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SUMS OF FRACTIONAL PARTS AND SUM OF RESTRICTED DIVISORS OF A NUMBER

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Abstract. Let us consider a strictly increasing sequence of positive integers a_n such that $A(x)$ is the distribution function of the sequence. That is, $A(x) = \sum_{a_n \leq x} 1$. We study the sum $\sum_{a_n \leq x} a_n^s \left\{ \frac{x}{a_n} \right\}$ and apply this formula in the study of the sum of a-divisors of a number. The distribution functions $A(x)$ considered are very general. The methods used are very elementary.

1. Introduction and Main Results

Let us consider a strictly increasing sequence a_n of positive integers. We shall denote a positive integer in this sequence a. Let $A(x)$ be the number of a not exceeding x. That is, $A(x)$ is the distribution function of the sequence a_n, $A(x) = \sum_{a \leq x} 1$. In this article we study the sum $\sum_{a \leq x} a^s \left\{ \frac{x}{a} \right\}$, where $s \geq 1$. We also study the sum of s-th powers of divisors a of n. That is, $\sigma_{a,s}(n) = \sum_{a \mid n} a^s$. If n has not divisors $a$we put $\sigma_{a,s}(n) = 0$. The distribution functions $A(x)$ considered are very general (see below).

We shall need the following well-known theorem (Abel summation).

Theorem 1.1. *Let c_n $(n \geq 1)$ be a sequence of real numbers. Let us consider the function*

$$A(x) = \sum_{n \leq x} c_n.$$

Suppose that $f(x)$ has a continuous derivative $f'(x)$ on the interval $[1, \infty]$. The following formula holds

$$\sum_{n \leq x} c_n f(n) = A(x) f(x) - \int_1^x A(t) f'(t)\ dt.$$

Proof. See [3], (Chapter XXII). □

We also shall need the following definition.

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Definition 1.2. Let us consider a positive function $f(x)$ such that $f'(x)$ is positive, strictly decreasing and $\lim_{x\to\infty} f(x) = \infty$. The function $f(x)$ is of slow increase if and only if the following limit holds

$$\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0. \tag{1.1}$$

Typical functions of slow increase are $\log x$, $\log\log x$, $\frac{\log x}{\log\log x}$, etc. The functions of slow increase are studied in [8]. We shall need the following properties of the functions of slow increase,

$$\lim_{x\to\infty} \frac{f(x)}{x^\alpha} = 0$$

for all $\alpha > 0$ and

$$\lim_{x\to\infty} \frac{f(Cx)}{f(x)} = 1 \tag{1.2}$$

for all $C > 0$.

Note that

$$\sum_{a\le x} a^s \left\{\frac{x}{a}\right\} = x\sum_{a\le x} a^{s-1} - \sum_{a\le x} a^s \left\lfloor\frac{x}{a}\right\rfloor \qquad (s \ge 1). \tag{1.3}$$

We define the function

$$G(x) = \sum_{a\le x} a^s \qquad (s \ge 0). \tag{1.4}$$

We have the following general theorem.

Theorem 1.3. *We have the equation*

$$\sum_{\frac{x}{k}<a\le x} a^s \left\lfloor\frac{x}{a}\right\rfloor = \left(\sum_{j=1}^{k} G\left(\frac{x}{j}\right)\right) - kG\left(\frac{x}{k}\right). \tag{1.5}$$

Proof. Note that if $\frac{x}{j+1} < a \le \frac{x}{j}$ then $\left\lfloor\frac{x}{a}\right\rfloor = j$. Consequently

$$\sum_{\frac{x}{k}<a\le x} a^s \left\lfloor\frac{x}{a}\right\rfloor = \sum_{j=1}^{k-1} j\left(G\left(\frac{x}{j}\right) - G\left(\frac{x}{j+1}\right)\right)$$
$$= \left(\sum_{j=1}^{k} G\left(\frac{x}{j}\right)\right) - kG\left(\frac{x}{k}\right).$$

□

Theorem 1.4. *Suppose that $c > 0$, $0 < \alpha \le 1$ and $f(x)$ is a function of slow increase. Suppose that $A(x) \sim h(x)$, where either $h(x) = cx^\alpha$ or $h(x) = \frac{x^\alpha}{f(x)}$. The following formula holds*

$$G(x) = \sum_{a\le x} a^s \sim \frac{\alpha}{s+\alpha} x^s A(x) \qquad (s \ge 0). \tag{1.6}$$

Proof. By Theorem 1.1 we have

$$\begin{aligned} G(x) &= \sum_{a\leq x} a^s = A(x)x^s - s\int_1^x A(t)t^{s-1}\,dt \\ &= A(x)x^s\left(1 - \frac{s\int_1^x A(t)t^{s-1}\,dt}{x^s A(x)}\right) = \frac{\alpha}{s+\alpha}x^s A(x) + o\left(x^s A(x)\right). \end{aligned}$$

Since (L'Hospital's rule and equation (1.1))

$$\lim_{x\to\infty} \frac{\int_1^x h(t)t^{s-1}\,dt}{x^s h(x)} = \frac{1}{s+\alpha}$$

and hence

$$\lim_{x\to\infty} \frac{\int_1^x o(1)h(t)t^{s-1}\,dt}{x^s h(x)} = 0.$$

□

Theorem 1.5. *We have*

$$\begin{aligned} \sum_{\frac{x}{k}<a\leq x} a^s \left\lfloor \frac{x}{a} \right\rfloor &= \left(\sum_{j=1}^k \frac{1}{j^{s+\alpha}} - \frac{k}{k^{s+\alpha}}\right) G(x) + o\left(G(x)\right) \\ &= \left(\sum_{j=1}^k \frac{1}{j^{s+\alpha}} - \frac{k}{k^{s+\alpha}}\right) \frac{\alpha}{s+\alpha} x^s A(x) + o\left(x^s A(x)\right) \quad (s\geq 1). \qquad (1.7) \end{aligned}$$

Proof. Equation (1.7) is an immediate consequence of equations (1.5), (1.6) and (1.2). □

Theorem 1.6. *The following formulas hold.*

$$x\sum_{a\leq x} a^{s-1} \sim \frac{\alpha}{s-1+\alpha} x^s A(x) \qquad (s\geq 1), \qquad (1.8)$$

$$x\sum_{a\leq \frac{x}{k}} a^{s-1} \sim \frac{\alpha}{(s-1+\alpha)k^{s-1+\alpha}} x^s A(x) \qquad (s\geq 1). \qquad (1.9)$$

Proof. Equations (1.8) and (1.9) are an immediate consequence of equation (1.6). □

Theorem 1.7. *The following formula holds.*

$$\begin{aligned} &\sum_{\frac{x}{k}<a\leq x} a^s \left\{ \frac{x}{a} \right\} \\ &= \left(\frac{\alpha}{s-1+\alpha} - \frac{\alpha}{(s-1+\alpha)(s+\alpha)}\frac{1}{k^{s-1+\alpha}} - \frac{\alpha}{s+\alpha}\sum_{j=1}^k \frac{1}{j^{s+\alpha}}\right) x^s A(x) \\ &+ \ o\left(x^s A(x)\right). \qquad (1.10) \end{aligned}$$

Proof. Equation (1.10) is an immediate consequence of equations (1.3), (1.8), (1.9) and (1.7). □

Theorem 1.8. *Suppose that*

$$\sum_{\frac{x}{k}<a\leq x} a^s \left\{\frac{x}{a}\right\} = g(k)x^s A(x) + o(x^s A(x)) \qquad (k \geq 2)$$

and $\lim_{k\to\infty} g(k) = l > 0$*. The following formula holds.*

$$\sum_{a\leq x} a^s \left\{\frac{x}{a}\right\} = lx^s A(x) + o(x^s A(x)).$$

Proof. We have

$$\begin{aligned}\sum_{a\leq x} a^s \left\{\frac{x}{a}\right\} &= \frac{\sum_{a\leq\frac{x}{k}} a^s \left\{\frac{x}{a}\right\}}{G\left(\frac{x}{k}\right)} \frac{G\left(\frac{x}{k}\right)}{G(x)} \frac{G(x)}{x^s A(x)} x^s A(x) \\ &+ \quad (g(k)-l)x^s A(x) + lx^s A(x) + o(x^s A(x)).\end{aligned}$$

That is,

$$\frac{\sum_{a\leq x} a^s \left\{\frac{x}{a}\right\}}{x^s A(x)} - l = \frac{\sum_{a\leq\frac{x}{k}} a^s \left\{\frac{x}{a}\right\}}{G\left(\frac{x}{k}\right)} \frac{G\left(\frac{x}{k}\right)}{G(x)} \frac{G(x)}{x^s A(x)} + (g(k)-l) + o(1).$$

Note that

$$0 \leq \frac{\sum_{a\leq\frac{x}{k}} a^s \left\{\frac{x}{a}\right\}}{G\left(\frac{x}{k}\right)} \leq 1$$

and

$$\frac{G\left(\frac{x}{k}\right)}{G(x)} \sim \frac{1}{k^{s+\alpha}}.$$

Therefore given $\epsilon > 0$ arbitrarily small there exists k sufficiently large such that if $x \geq x_\epsilon$ then we have

$$\left|\frac{\sum_{a\leq x} a^s \left\{\frac{x}{a}\right\}}{x^s A(x)} - l\right| \leq \epsilon + \epsilon + \epsilon = 3\epsilon \qquad (x \geq x_\epsilon).$$

□

Now, we can prove our two main theorems.

Theorem 1.9. *Suppose that* $c > 0$*,* $0 < \alpha \leq 1$ *and* $f(x)$ *is a function of slow increase. Suppose that either* $A(x) \sim cx^\alpha$ *or* $A(x) \sim \frac{x^\alpha}{f(x)}$*. The following formula holds.*

$$\sum_{a\leq x} a^s \left\{\frac{x}{a}\right\} = \left(\frac{\alpha}{s-1+\alpha} - \frac{\alpha}{s+\alpha}\zeta(s+\alpha)\right) x^s A(x) + o(x^s A(x)), \qquad (1.11)$$

where $s \geq 1$*.*

Proof. Equation (1.11) is an immediate consequence of equation (1.10) and Theorem 1.8. □

Remark 1.10. By use of equations (1.10) and (1.11) we can obtain asymptotic formulas for the sum

$$\sum_{a \leq \frac{x}{k}} a^s \left\{ \frac{x}{a} \right\}.$$

Theorem 1.11. *Suppose that $c > 0$, $0 < \alpha \leq 1$ and $f(x)$ is a function of slow increase. Suppose that either $A(x) \sim cx^\alpha$ or $A(x) \sim \frac{x^\alpha}{f(x)}$. The following formula holds.*

$$\sum_{n \leq x} \sigma_{a,s}(n) = \frac{\alpha}{s+\alpha} \zeta(s+\alpha) x^s A(x) + o\left(x^s A(x)\right), \tag{1.12}$$

where $s \geq 1$.

Proof. We have

$$\sum_{n \leq x} \sigma_{a,s}(n) = \sum_{a \leq x} a^s \left\lfloor \frac{x}{a} \right\rfloor = x \sum_{a \leq x} a^{s-1} - \sum_{a \leq x} a^s \left\{ \frac{x}{a} \right\}.$$

Now, the proof of equation (1.12) is an immediate consequence of equations (1.8) and (1.11). □

Example 1.12. There are many sequences in number theory such that $A(x) \sim cx$ $(c > 0)$. That is, sequences with positive density. The sequence a of all positive integers. The sequence a of integers in arithmetic progression. The sequence a of h-free numbers $(h \geq 2)$, where $A(x) \sim \frac{1}{\zeta(h)} x$ (see, for example, [6]). In particular, for the sequence of square-free numbers we have $A(x) \sim \frac{6}{\pi^2} x$. Etc.

Example 1.13. There are many sequences in number theory such that $A(x) \sim cx^\alpha$ $(c > 0)$ $(0 < \alpha < 1)$. The sequence a of k-th powers $(k \geq 2)$ where $A(x) \sim x^{\frac{1}{k}}$. The sequence a of all perfect powers where $A(x) \sim x^{\frac{1}{2}}$ (see [5]). The sequence a of h-full numbers $(h \geq 2)$, since that $A(x) \sim cx^{\frac{1}{h}}$, where the constant c depends of h (see, for example, either [4] or [7], for elementary methods). Etc.

Example 1.14. There exist infinite sequences of positive integers in number theory such that $A(x) \sim \frac{x^\alpha}{f(x)}$, where $0 < \alpha \leq 1$ and $f(x)$ is a function of slow increase. The sequence of prime numbers, the sequence of prime powers, the sequence of numbers with exactly h prime factors in their prime factorization and infinite sequences of composite numbers with certain restrictions on their prime factorization (see [9]). Etc.

Now, we obtain some applications of equation (1.12).

If a is the sequence of all positive integers n then $A(x) \sim x$. That is, $c = 1$ and $\alpha = 1$. Therefore if $s = 1$ then equation (1.12) becomes

$$\sum_{n \leq x} \sigma_{a,1}(n) = \sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 + o\left(x^2\right)$$

as it is well known. In this case $\sigma_{a,1}(n) = \sigma(n)$ is the sum of all positive divisors of n.

If a is the sequence of all positive square-free then $A(x) \sim \frac{6}{\pi^2}x$. That is, $c = \frac{6}{\pi^2}$ and $\alpha = 1$. Therefore if $s = 1$ then equation (1.12) becomes

$$\sum_{n \leq x} \sigma_{a,1}(n) = \frac{1}{2}x^2 + o\left(x^2\right).$$

In this case $\sigma_{a,1}(n)$ is the sum of all positive square-free divisors of n.

Let us consider the prime factorization of a positive integer $n = p_1^{r_1} \cdots p_k^{r_k}$, where $p_1, \dots p_k$ are the distinct primes in the prime factorization and $r_1, \dots r_k$ are the multiplicities.

If a is the sequence of all positive primes p then (prime number theorem) $A(x) = \pi(x) \sim \frac{x}{\log x}$. That is, $\alpha = 1$ and $f(x) = \log x$. Therefore if $s = 1$ then (1.12) becomes

$$\sum_{n \leq x} \sigma_{a,1}(n) = \frac{\pi^2}{12}\frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right)$$

as it is well known. In this case $\sigma_{a,1}(n) = p_1 + \cdots + p_k$. That is, $\sigma_{a,1}(n)$ is the sum of all positive prime divisors of n. This arithmetical function was studied by Hall [2].

If a is the sequence of primes powers then it is well known that $A(x) \sim \frac{x}{\log x}$, since the prime powers with exponent greater than or equal to 2 are square-full numbers and the number of square-full numbers not exceeding x is $O(\sqrt{x})$. Therefore is this case we also have $\alpha = 1$ and $f(x) = \log x$ and consequently if $s = 1$ then (1.12) becomes

$$\sum_{n \leq x} \sigma_{a,1}(n) = \frac{\pi^2}{12}\frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right)$$

That is, the same result that the Hall function. In this case

$$\sigma_{a,1}(n) = \sum_{i=1}^{k}\left(p_i + p_i^2 + \cdots + p_i^{r_i}\right) = \sum_{i=1}^{k} p_i \frac{p_i^{r_i} - 1}{p_i - 1}.$$

That is, $\sigma_{a,1}(n)$ is the sum of all prime powers divisors of n. This result can be surprising and it implies that another arithmetical functions, whose size is between the sizes of these two, have the same asymptotic behavior. For example, the arithmetical function of Alladi and Erdős [1] defined by $B_1(n) = r_1p_1 + r_2p_2 + \cdots + r_kp_k$ also satisfies (as it is well known)

$$\sum_{n \leq x} B_1(n) = \frac{\pi^2}{12}\frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right).$$

Analogously for the arithmetical function $B_2(n) = p_1^{r_1} + p_2^{r_2} + \cdots + p_k^{r_k}$ we also have

$$\sum_{n \leq x} B_2(n) = \frac{\pi^2}{12}\frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right).$$

If N is a positive integer we define $r_{a,N}$ the unique nonnegative integer remainder of the division $\frac{N}{a}$. That is, $N = aq + r_{a,N}$ and therefore $0 \leq r_{a,N} \leq a-1$.

We have the following theorem.

Theorem 1.15. *Suppose that $c > 0$, $0 < \alpha \leq 1$ and $f(x)$ is a function of slow increase. Suppose that either $A(x) \sim cx^{\alpha}$ or $A(x) \sim \frac{x^{\alpha}}{f(x)}$. The following formula holds*

$$\sum_{a \leq N} r_{a,N} = \left(1 - \frac{\alpha}{1+\alpha}\zeta(1+\alpha)\right) NA(N) + o(NA(N)). \tag{1.13}$$

Proof. It is an immediate consequence of Theorem 1.9, since $r_{a,N} = a\left\{\frac{N}{a}\right\}$. □

If $\alpha = 1$ then equation (1.13) becomes.

$$\sum_{a \leq N} r_{a,N} = \left(1 - \frac{\pi^2}{12}\right) NA(N) + o(NA(N)).$$

For example, if $a = n$ is the sequence of positive integers then we have

$$\sum_{n \leq N} r_{n,N} = \left(1 - \frac{\pi^2}{12}\right) N^2 + o(N^2)$$

and if $a = p$ is the sequence of primes then we have

$$\sum_{p \leq N} r_{p,N} = \left(1 - \frac{\pi^2}{12}\right) \frac{N^2}{\log N} + o\left(\frac{N^2}{\log N}\right).$$

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