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GENERALIZED LIÉNARD EQUATION VIA LOCAL DERIVATIVE

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Abstract. This work investigates the generalized Liénard equation using the Generalized Local Derivative framework, defined by a kernel allowing timescale variations and local memory effects. Sufficient conditions for the existence of periodic solutions are established by constructing invariant regions in the phase plane and applying Brouwer's Fixed-Point Theorem. The analysis extends to systems with bounded velocities and repulsive forces. Model consistency with classical calculations is verfied when the kernel is unitary. Numerical simulations of a generalized Van der Pol oscillator illustrate how the kernel modulates energy dissipation, potentially triggering limit cycle collapse or explosion.

Keywords. Local Generalized Derivative, Liénard Equations, Limit Cycles, Stability of Periodic Solutions, Van der Pol Oscillator
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1. Introduction and preliminaries

The study of the Liénard equation is fundamental in nonlinear oscillation theory, particularly in characterizing limit cycles [1]. While classical and fractional calculus frameworks have been widely applied to these models, traditional fractional derivatives often present limitations regarding the chain rule or the recovery of standard integer-order derivatives. To overcome these restrictions and effectively model memory effects, variable timescales, and non-homogeneous media, we utilize the Local Generalized Derivative defined by $N_F^\alpha f(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon F(t,\alpha))-f(t)}{\epsilon}$ (where $F(t,\alpha)$ acts as a scale kernel [12], also consult [7, 16]). The aforementioned local operators and others from the literature have become a useful tool in solving various modeling problems; interested readers can consult [5, 6, 7, 11, 13].

In this work, we consider the generalized Liénard equation:

$$N_F^\alpha[\varphi(N_F^\alpha x)] + f(x, N_F^\alpha x)N_F^\alpha x + g(x) = e(t), \tag{1.1}$$

here, $\varphi : (-\rho, \rho) \to (-\omega, \omega)$ is an odd increasing homeomorphism representing nonlinear acceleration terms, which encompasses classic formulations such as the

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p-Laplacian $\varphi(s) = |s|^{p-2}s$, prescribed curvature $\varphi(s) = s/\sqrt{1+s^2}$, and relativistic acceleration $\varphi(s) = s/\sqrt{1-s^2}$ for $\rho, \omega \in (0, +\infty]$. The damping function $f(x, y)$ satisfies standard continuity and local Lipschitz conditions to ensure existence and uniqueness along with the sign condition: there exist $a < 0 < b$, such that $f(x, y) > 0$ for $x > b, y > 0$ and for $x < a, y < 0$. To prevent vertical asymptotes, when $\omega = +\infty$, there exist $L, D > 0$ and a continuous function $T(x)$ such that $|f(x, y)| \leq LT(x)|y|$ for $|y| \leq D$. The restitution function $g(x)$ satisfies $g(x) > E_+$ for large positive x and $g(x) < E_-$ for small negative x, where $E_+ = \max e(t)$ and $E_- = \min e(t)$ for a continuous T-periodic forcing term $e(t)$, with $G(x) = \int_0^x g(s)ds$ satisfying $G(x) - E_\pm x \to \infty$ as $x \to \pm\infty$. An auxiliary function $h(y) := \frac{y}{\varphi^{-1}(y)}$ is introduced to facilitate slope computations in the phase plane. When $F \equiv 1$ and $f(x, y) = f(x)$, equation (1.1) reduces to the classical Liénard equation $x'' + f(x)x' + g(x) = 0$.

The primary objective of this paper is to re-evaluate this generalized system under the operator N_F^α, establishing sufficient conditions for the existence of harmonic solutions and examining how the kernel $F(t, \alpha)$ modulates energy dissipation, stability, and timescales.

2. First Results

The equivalent system in the phase plane (x, y), with $y = \varphi(N_F^\alpha x)$, is:

$$\begin{cases} \dot{x} = \frac{1}{F(t,\alpha)}\varphi^{-1}(y), \\ \dot{y} = \frac{1}{F(t,\alpha)}[-f(x, y)\varphi^{-1}(y) - g(x) + e(t)]. \end{cases} \tag{2.1}$$

Lemma 2.1. *Let $a < 0 < b$ be constants and suppose that there exist two smooth functions $\phi(x)$ and $\psi(x)$, such that:*

i) $0 < \phi(x) < \omega$ for all $x \leq a$ and $-\omega < \psi(x) < 0$ for all $x \geq b$.

ii) Upper damping condition. For $x \leq a$:

$$f(x, \phi(x))\phi(x) > -\phi'(x)\phi(x) - (g(x) - E_+)h(\phi(x)).$$

iii) Lower damping condition. For $x \geq b$:

$$f(x, \psi(x))\psi(x) < -\psi'(x)\psi(x) - (g(x) - E_-)h(\psi(x)),$$

where $h(y) = y/\varphi^{-1}(y)$. Then, the orbits of the generalized system enter the regions bounded by $y = \phi(x)$ for $x \leq a$ and $y = \psi(x)$ for $x \geq b$.

Proof. Computing the trajectory slope in the phase plane yields $dy/dx = \dot{y}/\dot{x} = -f(x, y) - [g(x) - e(t)]/\varphi^{-1}(y)$, where the kernel $F(t, \alpha)$ cancels out. Evaluating this expression along $y = \phi(x)$ and applying hypothesis ii) together with $e(t) \leq E_+$ and $h(\phi(x)) > 0$ gives $\frac{dy}{dx}\big|_{y=\phi(x)} < \phi'(x) + \frac{e(t)-E_+}{\varphi^{-1}(\phi(x))} \leq \phi'(x)$. This strictly forces trajectories to cross the curve $y = \phi(x)$ inward toward the x-axis. □

Lemma 2.2. *Suppose $\omega = +\infty$ and that $F(t, \alpha)$ is a continuous, T-periodic, and strictly positive function ($F(t, \alpha) \geq F_{min} > 0$). If there exists a continuous function $T(x)$ and constants $L, D, M > 0$ with $M \geq \max\{-a, b\}$, such that: $|f(x, y)| \leq LT(x)|y|$ for $x \in [-M, M]$ and $|y| \geq D$. Then, no orbit of the*

generalized system starting in the interval $x \in [-M, M]$ can escape to infinity in finite time (there are no vertical asymptotes).

Proof. Boundedness of $g(x)$ and $e(t)$ on $[-M, M]$ yields the linear differential inequality $|dy/dx| \leq LT(x)|y| + C$, which by Gronwall's Lemma guarantees that $y(x)$ remains bounded on the compact interval $[-M, M]$. Furthermore, because $y(x)$ is bounded and $F(t, \alpha) \geq F_{min} > 0$, the transit time $\Delta t = \int_{-M}^{M} \frac{F(t,\alpha)}{\varphi^{-1}(y(x))} dx$ is finite, preventing finite-time blow-up. □

While the previous Lemmas ensure that orbits do not escape "up/down" or to "infinity", the following three Lemmas guarantee that orbits "rotate" clockwise and remain confined laterally.

The following lemma is fundamental for cases such as that of the prescribed curvature equation, in which the velocity is bounded within a strip.

Lemma 2.3. *Let $\omega < +\infty$. Suppose there exist constants $M > 0$, $\delta_\omega > 0$, and $\delta_{-\omega} > 0$ such that for all $x \in [-M, M]$, the following holds uniformly:*

$$\lim_{y \to \omega^-} f(x, y) = \delta_\omega \quad y \quad \lim_{y \to -\omega^+} f(x, y) = \delta_{-\omega}.$$

Then, no orbit of the generalized system can escape the strip $\mathbb{R} \times (-\omega, \omega)$.

Proof. Near the boundary $y \to \omega^-$, the inverse homeomorphism $\varphi^{-1}(y) \to +\infty$, making the negative damping term $-f(x, y)\varphi^{-1}(y)$ dominate the bounded forces $g(x)$ and $e(t)$ in $\dot{y} = \frac{1}{F(t,\alpha)}[-f(x, y)\varphi^{-1}(y) - g(x) + e(t)]$. Since $F(t, \alpha) > 0$, $\dot{y}$ becomes strictly negative near $y = \omega$ (and strictly positive near $y = -\omega$), confining all trajectories within $(-\omega, \omega)$. □

The following result allows the non-autonomous system to be controlled by autonomous "ceiling" and "floor" systems.

Lemma 2.4. *Let us define the slopes of the generalized system (dy/dx) and the auxiliary autonomous systems $(dy_\pm/dx)$:*

$$\frac{dy_\pm}{dx} = -f(x, y) - \frac{g(x) - E_\pm}{\varphi^{-1}(y)}.$$

Then, the following comparison inequalities hold: $\frac{dy}{dx} \leq \frac{dy_+}{dx}$ for all $y > 0$ and $\frac{dy}{dx} \leq \frac{dy_-}{dx}$, for all $y < 0$.

Proof. Subtracting the slope expressions yields $dy/dx - dy_+/dx = [e(t) - E_+]/\varphi^{-1}(y)$. Since, $e(t) \leq E_+$ by definition and $\varphi^{-1}(y) > 0$ for $y > 0$, the difference is non-positive, establishing $dy/dx \leq dy_+/dx$. The proof for $y < 0$ follows symmetrically using $e(t) \geq E_-$. □

Below we present a result that ensures that the trajectories do not escape horizontally and return to the vertical control lines.

Lemma 2.5. *Suppose $\omega = +\infty$ and that $G(x) - E_+ x \to \infty$, when $x \to \infty$. If $f(x, y) > 0$ for $x > b$ (where b is the damping bound), then any path of the upper*

autonomous system starting at (b, y_b) with $y_b > 0$ must intercept the positive x axis and then return to the line $x = b$ with a value $y < 0$.

Proof. Along trajectories of the auxiliary system, the derivative of $H_+(x, y) = \int_0^y \varphi^{-1}(s)ds + G(x) - E_+x$ is $\dot{H}_+ = -\frac{f(x,y)}{F(t,\alpha)}[\varphi^{-1}(y)]^2 < 0$ for $x > b$. Since H_+ is strictly decreasing and its level sets are closed by hypothesis, orbits are trapped within bounded energy contours, forcing a clockwise rotation across the x-axis back to $x = b$. □

Theorem 2.6. *Let $\omega = +\infty$. Suppose the following conditions are met:*

H1) *Sign of $g(x)$. There exist constants $a < 0 < b$ such that $g(x) > E_+$ for $x > b$ and $g(x) < E_-$ for $x < a$, where $E_+ = \max e(t)$ and $E_- = \min e(t)$.*

H2) *Damping $f(x, y) > 0$ for $(x, y) \in (b, +\infty) \times \mathbb{R}_+$ and for $(x, y) \in (-\infty, a) \times \mathbb{R}_-$.*

H3) *Control Functions. There are smooth functions $\phi(x) > 0$ and $\psi(x) < 0$, such that for $x \le a$:*

$$f(x, \phi(x))\phi(x) > -\phi'(x)\phi(x) - (g(x) - E_+)h(\phi(x)),$$

and for $x \ge b$:

$$f(x, \psi(x))\psi(x) < -\psi'(x)\psi(x) - (g(x) - E_-)h(\psi(x)),$$

where $h(y) = y/\varphi^{-1}(y)$.

H4) *$G(x) - E_\pm x \to \infty$ when $x \to \pm\infty$.*

H5) *$F(t, \alpha)$ is continuous, T-periodic and strictly positive. Therefore, the equation (1.1) has at least one T-periodic solution.*

Proof. The slope of the trajectories of the system (2.1) is obtained by dividing $\dot{y}$ by $\dot{x}$:

$$\frac{dy}{dx} = \frac{\frac{1}{F(t,\alpha)}[-f(x, y)\varphi^{-1}(y) - g(x) + e(t)]}{\frac{1}{F(t,\alpha)}\varphi^{-1}(y)} = -f(x, y) - \frac{g(x) - e(t)}{\varphi^{-1}(y)}.$$

Note that the factor $F(t, \alpha)$ cancels out, which means that the direction of the vector field at each point (x, y) is ultimately:

$$\frac{dy}{dx} = -f(x, y) - \frac{g(x) - e(t)}{\varphi^{-1}(y)}.$$

Let us evaluate the slope on the curve $y = \phi(x)$ for $x \le a$ (Lemma 2.1),

$$\frac{dy}{dx}\Big|_{y=\phi(x)} = \frac{1}{\phi(x)}[-f(x, \phi(x))\phi(x) - (g(x) - e(t))h(\phi(x))].$$

Using condition H3) and the fact that $e(t) \le E_+$, we obtain:

$$\frac{dy}{dx}\Big|_{y=\phi(x)} < \phi'(x) + \frac{e(t) - E_+}{\varphi^{-1}(\phi(x))} \le \phi'(x).$$

This shows that any orbit that touches the curve $y = \phi(x)$ will enter the region bounded by it towards the x axis. Because $F(t, \alpha)$ is positive and bounded on $[0, T]$, the system cannot exhibit finite-time bursts if $|f(x, y)| \le LT(x)|y|$ holds (Lemma 2.2). The solutions can be continued for all $t \in [0, T]$. To ensure that

the orbits "rotate" clockwise, we compare the system to averaged autonomous systems. We define the energy:

$$H_+(x,y) = \int_0^y \varphi^{-1}(s)ds + G(x) - E_+x.$$

The time derivative of the energy along the trajectories of the system (2.1) is:

$$\dot{H}_+ = \frac{\partial H_+}{\partial x}\dot{x} + \frac{\partial H_+}{\partial y}\dot{y} = \cdots = -\frac{f(x,y)}{F(t,\alpha)}[\varphi^{-1}(y)]^2.$$

Since $F(t,\alpha) > 0$ and $f(x,y) > 0$ in the external regions, H_+ strictly decreases. This guarantees that the trajectories cross the axes and return to the origin, completing a clockwise cycle. We have constructed a closed region K (a "disk" in the phase plane) whose boundaries are the curves ϕ, ψ and energy levels $H_\pm$. From this, we have that the system's flux points inward toward K along its entire boundary. Let us define the Poincaré map $P : K \to K$ such that $P(z_0) = z(T; z_0)$. Since K is invariant and the system is continuous, Brouwer's Theorem guarantees the existence of a fixed point $z^* \in K$. The solution starting at z^* is, by definition, T-periodic. □

We will now present a concrete example applying Theorem (2.6) to a generalized Van der Pol oscillator. This example illustrates how the kernel $F(t,\alpha)$ modifies the classical dynamics and allows us to visualize the invariant region K.

Example 2.7. (The generalized Van der Pol oscillator)

We consider a version of the Van der Pol oscillator adapted to the equation (1.1):

$$N_F^\alpha[N_F^\alpha x] + \mu(x^2-1)N_F^\alpha x + x = A\cos(\omega t),$$

where the functions involved are: Acceleration (φ): In this case, we use the identity $\varphi(s) = s$, which implies $\varphi^{-1}(y) = y$ and $h(y) = 1$; Damping $f(x,y) = \mu(x^2-1)$: It satisfies the sign condition for $|x| > 1$ (where $b = 1, a = -1$); Restitution $g(x) = x$: Satisfies $G(x) = x^2/2$, fulfilling $G(x) - E_\pm x \to \infty$; Kernel: We choose an exponential-periodic kernel to model time variability, for example, $F(t,\alpha) = e^{\alpha \sin(\omega t)}$; Visualization of the Phase Plane and Invariant Region K.

The invariant region K is constructed using the barriers of Lemma 2.1 and the energy level curves of Lemma 2.5.

Next, we present a result that covers the bounded velocity cases ($\omega < +\infty$), specific for models such as prescribed curvature ($\varphi(s) = s/\sqrt{1+s^2}$) or relativistic dynamics ($\varphi(s) = s/\sqrt{1-s^2}$), where the velocity is confined to a strip, using the already proven Lemma 2.3.

Theorem 2.8. *Let $\omega < +\infty$ and assume the following conditions hold:*

Kernel $F(t,\alpha)$ continuous, T non-periodic, and $F(t,\alpha) \geq F_{min} > 0$.

Sign of $g(x)$. There exist $a < 0 < b$ such that $g(x) > E_+$ for $x > b$ and $g(x) < E_-$ for $x < a$.

Boundary damping. There exist $\delta_\omega, \delta_{-\omega} > 0$ such that $\lim_{y\to\omega^-} f(x,y) = \delta_\omega$ and $\lim_{y\to-\omega^+} f(x,y) = \delta_{-\omega}$ uniformly at x. The conditions of the control functions

$\phi(x)$ and $\psi(x)$ of Lemma 2.1 are satisfied within the strip $(-\omega, \omega)$. Therefore, the equation (1.1) has at least one T-periodic solution in the strip $\mathbb{R} \times (-\omega, \omega)$.

Proof. By Lemma 2.3, the time derivative $\dot{y} = \frac{1}{F(t,\alpha)}[-f(x,y)\varphi^{-1}(y) - g(x) + e(t)]$ becomes negative when $y \to \omega^-$ because the term $-f\varphi^{-1}(y)$ dominates since $\varphi^{-1}(y) \to \infty$ and $f \to \delta_\omega$. This guarantees that no solution leaves the vertical strip. Because $g(x)$ overcomes the external force $e(t)$ outside the interval $[a, b]$, the trajectories in the half-planes $x > b$ and $x < a$ are forced to return to the origin. Let us construct the region K. Let $K = \{(x, y) : x \in [x_{min}, x_{max}], y \in [-\omega+\epsilon, \omega-\epsilon]\}$. The lateral boundaries are fixed by energy levels $H_\pm$ which, since ω is finite, intersect the vertical lines $x = a$ and $x = b$. The flux of the system (2.1) points inward along its entire boundary. Applying Brouwer's Theorem to the Poincaré map $P : K \to K$ guarantees the existence of the periodic solution. □

In the next result, $g(x)$ acts as a force that moves the particle away from equilibrium ("repulsive force"), which requires a larger compensation of the damping term to maintain periodicity.

Theorem 2.9. *Suppose $\omega = +\infty$ and that the conditions on $g(x)$ are reversed:*

Sign Reversal. There exist $a < 0 < b$ such that $g(x) < E_-$ for $x > b$ and $g(x) > E_+$ for $x < a$.

Strong Damping. $f(x, y)$ must be sufficiently large such that the control functions $\phi(x)$ and $\psi(x)$ of Lemma 2.1 still exist, but now satisfying $f(x, \phi(x))\phi(x) > -\phi'(x)\phi(x) - (g(x) - E_-)h(\phi(x))$ for $x \le a$.

Continuous Kernel $F(t, \alpha) > 0$ and T-periodic.

Therefore, there exists at least one T-periodic solution.

Proof. Applying Lemma 2.1, the slope dy/dx on $y = \phi(x)$ satisfies $dy/dx < \phi'(x)$, forcing trajectories inward toward the origin. Lemma 2.2 rules out finite-time blow-up, ensuring continuation on $[0, T]$. Furthermore, by Lemma 2.5, the generalized energy H_+ strictly decreases ($\dot{H}_+ = -\frac{f(x,y)}{F(t,\alpha)}[\varphi^{-1}(y)]^2 < 0$), forcing orbits outside $x \in [a, b]$ to rotate clockwise across the axes. This constructs a closed, invariant disk-like region K in the phase plane. Applying Brouwer's Fixed-Point Theorem to the continuous Poincaré map $P : K \to K$, $P(z_0) = z(T; z_0)$, guarantees the existence of a T-periodic solution $z^* \in K$. □

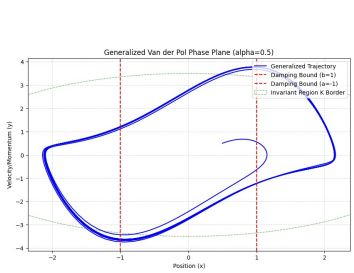
2.1. **Numerical Simulations and Dissipation Dynamics.** To illustrate the theoretical results established in Theorem 2.9, we consider the generalized Van der Pol oscillator:

$$N_F^\alpha[N_F^\alpha x] + \mu(x^2 - 1)N_F^\alpha x + x = A\cos(\omega t), \tag{2.2}$$

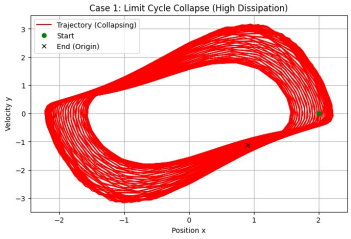
where $\varphi(s) = s$, $f(x, y) = \mu(x^2 - 1)$, $g(x) = x$, and $e(t) = A\cos(\omega t)$. In this setting, the invariant region K is bounded by the damping lines $x = \pm 1$ and the closed energy levels of $H_+(x, y)$.

Figure 1 demonstrates how the kernel $F(t, \alpha)$ modulates energy dissipation and governs the qualitative behavior of the oscillations. Under a bounded periodic kernel $F(t, \alpha) = e^{\alpha \sin(\omega t)}$, the trajectory converges toward a stable, asymmetric limit cycle where internal phase speed varies continuously. Conversely,

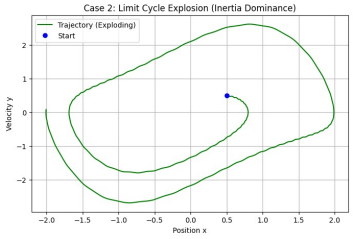
when $F(t,\alpha) \to 0$, hyper-dissipation occurs because the effective friction ratio $-\frac{f(x,y)}{F(t,\alpha)}$ grows excessively in positive damping regions ($|x| > 1$), causing the limit cycle to collapse toward the globally asymptotically stable origin. Finally, when $F(t,\alpha) \to \infty$, the system's braking capacity vanishes, producing hypo-dissipation that allows inertia to dominate and triggers an amplitude explosion where orbits spiral outward uncontrollably.



(a) Stable limit cycle



(b) Hyper-dissipative collapse



(c) Hypo-dissipative explosion

Fig. 1. Phase plane dynamics of the generalized Van der Pol oscillator (2.2) under different kernel behaviors: (a) periodic modulation producing a stable cycle, (b) collapse toward the origin as $F(t,\alpha) \to 0$, and (c) amplitude explosion as $F(t,\alpha) \to \infty$.

3. Consistency with Classical Models and Time Properties

This section focuses on demonstrating that the generalized local derivative is a consistent extension of ordinary calculus, recovering the classical Liénard system when the kernel $F(t,\alpha) = 1$. It also analyzes how the kernel affects the perception of internal time and the period of the solutions.

Theorem 3.1. *Let the generalized system (2.1) be defined in the phase plane (x,y). If the kernel satisfies $F(t,\alpha) \equiv 1$ for all t and α, then: The system (2.1) reduces exactly to the classical Liénard system studied by [3]. All the existence conditions (Lemmas 2.1-2.5 and Theorem 2.6) coincide with those of ordinary differential calculus.*

Proof. By definition, the generalized local derivative is $N_F^\alpha x(t) = F(t,\alpha)\dot{x}(t)$. If $F(t,\alpha) = 1$, then $N_F^\alpha x(t) = \dot{x}(t)$, recovering the first-order ordinary derivative.

Substituting $F(t,\alpha) = 1$ into the equations of motion (2.1):

$$\dot{x} = \frac{1}{1}\varphi^{-1}(y) = \varphi^{-1}(y),$$

$$\dot{y} = \frac{1}{1}[-f(x,y)\varphi^{-1}(y) - g(x) + e(t)] = -f(x,y)\varphi^{-1}(y) - g(x) + e(t).$$

This pair of equations defines the classical flow in the Liénard plane. The slope of the dy/dx trajectories for the generalized system is $-f(x,y) - \frac{g(x)-e(t)}{\varphi^{-1}(y)}$. Since this expression is independent of $F(t,\alpha)$, the topology of the vector field

(isoclinic curves and flow direction) is identical to that of the classical case, when $F = 1$. The energy functions $H_\pm$ and the barriers $\phi(x), \psi(x)$ do not depend on F in their static definition. With $F = 1$, the energy dissipation rate $\dot{H}_+$ reduces to $-f(x, y)[\varphi^{-1}(y)]^2$, which is the standard expression for the stability of Liénard systems. □

The use of N_F^α allows modeling non-homogeneous media or variable time scales using the kernel $F(t, \alpha)$.

Remark 3.2. In the Liénard system with generalized derivative N_F^α, the time T required to complete a closed path Γ in the phase plane is not a standard line integral over x, but rather an integral dependent on the kernel F and the nonlinear velocity $\varphi^{-1}(y)$. That is, if $(x(t), y(t))$ is a T-periodic solution, the period T satisfies:

$$T = \int_\Gamma \frac{F(t, \alpha)}{\varphi^{-1}(y)} dx,$$

where this expression should be understood as a Stieltjes integral or a reparametrization of time. To demonstrate this, let us follow the scheme below.

Starting from the definition of the generalized local derivative $N_F^\alpha x = F(t, \alpha)\dot{x}$ and the system's phase-plane relation $N_F^\alpha x = \varphi^{-1}(y)$, we have:

$$F(t, \alpha)\frac{dx}{dt} = \varphi^{-1}(y).$$

To obtain the infinitesimal time dt in terms of the state variables, we solve for it in the equation above:

$$dt = \frac{F(t, \alpha)}{\varphi^{-1}(y)} dx.$$

For a periodic solution, we integrate both sides over a complete period $[0, T]$. This corresponds to the point (x, y) traversing the closed path Γ once in the phase plane:

$$\int_0^T dt = \oint_\Gamma \frac{F(t, \alpha)}{\varphi^{-1}(y)} dx.$$

This integral is well-defined under the following conditions: Singularities. The denominator $\varphi^{-1}(y)$ vanishes when $y = 0$ (crossings of the x axis). However, at these points $dx/dt = 0$, and the analysis is performed by taking the limit of the path or by changing the integration variable to dy using the second equation of the system; Time Dependence: Unlike the classical case where $F \equiv 1$, here the numerator $F(t, \alpha)$ depends explicitly on t. This implies that the integral is actually an integral equation for T, since t varies as the Γ curve is traversed; Continuity: Since $F(t, \alpha) \geq F_{min} > 0$ is assumed to be continuous, the integral does not diverge and guarantees that the transit time is finite.

Remark 3.3. As mentioned earlier, if α changes in such a way that $F(t, \alpha)$ decreases (for example, when $\alpha \to 0$ in certain kernels), the value of dt for the same offset dx is smaller. This confirms that the system's internal "clock" speeds up or slows down depending on the order of the derivative and the chosen kernel.

Theorem 3.4. *Suppose the kernel has the form $F(t,\alpha) = \lambda(\alpha)\cdot\sigma(t)$, where $\sigma(t)$ is a scaling function and $\lambda(\alpha)$ is an increasing function such that $\lambda(\alpha)\to 0$ when $\alpha\to 0$. For a periodic solution $x_\alpha(t)$ of the system (2.1), the period T of the external force remains constant, but the internal "phase velocity" accelerates as $\lambda(\alpha)$ decreases. If $f(x,y) = f(x)$, the amplitude of the solution is inversely correlated with the minimum values of $F(t,\alpha)$.*

Proof. The time required to complete one oscillation in the phase plane is given by the integral of the system's period (2.1):

$$T = \oint_\Gamma \frac{F(t,\alpha)}{\varphi^{-1}(y)}dx.$$

For the period T to remain constant against changes in α (due to the external force $e(t)$), the trajectory (x,y) in the phase plane must be adjusted. Since $\dot{x} = \frac{\varphi^{-1}(y)}{F(t,\alpha)}$, if $\alpha\to 0$ and therefore $F\to 0$, the orbital velocity $\dot{x}$ tends to infinity for any $y\neq 0$. This implies that the solution traverses the cycle in the phase plane much faster in terms of system time. The energy variation over one cycle is:

$$\Delta H = \int_0^T -\frac{f(x,y)}{F(t,\alpha)}[\varphi^{-1}(y)]^2 dt.$$

As $\alpha\to 0$ (F decreases), the effective damping term f/F increases dramatically. This increases the energy dissipation per cycle, which tends to "shrink" the limiting cycle and reduce the maximum amplitude of the oscillation $x(t)$.
If $\alpha\to 1$ ($F\to 1$), the system converges to classical smooth harmonic behavior.
If $\alpha\to 0$ ($F\to 0$), the system becomes extremely dissipative or hypersensitive to variations in $e(t)$, which can collapse the harmonic solution to an equilibrium point or generate relaxation oscillations of very small amplitude. □

4. Stability of Periodic Solutions

Unlike global fractional derivatives (such as Caputo or Riemann-Liouville), the derivative N_F^α is local and preserves properties of standard differential calculus (such as the chain rule), allowing the use of the Levinson-Smith Stability Criterion and Floquet Exponent analysis adapted to the kernel $F(t,\alpha)$.

Theorem 4.1. *Let $x^*(t)$ be a T-periodic solution of the generalized equation:*

$$N_F^\alpha[N_F^\alpha x] + f(x)N_F^\alpha x + g(x) = e(t).$$

Suppose that the existence conditions of Theorem 2.6 are met. The solution $x^*(t)$ is orbitally asymptotically stable if the following condition on the kernel-weighted average damping is satisfied:

$$\int_0^T \frac{f(x^*(t))}{F(t,\alpha)}dt > 0.$$

Proof. We consider the equivalent system (2.1) with $\varphi(s) = s$ to simplify the exposition (standard oscillator case):

$$\begin{cases} \dot{x} = \frac{1}{F(t,\alpha)} y, \\ \dot{y} = \frac{1}{F(t,\alpha)} [-f(x)y - g(x) + e(t)]. \end{cases}$$

Let $\mathbf{z}^*(t) = (x^*(t), y^*(t))$ be the periodic solution. We introduce a small perturbation $\mathbf{u}(t) = \mathbf{z}(t) - \mathbf{z}^*(t)$. The linearized system is:

$$\dot{\mathbf{u}} = \mathbf{J}(t, \alpha)\mathbf{u},$$

where $\mathbf{J}(t, \alpha)$ is the Jacobian matrix of the system evaluated in the periodic solution. Let us calculate the partial derivatives of the vector field $\mathbf{V}(x, y, t) = \left(\frac{y}{F}, \frac{-f(x)y - g(x) + e(t)}{F}\right)$:

$$\frac{\partial V_1}{\partial x} = 0, \quad \frac{\partial V_1}{\partial y} = \frac{1}{F(t,\alpha)},$$

$$\frac{\partial V_2}{\partial x} = \frac{-f'(x)y - g'(x)}{F(t,\alpha)}, \quad \frac{\partial V_2}{\partial y} = -\frac{f(x)}{F(t,\alpha)}.$$

Therefore, the matrix is:

$$\mathbf{J}(t, \alpha) = \frac{1}{F(t,\alpha)} \begin{pmatrix} 0 & 1 \\ -f'(x^*)y^* - g'(x^*) & -f(x^*) \end{pmatrix}.$$

The stability of a periodic orbit in a 2-dimensional system is determined by its Floquet multipliers. One of the multipliers is always 1 (corresponding to the direction of flow). The other multiplier, μ_2, is given by the integral of the trace of the Jacobian matrix:

$$\mu_2 = \exp\left(\int_0^T \operatorname{tr}(\mathbf{J}(t, \alpha))dt\right).$$

The trace of the Jacobian matrix is simply the sum of the diagonal elements:

$$\operatorname{tr}(\mathbf{J}(t, \alpha)) = 0 + \left(-\frac{f(x^*(t))}{F(t,\alpha)}\right) = -\frac{f(x^*(t))}{F(t,\alpha)}.$$

For the solution to be orbitally asymptotically stable, the second Floquet multiplier must satisfy $|\mu_2| < 1$. This requires the exponent to be negative:

$$\int_0^T -\frac{f(x^*(t))}{F(t,\alpha)}dt < 0 \implies \int_0^T \frac{f(x^*(t))}{F(t,\alpha)}dt > 0.$$

□

Remark 4.2. This result demonstrates that stability depends not only on the damping function $f(x)$, but also on its interaction with the kernel $F(t, \alpha)$.

If $F(t, \alpha)$ is small in regions where $f(x) > 0$, stability is strengthened (faster attraction towards the cycle).

If $F(t, \alpha)$ is small in regions where $f(x) < 0$ (near the origin), the instability at the origin is accentuated, which could push the solutions towards an external limit cycle more forcefully.

Theorem 4.3. *If $e(t) = 0$ and $g(0) = 0$, the origin $(0, 0)$ is locally asymptotically stable if:*

$f(0) > 0$, $g'(0) > 0$ and $F(t, \alpha) > 0$ for all t.

Proof. We define the standard Lyapunov function (Energy):

$$V(x, y) = \frac{1}{2}y^2 + G(x).$$

Let us calculate its time derivative along the trajectories of the generalized system (2.1):

$$\dot{V} = G'(x)\dot{x} + y\dot{y} = g(x)\left(\frac{y}{F}\right) + y\left(\frac{-f(x)y - g(x)}{F}\right),$$

$$\dot{V} = \frac{g(x)y - f(x)y^2 - g(x)y}{F(t, \alpha)} = -\frac{f(x)y^2}{F(t, \alpha)}.$$

If $f(0) > 0$, then in a neighborhood of the origin $\dot{V} \leq 0$. Since $F(t, \alpha) > 0$, the dissipation is effective. By LaSalle's Invariance Principle, the origin is asymptotically stable. □

Remark 4.4. The kernel $F(t, \alpha)$ does not act merely as a multiplicative constant; rather, it defines the underlying time metric governing energy dissipation and recovery. Consequently, both orbital and Lyapunov stability are deeply shaped by the kernel structure. First, regarding orbital stability, the stability condition depends on the kernel-weighted integral $\int_0^T \frac{f(x^*(t))}{F(t,\alpha)}dt > 0$, where $F(t, \alpha)$ functions as a dissipation filter. If $F(t, \alpha)$ becomes sufficiently small when $x^*(t)$ enters positive damping regions ($f(x) > 0$), the integral increases, potentially stabilizing solutions that would remain unstable in standard calculus. Conversely, if $F(t, \alpha)$ takes large values in positive damping regions, the ratio f/F vanishes, leading the limit cycle to lose stability and become repulsive. Second, concerning Lyapunov stability, the energy decay rate $\dot{V} = -\frac{f(x)y^2}{F(t,\alpha)}$ dictates the convergence speed. When $F(t, \alpha) \to 0$ as $t \to \infty$, convergence to equilibrium accelerates dramatically. On the other hand, if $F(t, \alpha) \to \infty$, trajectories approach the origin slowly, mimicking quasi-conservative dynamics. From a physical perspective, variations in the derivative order α modify the system memory or viscosity, altering the overall stability margin. Third, a periodic kernel $F(t, \alpha)$ can induce kernel resonance with the harmonic oscillation. Specifically, an in-phase kernel that reaches its minimum when velocity y is maximized yields maximum dissipation and rigid limit cycles, whereas an out-of-phase kernel minimizes dissipation at peak velocity, producing wider and less stable cycles.

Therefore, we can state that stability is not an intrinsic property of the functions f and g, but rather an emergent property of the interaction between the oscillator structure and the kernel of the generalized local derivative. This allows us to "tune" the stability of a physical mechanism simply by adjusting the parameter α or the form of $F(t, \alpha)$.

5. Synthesis and relationships with current references

The results established in this work extend classical Liénard theory while synthesizing recent advances in nonlinear differential operators. First, regarding the geometry of operators and domain boundaries, relativistic acceleration and prescribed curvature introduce intrinsic singularities since velocity is constrained to an open interval $(-a, a)$ ([9] and [2]). While previous studies focused primarily on boundary conditions to prevent trajectories from reaching domain edges, the present framework clarifies how this geometric constraint interacts with damping $f(x)$ to force limit cycles. Second, concerning existence and multiplicity ([10]), the present energy-based approach complements topological degree methods by establishing upper bounds on cycle count and providing sufficient conditions for orbital stability under generalized derivatives. Third, this formulation extends p-Laplacian unification results by demonstrating that comparison theorems and uniqueness criteria remain valid in the presence of local generalized derivatives, thereby connecting abstract operator theory with relativistic mechanical systems ([14]). Finally, the analysis of non-autonomous systems shows that the Liénard structure under local operators preserves harmonic solutions under periodic forcing ([3]), where our growth estimates at infinity ensure that required dissipation conditions hold across broader classes of forced oscillators.

6. Conclusions

This work demonstrates that the Generalized Local Derivative framework (N_F^α) provides a consistent extension of ordinary calculus for Liénard-type systems, recovering classical dynamics when $F(t, \alpha) \equiv 1$. By constructing invariant regions in the phase plane and applying Brouwer's Fixed-Point Theorem, we established sufficient conditions for the existence of harmonic solutions across systems with standard, bounded, or repulsive dynamics. Crucially, the kernel $F(t, \alpha)$ acts as a dissipation regulator: decreasing F induces hyper-dissipation and limit cycle collapse, whereas large kernel values weaken braking capacity and trigger amplitude explosions. Furthermore, the kernel structure alters the internal time scale without shifting the external forcing period, allowing local memory and time-variance to deform phase trajectories. The adapted Levinson-Smith criterion confirms that orbital stability is governed by the kernel-weighted average damping over a period. These findings offer a practical analytical tool for generalized nonlinear oscillators, opening future directions for other recently proposed local operators.

All of the above opens up new lines of research using other local operators defined in recent years, such as those defined in [4, 8, 15].

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