-N} \sim d^{-n}$

$(N-n>0)$. Thus

$$|\varphi_\tau * \chi_B(y)| \le C_{n,N}\tau^{N-n}\int_B (\tau+|y-z|)^{-N}dz$$
$$\le C_{n,N}R^n \sup_{z\in B}\tau^{N-n}(\tau+|y-z|)^{-N} \le C_{n,N}R^n \sup_{z\in B}|y-z|^{-n},$$

and $|y-z|\gtrsim |x|$ once $|y-x|<\tau$ and $|x|\ge 2R$, giving the second bound.

Lemma 1.16. *Let $\alpha\in(0,n)$, $p(\cdot),q(\cdot)\in C^{\log}(\mathbb{R}^n)$, $p\le q$, $1<p_-\le p_+<\infty$, $1<q_-\le q_+<\infty$, $w\in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)\cap\mathcal{A}_{q(\cdot)}(\mathbb{R}^n)$, and let a,b satisfy (2.6)–(2.7) with $wv_a\in\mathcal{A}_{q(\cdot)}(\mathbb{R}^n)$, $wv_b\in\mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$. Then there exists $C>0$ such that $\|I_\alpha f\|_{L^{q(\cdot)}(w,a)}\le C\|f\|_{L^{p(\cdot)}(w,b)}$, $f\in L^{p(\cdot)}(w,b)$, where $\|f\|_{L^{r(\cdot)}(w,c)}:=\|fv_cw\|_{L^{r(\cdot)}}$.*

Lemma 1.16 extends Theorem 1.1 and Theorem 2.6 of [27] (itself equivalent to [28]) by the same weighted Calderón–Zygmund-type argument used for the classical Stein–Weiss inequality [2, 3], once Definition 1.3 and Assumption 1.10 are in force; the proof, being a routine but lengthy adaptation, is omitted.

Proposition 1.17. *Under the hypotheses of Lemma 1.16, there exists $C>0$ such that*

$$\|\chi_B\|_{L^{p'(\cdot)}(w^{-1},-b)}\|\chi_B\|_{L^{q(\cdot)}(w,a)}\le C|B|^{1-\alpha/n}$$

for every ball B.

Proof. As in Lemma 2.7 of [27], apply Lemma 1.16 to $f=\chi_B v_{-b}w^{-1}$ and duality between $L^{p(\cdot)}(w,b)$ and $L^{p'(\cdot)}(w^{-1},-b)$.

Corollary 1.18. *There exist $C_0,C_1>0$ such that, for every $\kappa\ge j\ge 0$ and every $r(\cdot)\in P^{\log}(\mathbb{R}^n)$, $c\in\mathbb{R}$ admissible for w,*

$$C_0 2^{(\kappa-j)c} \le \frac{\|\chi_\kappa v_c\|_{L^{r(\cdot)}(w)}/\|\chi_j v_c\|_{L^{r(\cdot)}(w)}}{\|\chi_\kappa\|_{L^{r(\cdot)}(w)}/\|\chi_j\|_{L^{r(\cdot)}(w)}} \le C_1 2^{(\kappa-j)c}.$$

Proof. Immediate from applying Lemma 1.11 at $R=2^\kappa$ and $R=2^j$ and taking the ratio.

Lemma 1.19. *Under the hypotheses of Lemma 1.16: for $j\le\kappa-2$, $\gamma\in(1,q_-)$,*

$$\|\chi_\kappa I_\alpha(f\chi_j)\|_{L^{q(\cdot)}(w,a)}\le C2^{-(\kappa-j)(n-\alpha-a)}2^{(\kappa-j)n/\gamma}\|f\chi_j\|_{L^{p(\cdot)}(w,b)};$$

for $j\ge\kappa+2$, $\gamma\in(q_+,\infty)$,

$$\|2^{\kappa\beta}\chi_\kappa I_\alpha(f\chi_j)\|_{L^{q(\cdot)}(w,a)}\le C2^{(\kappa-j)(a+n/\gamma+\beta)}\|2^{j\beta}f\chi_j\|_{L^{p(\cdot)}(w,b)}.$$

Lemma 1.19 is the weighted form, valid on $L^{q(\cdot)}(w)$ by Corollary 1.18, of the vector-valued singular integral estimate of Ho [23, Prop. 2.5, Lemma 6.3], combined with the generalized Hölder inequality of Lemma 1.4(a) and Proposition 1.17, exactly as in the proof of term I/III of Theorem 1.2 in [27]; we omit the routine details.

2. Main results

Throughout this section $\alpha\in(0,n)$ is fixed, $p(\cdot),q(\cdot)\in C^{\log}(\mathbb{R}^n)$ satisfy $p(x)\le q(x)$ for every x with $1<p_-\le p_+<\infty$, $1<q_-\le q_+<\infty$, and $w\in\mathcal{A}_{p(\cdot)}(\mathbb{R}^n)\cap\mathcal{A}_{q(\cdot)}(\mathbb{R}^n)$.

2.1. **A ball-membership lemma.** Ho's Lemma 2.9 in [27] tests membership of a ball's characteristic function in the power-weighted Herz space. The following statement establishes the corresponding property for the scale of Definition 1.12; Assumption 1.14, Proposition 1.15, Lemma 1.11, and the basic monotonicity of the weighted norm (Definition 1.3) are used throughout the proof.

Lemma 2.1. *Let* $\mathrm{p}(\cdot) \in P^{\log}(\mathbb{R}^n)$, $1 < \mathrm{p}_- \le \mathrm{p}_+ < \infty$, $w \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ *satisfying Assumption 1.14 with exponent* σ_w, $\beta(\cdot) \in L^\infty(\mathbb{R}^n)$ *log-Hölder at* $0, \infty$, $\Gamma \ge 0$, $\upsilon \in (0,\infty)$, $j \in \mathbb{N}$, $B = B(0, 2^j)$. *Then*

$$\textit{(1) } b \ge 0 \implies \chi_B \in K^{\beta(\cdot),b,\upsilon}_{\Gamma,\mathrm{p}(\cdot)}(w);$$

$$\textit{(2) } b \in \Big(-\frac{n}{\mathrm{p}_+}, 0\Big),\ w v_b \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n) \implies \chi_B \in K^{\beta(\cdot),b,\upsilon}_{\Gamma,\mathrm{p}(\cdot)}(w);$$

$$\textit{(3) } \big[\textit{(1) or (2)}\big] \ \wedge\ \beta_\infty + b < n - \sigma_w \implies \chi_B \in HK^{\beta(\cdot),b,\upsilon}_{\Gamma,\mathrm{p}(\cdot)}(w).$$

Proof. Let $(c_\kappa)_{\kappa\ge 0} \subset [0,\infty)$ with $\sum_{\kappa=0}^\infty c_\kappa^\upsilon < \infty$. Since $0 \le 2^{-m_0\Gamma} \le 1$ for $\Gamma \ge 0$, $m_0 \ge 0$, Lemma 1.13 gives

$$\sup_{\vartheta>0}\sup_{m_0\ge 0} 2^{-m_0\Gamma}\Big(\sum_{\kappa=0}^{m_0} c_\kappa^{\upsilon(1+\vartheta)}\Big)^{1/(\upsilon(1+\vartheta))} \le \Big(\sum_{\kappa=0}^\infty c_\kappa^\upsilon\Big)^{1/\upsilon}. \tag{$*$}$$

It therefore suffices, in each of (1)–(3), to show that $c_\kappa := 2^{\kappa\phi(\kappa)}\|\mu\,\chi_\kappa v_b\|_{L^{\mathrm{p}(\cdot)}(w)}$ ($\mu = \chi_B$ in (1),(2); $\mu = \mathcal{G}_N\chi_B$ in (3)) satisfies $\sum_{\kappa=0}^\infty c_\kappa^\upsilon < \infty$.

(1) $b \ge 0$. For $0 \le \kappa \le j$ and $x \in \mathcal{A}_\kappa$, $|x| < 2^\kappa$, and since $t \mapsto t^b$ is nondecreasing on $(0,\infty)$ for $b \ge 0$,

$$v_b(x) = |x|^b < (2^\kappa)^b = 2^{\kappa b}, \qquad x \in \mathcal{A}_\kappa,$$

so $\chi_\kappa v_b \le 2^{\kappa b}\chi_\kappa$ pointwise on $\mathbb{R}^n$. Since $0 \le f \le g$ a.e. implies $\|f\|_{L^{\mathrm{p}(\cdot)}(w)} \le \|g\|_{L^{\mathrm{p}(\cdot)}(w)}$ directly from the Luxemburg infimum,

$$\|\chi_B\chi_\kappa v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le 2^{\kappa b}\|\chi_\kappa\|_{L^{\mathrm{p}(\cdot)}(w)}, \qquad 0 \le \kappa \le j.$$

For $\kappa > j$, $\mathcal{A}_\kappa \cap B(0,2^j) = \emptyset$, so $\chi_B\chi_\kappa v_b \equiv 0$ and $c_\kappa = 0$. For $0 \le \kappa \le j$, $\mathcal{A}_\kappa \subset B(0,2^j)$, so, by the monotonicity in Definition 1.3, with $M_j := \|\chi_{B(0,2^j)}\|_{L^{\mathrm{p}(\cdot)}(w)} < \infty$,

$$\|\chi_\kappa\|_{L^{\mathrm{p}(\cdot)}(w)} \le M_j, \qquad c_\kappa = 2^{\kappa\phi(\kappa)}\|\chi_B\chi_\kappa v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le 2^{\kappa(\phi(\kappa)+b)}M_j, \quad 0 \le \kappa \le j.$$

Since $\phi(0) = \beta(0)$ and $\phi(\kappa) = \beta_\infty$ for $1 \le \kappa \le j$,

$$\sum_{\kappa=0}^j c_\kappa^\upsilon \le M_j^\upsilon\Big(1 + \sum_{\kappa=1}^j 2^{\kappa(\beta_\infty+b)\upsilon}\Big) \le M_j^\upsilon(j+1)\max\big(1, 2^{j(\beta_\infty+b)\upsilon}\big) =: A_j < \infty,$$

using, for a geometric sum with j terms of ratio $r = 2^{(\beta_\infty+b)\upsilon}$, the elementary bound $\sum_{\kappa=1}^j r^\kappa \le j\max(1, r^j)$ (each term is $\le \max(1,r^j)$: if $r \ge 1$ the terms increase up to r^j; if $r < 1$ each term is ≤ 1). Thus $\sum_{\kappa=0}^\infty c_\kappa^\upsilon = \sum_{\kappa=0}^j c_\kappa^\upsilon \le A_j < \infty$, and $(*)$ gives $\chi_B \in K^{\beta(\cdot),b,\upsilon}_{\Gamma,\mathrm{p}(\cdot)}(w)$.

(2) $b \in (-n/\mathrm{p}_+, 0)$, $wv_b \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$. For $0 \le \kappa \le j$, $\mathcal{A}_\kappa \subset B(0,2^\kappa) =: \mathcal{B}_\kappa$, so $\chi_\kappa \le \chi_{\mathcal{B}_\kappa}$ pointwise, and

$$\|\chi_B \chi_\kappa v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le \|\chi_\kappa v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le \|\chi_{\mathcal{B}_\kappa} v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \overset{\text{Lem. 1.11}, R=2^\kappa}{\le} C_1 2^{\kappa b} \|\chi_{\mathcal{B}_\kappa}\|_{L^{\mathrm{p}(\cdot)}(w)}.$$

Since $\kappa \le j$, $\mathcal{B}_\kappa \subset B(0,2^j)$, so, by the monotonicity in Definition 1.3, $\|\chi_{\mathcal{B}_\kappa}\|_{L^{\mathrm{p}(\cdot)}(w)} \le M_j$, and $c_\kappa = 0$ for $\kappa > j$ as in (1). Hence

$$c_\kappa = 2^{\kappa\phi(\kappa)} \|\chi_B \chi_\kappa v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le C_1 M_j \, 2^{\kappa(\phi(\kappa)+b)}, \qquad 0 \le \kappa \le j,$$

and, exactly as in (1),

$$\sum_{\kappa=0}^{j} c_\kappa^\upsilon \le C_1^\upsilon M_j^\upsilon (j+1) \max\left(1, 2^{j(\beta_\infty + b)\upsilon}\right) =: B_j < \infty,$$

so $\sum_{\kappa=0}^{\infty} c_\kappa^\upsilon = \sum_{\kappa=0}^{j} c_\kappa^\upsilon \le B_j < \infty$, and $(*)$ gives $\chi_B \in K^{\beta(\cdot),b,\upsilon)}_{\Gamma,\mathrm{p}(\cdot)}(w)$.

(3) Suppose (1) or (2) holds, and $\beta_\infty + b < n - \sigma_w$. Set $\mu := \mathcal{G}_N \chi_B$, $R := 2^j$.

Range $0 \le \kappa \le j+1$. By Proposition 1.15, $\mu \le C_N$ pointwise on $\mathbb{R}^n$, so

$$\chi_\kappa \mu \, v_b \le C_N \chi_\kappa v_b \quad \text{pointwise}, \qquad \|\chi_\kappa \mu \, v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le C_N \|\chi_\kappa v_b\|_{L^{\mathrm{p}(\cdot)}(w)}.$$

As in (2), $\chi_\kappa \le \chi_{\mathcal{B}_\kappa}$, and, by Lem. 1.11 with $R = 2^\kappa$ followed by Assumption 1.14 (valid since $2^\kappa \ge 1$),

$$\|\chi_\kappa v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le \|\chi_{\mathcal{B}_\kappa} v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le C_1 2^{\kappa b} \|\chi_{\mathcal{B}_\kappa}\|_{L^{\mathrm{p}(\cdot)}(w)} \le C_1 C_2 2^{\kappa(b+\sigma_w)}.$$

Hence

$$c_\kappa = 2^{\kappa\phi(\kappa)} \|\chi_\kappa \mu \, v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le C_N C_1 C_2 \, 2^{\kappa(\phi(\kappa)+b+\sigma_w)}, \qquad 0 \le \kappa \le j+1,$$

and, exactly as in (1),

$$\sum_{\kappa=0}^{j+1} c_\kappa^\upsilon \le (C_N C_1 C_2)^\upsilon (j+2) \max\left(1, 2^{(j+1)(\beta_\infty + b + \sigma_w)\upsilon}\right) =: N_1(j) < \infty.$$

Range $\kappa \ge j+2$. For $x \in \mathcal{A}_\kappa$, $|x| \ge 2^{\kappa-1}$; since $\kappa \ge j+2$, $2^{\kappa-1} \ge 2^{j+1} = 2R$, so Prop. 1.15 gives

$$\mu(x) = \mathcal{G}_N \chi_B(x) \le C_{n,N} R^n |x|^{-n} \le C_{n,N} R^n \left(2^{\kappa-1}\right)^{-n} = C_{n,N} 2^n \, 2^{jn} 2^{-\kappa n} =: C_4 2^{(j-\kappa)n}, \qquad x \in \mathcal{A}_\kappa,$$

so $\chi_\kappa \mu \le C_4 2^{(j-\kappa)n} \chi_\kappa$ pointwise, and (with the same bound $\|\chi_\kappa v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le C_1 C_2 2^{\kappa(b+\sigma_w)}$ established above)

$$\|\chi_\kappa \mu \, v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le C_4 2^{(j-\kappa)n} \|\chi_\kappa v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le C_4 C_1 C_2 \, 2^{jn} 2^{\kappa(b+\sigma_w - n)},$$

$$c_\kappa = 2^{\kappa\beta_\infty} \|\chi_\kappa \mu \, v_b\|_{L^{\mathrm{p}(\cdot)}(w)} \le C_4 C_1 C_2 \, 2^{jn} \, 2^{\kappa\lambda}, \qquad \lambda := \beta_\infty + b + \sigma_w - n, \quad \kappa \ge j+2.$$

By hypothesis $\beta_\infty + b < n - \sigma_w \Leftrightarrow \lambda < 0$, so $2^{\lambda\upsilon} < 1$ and

$$\sum_{\kappa=j+2}^{\infty} c_\kappa^\upsilon \le (C_4 C_1 C_2)^\upsilon 2^{jn\upsilon} \sum_{\kappa=j+2}^{\infty} 2^{\kappa\lambda\upsilon} = (C_4 C_1 C_2)^\upsilon 2^{jn\upsilon} \frac{2^{(j+2)\lambda\upsilon}}{1 - 2^{\lambda\upsilon}} =: N_2(j) < \infty.$$

Total.

$$\sum_{\kappa=0}^{\infty} c_\kappa^{\upsilon} = \sum_{\kappa=0}^{j+1} c_\kappa^{\upsilon} + \sum_{\kappa=j+2}^{\infty} c_\kappa^{\upsilon} \le N_1(j) + N_2(j) < \infty,$$

and $(*)$ gives

$$\|\chi_B\|_{HK^{\beta(\cdot),b,\upsilon)}_{\Gamma,\mathrm{p}(\cdot)}(w)} = \|\mathcal{G}_N \chi_B\|_{K^{\beta(\cdot),b,\upsilon)}_{\Gamma,\mathrm{p}(\cdot)}(w)} \le \big(N_1(j) + N_2(j)\big)^{1/\upsilon} < \infty,$$

i.e. $\chi_B \in HK^{\beta(\cdot),b,\upsilon)}_{\Gamma,\mathrm{p}(\cdot)}(w)$.

2.2. **The main theorem.** We now state and prove the boundedness of I_α on the scale of Definition 1.12, using Lemmas 1.16–1.19 and Proposition 1.17 of §1.3 as inputs, with no intermediate base-case statement.

Theorem 2.2. *Let $\alpha \in (0,n)$, $\upsilon \in (0,\infty)$, $\beta \in \mathbb{R}$, $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$, $p(x) \le q(x)$ $\forall x$, $1 < p_- \le p_+ < \infty$, $1 < q_- \le q_+ < \infty$, $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n) \cap \mathcal{A}_{q(\cdot)}(\mathbb{R}^n)$ with wv_a, wv_b admissible as in Lemma 1.16. Suppose a, b satisfy (2.6)–(2.7),*

$$-\frac{n}{p_+} + \alpha - b < \beta < n - \frac{n}{p_-} - b, \tag{2.1}$$

and, fixing once and for all $\gamma_1 \in (1, q_-)$ with $\beta < n - n/\gamma_1 - \alpha - a$ and $\gamma_2 \in (q_+, \infty)$ with $-n/\gamma_2 - a < \beta$ (both exist by the previous display via (2.7)), set $\mu := a + n/\gamma_2 + \beta > 0$, and suppose

$$0 \le \Gamma < \mu = a + \beta + \frac{n}{\gamma_2}\Big(\le a + \beta + \frac{n}{q_+}\Big). \tag{2.2}$$

Then there is $C > 0$ such that, for every $f \in K^{\beta,b,\upsilon)}_{\Gamma,p(\cdot)}(w)$,

$$\|I_\alpha f\|_{K^{\beta,a,\upsilon)}_{\Gamma,q(\cdot)}(w)} \le C \|f\|_{K^{\beta,b,\upsilon)}_{\Gamma,p(\cdot)}(w)}.$$

Proof. Fix $\vartheta > 0$, $m_0 \ge 0$; set $\theta := \upsilon(1+\vartheta) \in (0,\infty)$, $d_j := 2^{j\beta}\|f\chi_j v_b\|_{L^{p(\cdot)}(w)}$, $\|f\|_\Gamma := \|f\|_{K^{\beta,b,\upsilon)}_{\Gamma,p(\cdot)}(w)}$. By definition, for every $M \ge 0$,

$$\Big(\sum_{j=0}^{M} d_j^\theta\Big)^{1/\theta} \le 2^{M\Gamma}\|f\|_\Gamma, \qquad \text{hence} \qquad d_i \le 2^{i\Gamma}\|f\|_\Gamma \ \ (i \ge 0), \tag{2.3}$$

the second by taking $M = i$ and discarding all but the i-th term.

Split $f = f_1 + f_2$ with $f_1 := \sum_{i=0}^{m_0+1} f\chi_i$ and $f_2 := \sum_{i \ge m_0+2} f\chi_i$. For $0 \le \kappa \le m_0$ introduce the index sets

$$J_\kappa^- := \{j \in \mathbb{Z} :\ 0 \le j \le \kappa - 2\}, \qquad J_\kappa^0 := \{j \in \mathbb{Z} :\ \max(\kappa - 1, 0) \le j \le \kappa + 1\},$$

$$J_\kappa^+ := \{j \in \mathbb{Z} :\ \kappa + 2 \le j \le m_0 + 1\},$$

(note that $\kappa + 1 \le m_0 + 1$, so $J_\kappa^0 \subset [0, m_0 + 1]$), and set

$$F_\kappa := \sum_{j \in J_\kappa^-} I_\alpha(f\chi_j), \qquad G_\kappa := \sum_{j \in J_\kappa^0} I_\alpha(f\chi_j), \qquad H_\kappa := \sum_{j \in J_\kappa^+} I_\alpha(f\chi_j),$$

with the convention that a sum over an empty set is zero. Then $I_\alpha f = F_\kappa + G_\kappa + H_\kappa + I_\alpha f_2$, and, by the $\theta_0 := \min(1,\theta)$-quasi-triangle inequality,

$$\sum_{\kappa=0}^{m_0} 2^{\kappa\beta\theta}\|\chi_\kappa I_\alpha f\|_{q,a}^\theta \;\le\; C\,(I + II + III + IV), \tag{2.4}$$

where $\|\cdot\|_{q,a} := \|\cdot\|_{L^{q(\cdot)}(w,a)}$ and

$$I := \sum_{\kappa=0}^{m_0} 2^{\kappa\beta\theta}\|\chi_\kappa F_\kappa\|_{q,a}^\theta, \qquad II := \sum_{\kappa=0}^{m_0} 2^{\kappa\beta\theta}\|\chi_\kappa G_\kappa\|_{q,a}^\theta,$$

$$III := \sum_{\kappa=0}^{m_0} 2^{\kappa\beta\theta}\|\chi_\kappa H_\kappa\|_{q,a}^\theta, \qquad IV := \sum_{\kappa=0}^{m_0} 2^{\kappa\beta\theta}\|\chi_\kappa I_\alpha f_2\|_{q,a}^\theta.$$

Term II. For $j \in J_\kappa^0$ we have $|\kappa - j| \le 1$, so Lemma 1.16 gives $\|2^{\kappa\beta}\chi_\kappa I_\alpha(f\chi_j)\|_{q,a} \le C2^{|\kappa-j||\beta|}d_j \le Cd_j$. Since J_κ^0 has at most three elements and each j belongs to at most three sets J_κ^0,

$$II \le C\sum_{\kappa=0}^{m_0}\sum_{j\in J_\kappa^0} d_j^\theta \le C\sum_{j=0}^{m_0+1} d_j^\theta \le C\,2^{(m_0+1)\Gamma\theta}\|f\|_\Gamma^\theta. \tag{2.5}$$

Term I. For $x \in \mathcal{A}_\kappa$ and $j \in J_\kappa^-$, $|I_\alpha(f\chi_j)(x)| \le C2^{-\kappa(n-\alpha)}\int_{\mathcal{A}_j}|f|$. Lemma 1.4(a), Proposition 1.17 and Corollary 1.18, applied as in the proof of [27, Theorem 3.1], together with Lemma 1.19, give

$$\|2^{\kappa\beta}\chi_\kappa I_\alpha(f\chi_j)\|_{q,a} \le C2^{-(\kappa-j)\rho}d_j, \qquad \rho := n - \frac{n}{\gamma_1} - \alpha - a - \beta > 0. \tag{2.6}$$

If $\theta \le 1$, we use subadditivity; if $\theta > 1$, we apply Hölder's inequality in j after writing $2^{-(\kappa-j)\rho} = 2^{-(\kappa-j)\rho/2}\,2^{-(\kappa-j)\rho/2}$. In both cases, with $\theta_1 \in \{\theta, \theta/2\}$ accordingly,

$$I \le C\sum_{j=0}^{m_0-2} d_j^\theta \sum_{\kappa=j+2}^{m_0} 2^{-(\kappa-j)\rho\theta_1} \le \frac{C}{1-2^{-\rho\theta_1}}\sum_{j=0}^{m_0} d_j^\theta \le C_\rho\,2^{m_0\Gamma\theta}\|f\|_\Gamma^\theta. \tag{2.7}$$

Term III. For $x \in \mathcal{A}_\kappa$ and $j \in J_\kappa^+$, $|I_\alpha(f\chi_j)(x)| \le C2^{-j(n-\alpha)}\int_{\mathcal{A}_j}|f|$; as for term I, Lemma 1.4(a), Proposition 1.17, Corollary 1.18, and Lemma 1.19 (with γ_2) give

$$\|2^{\kappa\beta}\chi_\kappa I_\alpha(f\chi_j)\|_{q,a} \le C2^{(\kappa-j)\mu}d_j, \qquad j \ge \kappa+2, \tag{2.8}$$

$\mu = a + n/\gamma_2 + \beta > 0$ by hypothesis. As for term I (either $\theta \le 1$ or $\theta \ge 1$, same manipulation with ρ replaced by μ and the roles of j,κ exchanged),

$$\begin{aligned} III &\le C\sum_{\kappa=0}^{m_0}\sum_{j=\kappa+2}^{m_0+1} 2^{(\kappa-j)\mu\theta_1}d_j^\theta \quad (\theta_1 = \theta \text{ or } \theta/2) \\ &= C\sum_{j=0}^{m_0+1} d_j^\theta \sum_{\kappa=0}^{j-2} 2^{(\kappa-j)\mu\theta_1} \;\le\; \frac{C}{1-2^{-\mu\theta_1}}\sum_{j=0}^{m_0+1} d_j^\theta, \end{aligned}$$

so, exactly as for term II,

$$III \le C_\mu\, 2^{(m_0+1)\Gamma\theta}\|f\|_\Gamma^\theta \le C'_\mu\, 2^{m_0\Gamma\theta}\|f\|_\Gamma^\theta. \tag{2.9}$$

Term IV (far tail). For $\kappa \le m_0 < m_0+2 \le i$, always $i \ge \kappa+2$, so Lemma 1.19 with γ_2 gives, as noted above,

$$\|2^{\kappa\beta}\chi_\kappa I_\alpha(f\chi_i)\|_{q,a} \le C2^{(\kappa-i)\mu}d_i, \qquad i \ge m_0+2.$$

By the θ_0-quasi-triangle inequality on i and, for the finite κ-sum, subadditivity/Hölder exactly as for term III,

$$\begin{aligned} IV &\le C\sum_{\kappa=0}^{m_0}\sum_{i\ge m_0+2} 2^{(\kappa-i)\mu\theta_1}d_i^\theta \\ &= C\sum_{i\ge m_0+2} d_i^\theta \sum_{\kappa=0}^{m_0} 2^{(\kappa-i)\mu\theta_1} \;\le\; \frac{C}{1-2^{-\mu\theta_1}}\sum_{i\ge m_0+2} 2^{(m_0-i)\mu\theta_1}d_i^\theta. \end{aligned}$$

By $d_i \le 2^{i\Gamma}\|f\|_\Gamma$ and $\Gamma < \mu \le \mu\theta_1/\theta$ (since $\theta_1/\theta \in \{1, 1/2\}$; take γ_2 closer to its lower endpoint q_+ if necessary so that $\mu\theta_1/\theta > \Gamma$ still holds — possible since $\mu \to a+\beta+n/q_+ > \Gamma$ as $\gamma_2 \downarrow q_+$, by hypothesis),

$$\sum_{i\ge m_0+2} 2^{(m_0-i)\mu\theta_1}d_i^\theta \le \|f\|_\Gamma^\theta 2^{m_0\mu\theta_1}\sum_{i\ge m_0+2} 2^{i(\Gamma\theta/\theta_1-\mu)\theta_1} = \|f\|_\Gamma^\theta \frac{2^{2(\Gamma\theta/\theta_1-\mu)\theta_1}}{1-2^{(\Gamma\theta/\theta_1-\mu)\theta_1}}\, 2^{m_0\Gamma\theta},$$

so

$$IV \le C''_\mu\, 2^{m_0\Gamma\theta}\|f\|_\Gamma^\theta. \tag{2.10}$$

Combination. Summing the four bounds,

$$\sum_{\kappa=0}^{m_0} 2^{\kappa\beta\theta}\|\chi_\kappa I_\alpha f\|_{q,a}^\theta \;\le\; C^*\, 2^{m_0\Gamma\theta}\|f\|_\Gamma^\theta,$$

C^* independent of ϑ, m_0 (all constants above depend only on $n, \alpha, a, b, \beta, \gamma_1, \gamma_2, p(\cdot), q(\cdot), [w]_{\mathcal{A}_{p(\cdot)}}, [w]_{\mathcal{A}_{q(\cdot)}}$, and are monotone/bounded as θ ranges over (υ, ∞)). Taking θ-th roots and multiplying by $2^{-m_0\Gamma}$,

$$2^{-m_0\Gamma}\Big(\sum_{\kappa=0}^{m_0} 2^{\kappa\beta\theta}\|\chi_\kappa I_\alpha f\|_{q,a}^\theta\Big)^{1/\theta} \;\le\; (C^*)^{1/\theta}\|f\|_\Gamma \;\le\; \max\big(1, (C^*)^{1/\upsilon}\big)\|f\|_\Gamma,$$

uniformly in $\vartheta > 0$, $m_0 \ge 0$. Taking $\sup_{\vartheta>0}\sup_{m_0\ge 0}$,

$$\|I_\alpha f\|_{K^{\beta,a,\upsilon)}_{\Gamma,q(\cdot)}(w)} \le C\|f\|_{K^{\beta,b,\upsilon)}_{\Gamma,p(\cdot)}(w)}.$$

Remark 2.3. Under the parameter setting $w \equiv 1$, $\Gamma = 0$, Theorem 2.2 recovers Theorem 1.2 of [27]; under the further specialization $\Gamma = 0$, $\beta \equiv$ const, $w \equiv 1$, Lemma 1.16 recovers Theorem 1.1 of [1].

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