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EXISTENCE AND UNIQUENESS RESULTS FOR L_Υ-PARABOLIC PROBLEMS

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Abstract. This paper investigates the existence and uniqueness results in Musielak-Orlicz spaces for some nonlinear parabolic Dirichlet problems. Specifically, the study focuses on a time-dependent partial differential equation governed by a Leray-Lions operator in divergence form, combined with an additional divergence term involving a continuous vector-valued function. The source term of the equation is defined by a Carathéodory function that satisfies specific growth conditions with respect to the unknown solution and the space-time variables.

Keywords. Nonlinear parabolic problems, Musielak-Orlicz space, Entropy solution, Renormalized solution, Existence, Uniqueness.
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1. Introduction

Let T be a positive constant, and we set $Q_T = \Omega \times (0,T)$, where Ω is a bounded Lipschitz domain of $\mathbb{R}^N$ $(N \geq 2)$. Suppose that Υ is a Musielak-Orlicz function satisfying the fundamental regularity conditions, and assume its conjugate function $\overline{\Upsilon}$ satisfies the Δ_2-condition.
In this paper we focus on proving the existence and uniqueness results for the following strongly nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial \vartheta}{\partial t} + A(\vartheta) - \operatorname{div}(\phi(\vartheta)) = F(x,t,\vartheta) & \text{in} \quad Q_T, \\ \vartheta(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ \vartheta(t=0) = \vartheta_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $A(\vartheta) = -\operatorname{div}(a(x,t,\vartheta,\nabla\vartheta))$ is a Leray-Lions operator defined on $D(A) \subset W_0^{1,x} L_\Upsilon(Q_T) \longmapsto W^{-1,x} L_{\overline{\Upsilon}}(Q_T)$, with the Carathéodory function $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ satisfies the following conditions :

$$|a(x,t,r,\xi)| \leq a_0(x,t) + \rho_1 \overline{\Upsilon}_x^{-1}(\delta(x,\rho_2|r|)) + \rho_1 \overline{\Upsilon}_x^{-1}(\Upsilon(x,\rho_3|\xi|)), \tag{1.2}$$

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for a.e $(x,t) \in Q_T$ and for all $(r,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where δ is a Musielak-Orlicz function such that $\delta \prec\prec \Upsilon$, $a_0(\cdot,\cdot) \in L_{\overline{\Upsilon}}(Q_T)$ and $\rho_j > 0$ for $j = 1,2,3$.

$$(a(x,t,r,\xi) - a(x,t,r,\xi')) \cdot (\xi - \xi') > 0 \quad \text{for any} \quad \xi \neq \xi'. \tag{1.3}$$

$$a(x,t,r,\xi) \cdot \xi \geq \alpha \left(\Upsilon(x,|\xi|) + \Upsilon(x,|r|)\right) \quad \text{for some constant} \quad \alpha > 0. \tag{1.4}$$

For the others terms, we assume that :

$$\phi : \mathbb{R} \longmapsto \mathbb{R}^N \quad \text{is a continuous function.} \tag{1.5}$$

The lower order term F is a Carathéodory function satisfying, for a.e. $(x,t) \in Q_T$ and for all $r \in \mathbb{R}$, the following condition :

$$|F(x,t,r)| \leq f(x,t) + \overline{\Upsilon}_x^{-1}(\psi(x,|r|)), \tag{1.6}$$

where ψ is a Musielak-Orlicz function such that $\psi(x,r) \prec\prec \lambda(r)\Upsilon(x,r)$, with we can take $\lambda(r) = \frac{1}{(1+r)^\theta}$ for any $r \geq 0$ and some $\theta > 1$.

$$f \in L^1(Q_T), \tag{1.7}$$

$$\vartheta_0 \in L^1(\Omega). \tag{1.8}$$

In the classical Lebesgue-Sobolev setting ($\Upsilon(x,r) = r^p$), Prignet [21] defined the concept of entropy solutions for nonlinear parabolic problems under very weak data assumptions.

In the framework of modular spaces, Mabdaoui et al. in [18] have studied the existence results for the entropy solutions for the parabolic problem (1.1) with $\phi = \phi(x,t,\vartheta)$, where the lower order term ϕ is a Carathéodory function satisfying only a growth condition and the right hand side $F(x,t,\vartheta) = f(x,t)$ belongs to $L^1(Q_T)$.
On the other hand, Elmahi and Meskine in [14] have established in the Orlicz spaces ($\Upsilon(x,r) = \Upsilon(r)$), the existence of entropy solutions for the class of nonlinear parabolic problem of the type

$$\frac{\partial \vartheta}{\partial t} + A(\vartheta) + g(x,t,\vartheta,\nabla\vartheta) = f \quad \text{in} \quad Q_T, \tag{1.9}$$

where g is a Carathéodory function satisfying the sign condition and

$$|g(x,t,s,\xi)| \leq b(|s|)(h(x,t) + \Upsilon(|\xi|)),$$

with b a positive function in $L^1(\mathbb{R}^+)$, $h \in L^1(Q_T)$, and $f \in L^1(Q_T)$. For further studies in this direction, we refer the reader to [19], [15], and [8].

In the Musielak-Orlicz setting, Talha et al. in [22] have studied the existence result for the equation (1.9), when f is a Radon measure belonging to $L^1(Q_T) + W^{-1,x}L_{\overline{\Upsilon}}(Q_T)$. we refer the reader to [1], [11], [13], [20], [12],[6], [7], [10] and [16] for more details.

The principal difficulty in this work stems from the lower order term F, whose growth, prescribed by condition (1.6), is critically linked to the Musielak-Orlicz function Υ. In this context, the coercivity condition (1.4) for the Leray-Lions operator A, and the function λ in condition (1.6) play a crucial role in establishing

the necessary a priori estimates and proving the existence of an entropy solution within the delicate framework of Musielak-Orlicz spaces.

This paper is structured as follows: In Section 2, we recall some definitions and properties of Musielak–Orlicz–Sobolev spaces. Section 3 presents some auxiliary lemmas that will be used to solve our problem. Section 4, is devoted to existence of entropy solution of problem (1.1). In the last section, we prove that any entropy solution is also a renormalized solution, which in turn establishes the uniqueness of these solutions.

2. Preliminaries

Throughout this paper, Ω denotes a bounded Lipschitz domain of $\mathbb{R}^N$ ($N \geq 2$), and let $\Upsilon(x,r) : \Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a function such that :

(i) For all $x \in \Omega$, the function $\Upsilon(x,\cdot)$ is an N-function, i.e. convex, continuous, strictly increasing with $\Upsilon(x,0) = 0$, $\Upsilon(x,r) > 0$ for all $r > 0$ and such that

$$\lim_{r\to 0} \sup_{x\in\Omega} \frac{\Upsilon(x,r)}{r} = 0 \quad \text{and} \quad \lim_{r\to\infty} \inf_{x\in\Omega} \frac{\Upsilon(x,r)}{r} = \infty. \tag{2.1}$$

(ii) For any $r \geq 0$, the function $\Upsilon(\cdot,r)$ is a measurable.

We say that a function $\Upsilon(x,r)$ is a Musielak–Orlicz function if it satisfies conditions (i) and (ii).

The conjugate of a Musielak-Orlicz function Υ in the sense of Fenchel-Young, defined by

$$\overline{\Upsilon}(x,r) = \sup_{s\geq 0}(sr - \Upsilon(x,s)) \quad \text{a.e. } x \in \Omega. \tag{2.2}$$

This implies the Fenchel-Young inequality

$$rs \leq \Upsilon(x,r) + \overline{\Upsilon}(x,s), \quad \text{for any} \quad r,s \geq 0 \quad \text{and a.e. } x \in \Omega. \tag{2.3}$$

A Musielak-Orlicz function Υ satisfies the Δ_2-condition, if there exist $k_0 > 0$, and a nonnegative function $h_0(\cdot) \in L^1(\Omega)$ such that

$$\Upsilon(x,2r) \leq k_0\Upsilon(x,r) + h_0(x) \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad r \geq 0. \tag{2.4}$$

We assume that the Musielak-Orlicz function $\Upsilon(x,r)$ satisfies the following fundamental regularity assumptions:

($\mathcal{M}_1$) : There exist a function $\gamma : [0,\frac{1}{2}] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$such that $\gamma(\cdot,s)$ and $\gamma(r,\cdot)$ are nondecreasing functions and for all $x,y \in \overline{\Omega}$ with $|x-y| \leq \frac{1}{2}$:

$$\Upsilon(x,s) \leq \gamma(|x-y|,s)\Upsilon(y,s) \quad \text{with} \quad \limsup_{\varepsilon\to 0^+} \gamma\left(\varepsilon, c\varepsilon^{-N}\right) < C,$$

for any constant $c > 0$ and for some real constant $C > 0$.

($\mathcal{M}_2$) : The Musielak-Orlicz function $\Upsilon(x,r)$ satisfies the Y-condition on a segment $[a,b]$ of the real line $\mathbb{R}$, i.e., there exist $1 \leq i \leq N$ such that the partial function, defined by $\Upsilon_i : x_i \in [a,b] \mapsto \Upsilon(x,r)$, satisfying one of the following conditions:

- Υ_i changes its monotonicity at some point $t_0 \in [a,b]$.

- Υ_i is monotonic on $[a, b]$.

Here, the i^{th} component of $x = (x_1, x_2, \ldots, x_N) \in \Omega$ is denoted by x_i.

Let $\Upsilon(x, r) : \Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a Musielak–Orlicz function satisfying $(\mathcal{M}_1)$. Then $\Upsilon(\cdot, r) \in L^1(\Omega)$ for every $r \geq 0$. Moreover, for any Musielak–Orlicz function $\psi(x, r)$ satisfying $\psi \prec\prec \Upsilon$ (that is, ψ grows essentially more slowly than Υ near infinity, uniformly with respect to x), for every $\varepsilon > 0$ there exists a nonnegative function $g_\varepsilon \in L^1(\Omega)$ such that

$$\psi(x, r) \leq \Upsilon(x, \varepsilon r) + g_\varepsilon(x) \quad \text{for all } r \geq 0 \text{ and a.e. } x \in \Omega. \tag{2.5}$$

The Musielak–Orlicz space $L_\Upsilon(\Omega)$ (respectively, $E_\Upsilon(\Omega)$) is defined as the set of all measurable functions $\vartheta : \Omega \longmapsto \mathbb{R}$ such that

$$\int_\Omega \Upsilon\Big(x, \frac{|\vartheta|}{\lambda}\Big)\, dx < \infty,$$

for some $\lambda > 0$ (respectively, for all $\lambda > 0$) equipped with Luxemburg norm

$$\|\vartheta\|_{\Upsilon,\Omega} = \inf\left\{\lambda > 0 \ : \int_\Omega \Upsilon\Big(x, \frac{|\vartheta|}{\lambda}\Big)\, dx \leq 1\right\}. \tag{2.6}$$

The Musielak-Orlicz-Sobolev spaces $W^1L_\Upsilon(\Omega)$ and $W^1E_\Upsilon(\Omega)$ are defined, respectively, as follows:

$$W^1L_\Upsilon(\Omega) = \{\vartheta \in L_\Upsilon(\Omega)), \quad \text{with } |\nabla\vartheta| \in L_\Upsilon(\Omega)\},$$

and

$$W^1E_\Upsilon(\Omega) = \{\vartheta \in E_\Upsilon(\Omega), \quad \text{with } |\nabla\vartheta| \in E_\Upsilon(\Omega)\}.$$

The norm on the space $W^1L_\Upsilon(\Omega)$ is defined by

$$\|\vartheta\|_{1,\Upsilon,\Omega} = \|\vartheta\|_{\Upsilon,\Omega} + \|\nabla\vartheta\|_{\Upsilon,\Omega}. \tag{2.7}$$

The space $W^1L_\Upsilon(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{1,\Upsilon,\Omega}$.

We say that $(\vartheta_n)_n$ converges modularly to ϑ in $W^1L_\Upsilon(\Omega)$ if there exists $\lambda > 0$ such that

$$\int_\Omega \Upsilon\Big(x, \frac{|\vartheta_n - \vartheta|}{\lambda}\Big)\, dx + \int_\Omega \Upsilon\Big(x, \frac{|\nabla\vartheta_n - \nabla\vartheta|}{\lambda}\Big)\, dx \longrightarrow 0 \quad \text{as } n \to \infty.$$

The inhomogeneous Musielak-Orlicz-Sobolev spaces are defined as follows,

$$W^{1,x}L_\Upsilon(Q_T) = \{\vartheta : Q_T \longmapsto \mathbb{R} : \vartheta \in L_\Upsilon(Q_T),\ |\nabla\vartheta| \in L_\Upsilon(Q_T)\}\,,$$

and

$$W^{1,x}E_\Upsilon(Q_T) = \{\vartheta : Q_T \longmapsto \mathbb{R} : \vartheta \in E_\Upsilon(Q_T),\ |\nabla\vartheta| \in E_\Upsilon(Q_T)\}\,,$$

where $\nabla\vartheta$ denotes the gradient with respect to the spatial variable x.

The space $W_0^{1,x}E_\Upsilon(Q_T)$ is defined as the closure of $\mathcal{D}(Q_T)$ with respect to the norm topology in $W^{1,x}L_\Upsilon(Q_T)$. If the function Υ satisfies the regularity condition $(\mathcal{M}_1)$, the closure of $\mathcal{D}(Q_T)$ with respect to the weak $\sigma(\Pi L_\Upsilon, \Pi E_{\overline{\Upsilon}})$-topology, denoted by $W_0^{1,x}L_\Upsilon(Q_T)$, coincides with the closure of $\mathcal{D}(Q_T)$ with respect to

modular convergence in $W^{1,x}L_\Upsilon(Q_T)$.
The dual space of $W_0^{1,x}E_\Upsilon(Q_T)$, denoted by $W^{-1,x}L_{\overline{\Upsilon}}(Q_T)$, is given by

$$W^{-1,x}L_{\overline{\Upsilon}}(Q_T) = \left\{w = h - \mathrm{div}F, \text{ with } h \in L_{\overline{\Upsilon}}(Q_T) \text{ and } F \in (L_{\overline{\Upsilon}}(Q_T))^N\right\}.$$

3. Some Technical Lemmas

The following technical lemmas play a crucial role in the proof of our main results.

Lemma 3.1 (see [5], Theorem 3). *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded Lipschitz domain, and let Υ be a Musielak-Orlicz function satisfying condition $(\mathcal{M}_1)$. Consequently, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_\Upsilon(\Omega)$ for the modular topology. That is, for any $\vartheta \in W_0^1L_\Upsilon(\Omega)$ there exists a sequence of functions $(\vartheta_n)_n \subset \mathcal{D}(\Omega)$ such that*

$$\vartheta_n \longrightarrow \vartheta \quad \text{modularly in } W_0^1L_\Upsilon(\Omega) \quad \text{as } n \to \infty.$$

Lemma 3.2 (see [4], Theorem 1.1). *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded Lipschitz domain, and let $\Upsilon(x,r)$ be a Musielak-Orlicz function that satisfies $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$. Then, there exists two constants $d_1 > 0$ and $d_2 > 0$ depending only on Ω and Υ such that*

$$\int_\Omega \Upsilon(x,|\vartheta|)\,dx \leq d_1 \int_\Omega \Upsilon(x, d_2|\nabla\vartheta|)\,dx \quad \text{for any } \vartheta \in W_0^1L_\Upsilon(\Omega). \tag{3.1}$$

Lemma 3.3 (see [2], Lemma 2.5). *Let $a < b \in \mathbb{R}$ and Ω be a bounded domain of $\mathbb{R}^N$ with the segment property, then*

$$\left\{\vartheta \in W_0^{1,x}L_\Upsilon(\Omega\times]a,b[) : \frac{\partial\vartheta}{\partial t} \in W_0^{-1,x}L_\Upsilon(\Omega\times]a,b[) + L^1(\Omega\times]a,b[)\right\} \subset \mathcal{C}(]a,b[, L^1(\Omega)).$$

Lemma 3.4 (see [8], Lemma 3.2). *Under assumptions (1.2)-(1.8), let $(\vartheta_n)_n$ be a sequence in $W_0^{1,x}L_\Upsilon(Q_T)$ uniformly bounded in $L^\infty(Q_T)$ such that :*

$$\vartheta_n \rightharpoonup \vartheta \quad \text{weakly in } W_0^{1,x}L_\Upsilon(Q_T) \quad \text{for } \sigma(\Pi L_\Upsilon, \Pi E_{\overline{\Upsilon}}),$$

$$(a(x,t,\vartheta_n,\nabla\vartheta_n))_n \quad \text{is uniformly bounded in } (L_{\overline{\Upsilon}}(Q_T))^N,$$

$$\lim_{\tau\to\infty}\lim_{n\to\infty}\int_{Q_T}(a(x,t,\vartheta_n,\nabla\vartheta_n) - a(x,t,\vartheta_n,\nabla\vartheta\chi^\tau))\cdot(\nabla\vartheta_n - \nabla\vartheta\chi^\tau)\,dx\,dt = 0,$$

where χ^τ is the characteristic function of $Q^\tau = \{(x,t) \in Q_T : |\nabla\vartheta| \leq \tau\}$. Then,

$$\nabla\vartheta_n \longrightarrow \nabla\vartheta \quad a.e \quad \text{in } Q_T,$$

$$\vartheta_n \longrightarrow \vartheta \quad \text{modulary in } W_0^{1,x}L_\Upsilon(Q_T),$$

$$a(x,t,\vartheta_n,\nabla\vartheta_n)\cdot\nabla\vartheta_n \longrightarrow a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla\vartheta \quad \text{strongly in } L^1(Q_T).$$

For any $k > 0$, we define the truncation function by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and its primitive by

$$\widetilde{T}_k(s) = \int_0^s T_k(r)\,dr = \begin{cases} \dfrac{s^2}{2} & \text{if} \quad |s| \le k, \\ k|s| - \dfrac{k^2}{2} & \text{if} \quad |s| \ge k, \end{cases}$$

it is pretty obvious that $\widetilde{T}_k(s) \ge 0$ and $\widetilde{T}_k(s) \le k|s|$.

4. Existence results

This section is devoted to the existence of entropy solutions for our problem. First of all, we present the definition of entropy solutions for the nonlinear parabolic problem (1.1).

Definition 4.1. A entropy solution of the nonlinear parabolic problem (1.1) is a measurable function ϑ defined on Q_T, satisfying :

$$T_k(\vartheta) \in W_0^{1,x} L_\Upsilon(Q_T) \text{ for any } k > 0, \tag{4.1}$$

$$\vartheta \in L^\infty(0,T;L^1(\Omega)), \tag{4.2}$$

$$F(x,t,\vartheta) \in L^1(Q_T), \tag{4.3}$$

$$\vartheta(x,0) = \vartheta_0(x) \quad \text{for a.e. } x \in \Omega, \tag{4.4}$$

$$\begin{aligned} &\int_{Q_T} \frac{\partial \vartheta}{\partial t} T_k(\vartheta - v)\,dx\,dt + \int_{Q_T} a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla T_k(\vartheta - v)\,dx\,dt \\ &+ \int_{Q_T} \phi(\vartheta)\cdot\nabla T_k(\vartheta - v)\,dx\,dt \le \int_{Q_T} F(x,t,\vartheta) T_k(\vartheta - v)\,dx\,dt, \end{aligned} \tag{4.5}$$

for every $v \in W_0^{1,x} L_\Upsilon(Q_T) \cap L^\infty(Q_T)$ such that $\dfrac{\partial v}{\partial t} \in W^{-1,x} L_{\overline{\Upsilon}}(Q_T) + L^1(Q_T)$.

Theorem 4.2. *Assuming that (1.2)-(1.8) hold true, then there exists at least one entropy solution for the nonlinear parabolic problem (1.1).*

Proof of theorem 4.2

Firstly, we consider the following approximate problem:

$$\begin{cases} \dfrac{\partial \vartheta_n}{\partial t} - \operatorname{div}(a_n(x,t,\vartheta_n,\nabla\vartheta_n) + \phi_n(\vartheta_n)) = F_n(x,t,\vartheta_n) & \text{in } Q_T, \\ \vartheta_n(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ \vartheta_n(t=0) = \vartheta_{0,n} & \text{in } \Omega, \end{cases} \tag{4.6}$$

where for any $n \in \mathbb{N}^*$ and any $(r,\xi) \in \mathbb{R} \times \mathbb{R}^N$

$$a_n(x,t,r,\xi) = a(x,t,T_n(r),\xi) \quad \text{a.e. } (x,t) \in Q_T, \tag{4.7}$$

$$F_n(x,t,r) = T_n(F(x,t,r)) \quad \text{a.e. } (x,t) \in Q_T, \tag{4.8}$$

$$\phi_n(r) = \phi(T_n(r)), \tag{4.9}$$

$$\vartheta_{0,n} = T_n(\vartheta_0). \tag{4.10}$$

Note that :

$$\vartheta_{0,n} \longrightarrow \vartheta_0 \quad \text{strongly in } L^1(\Omega), \tag{4.11}$$

$$|F_n(x,t,r)| \le n \quad \text{and} \quad |F_n(x,t,r)| \le |F(x,t,r)|. \tag{4.12}$$

Moreover, thanks to (1.5) we obtain

$$|\phi_n(r)| \le \sup_{|r|\le n} |\phi(r)| < \infty. \tag{4.13}$$

Since F_n is bounded and the Carathéodory function $a_n + \phi_n$ satisfying the assumptions of Theorem 4 in [15]. Hence, the approximate problem (4.6) admits at least one weak solution ϑ_n.

For each fixed $k > 0$, we will establish the existence of a measurable function ϑ and a subsequence still of $T_k(\vartheta_n)$ satisfies

$$\int_{Q_T} \Upsilon(x, |\nabla T_k(\vartheta_n|))\, dx\, dt \le kC_1, \tag{4.14}$$

$$T_k(\vartheta_n) \rightharpoonup T_k(\vartheta) \quad \text{weakly in } W_0^{1,x} L_\Upsilon(Q_T) \text{ for } \sigma(\Pi L_\Upsilon, \Pi E_{\overline{\Upsilon}}), \tag{4.15}$$

$$(a_n(x,t,T_k(\vartheta_n), \nabla T_k(\vartheta_n)))_n \quad \text{is uniformly bounded in } (L_{\overline{\Upsilon}}(Q_T))^N, \tag{4.16}$$

$$\vartheta \in L^\infty(0,T; L^1(\Omega)). \tag{4.17}$$

Let $B(t)$ be the continuous function defined by

$$B(t) = \int_0^t \lambda(\varepsilon r)\, dr \quad \text{for any } t \ge 0,$$

where $0 < \varepsilon < \min\left(1, \dfrac{\alpha}{C_0}\right)$ and $C_0 = \exp(||B(\cdot)||_{L^\infty(\mathbb{R})})$.

Fixed $\tau \in (0,T)$. By taking $T_k(\vartheta_n)\chi_{(0,\tau)} \exp(B(|\vartheta_n|))$ as a test function in (4.6), and by using (1.6) we obtain

$$\begin{aligned}
&\int_\Omega \Gamma_k(\vartheta_n(\tau))\, dx + \int_{Q_\tau} a_n(x,t,T_k(\vartheta_n), \nabla T_k(\vartheta_n)) \cdot \nabla T_k(\vartheta_n) \exp(B(|\vartheta_n|))\, dx\, dt \\
&+ \int_{Q_\tau} a_n(x,t,\vartheta_n, \nabla \vartheta_n) \cdot \nabla \vartheta_n \lambda(\varepsilon|\vartheta_n|)|T_k(\vartheta_n)| \exp(B(|\vartheta_n|))\, dx\, dt \\
&+ \int_{Q_\tau} \phi_n(\vartheta_n) \exp(B(|\vartheta_n|)) \cdot \nabla T_k(\vartheta_n)\, dx\, dt + \int_{Q_\tau} \phi_n(\vartheta_n)\lambda(\varepsilon|\vartheta_n|)|T_k(\vartheta_n)| \exp(B(|\vartheta_n|)) \cdot \nabla \vartheta_n\, dx\, dt \\
&\le \int_{Q_\tau} f T_k(\vartheta_n) \exp(B(|\vartheta_n|))\, dx\, dt + \int_\Omega \Gamma_k(\vartheta_n(0))\, dx \\
&\qquad + \int_{Q_\tau} \overline{\Upsilon}_x^{-1}(\psi(x,|\vartheta_n|)) T_k(\vartheta_n) \exp(B(|\vartheta_n|))\, dx\, dt,
\end{aligned} \tag{4.18}$$

where $\Gamma_k(s) = \displaystyle\int_0^s T_k(r) \exp(B(|r|))\, dr$. By the definition of $\Gamma_k(s)$, we deduce that

$$\int_\Omega \Gamma_k(\vartheta_n(\tau))\, dx \ge 0 \quad \text{and} \quad \int_\Omega \Gamma_k(\vartheta_n(0))\, dx \le kC_0 ||\vartheta_0||_{L^1(\Omega)}.$$

Moreover, we have

$$\int_{Q_\tau} f T_k(\vartheta_n) \exp(B(|\vartheta_n|))\, dx\, dt \le kC_0 ||f||_{L^1(Q_T)}. \tag{4.19}$$

We set

$$G_n(t) = \int_0^t \phi_n(r) \exp(B(|r|))\, dr,$$

and

$$K_n(t) = \int_0^t \phi_n(r)\lambda(\epsilon|r|)|T_k(r)| \exp(B(|r|))\, dr,$$

since $\phi(\cdot) \in C^0(\mathbb{R}, \mathbb{R}^N)$, we have $G_n(\vartheta_n) = K_n(\vartheta_n) = 0$ on $\partial\Omega$. Let $\vec{n}$ be the exterior normal vector to the boundary $\partial\Omega$. By Green's formula, we get

$$\begin{aligned} \int_\Omega \phi_n(\vartheta_n) \exp(B(|\vartheta_n|)) \cdot \nabla T_k(\vartheta_n)\, dx &= \int_\Omega \operatorname{div}(G_n(T_k(\vartheta_n)))\, dx \\ &= \int_{\partial\Omega} G_n(T_k(\vartheta_n)) \cdot \vec{n}\, d\sigma \\ &= 0, \end{aligned} \tag{4.20}$$

Similarly we deduce

$$\int_\Omega \phi_n(\vartheta_n)\lambda(\varepsilon|\vartheta_n|)|T_k(\vartheta_n)| \exp(B(|\vartheta_n|)) \cdot \nabla\vartheta_n\, dx = 0. \tag{4.21}$$

Returning to (4.18) and using (1.4), (2.3), and (2.5), we obtain

$$\begin{aligned} &\int_\Omega \Gamma_k(\vartheta_n(\tau))\, dx + \int_{Q_T} a_n(x,t,T_k(\vartheta_n), \nabla T_k(\vartheta_n))\, dx\, dt \\ &\quad + (\alpha - C_0\varepsilon) \int_{Q_T} \Upsilon(x, |\vartheta_n|)\lambda(\varepsilon|\vartheta_n|)|T_k(\vartheta_n)|\, dx\, dt \le kC_1. \end{aligned} \tag{4.22}$$

Since $(\alpha - C_0\varepsilon) > 0$, and from (1.4) we conclude the estimate result(4.14). Consequently, for any fixed $k > 0$, the sequence $(T_k(\vartheta_n))_n$ is uniformly bounded in $W_0^{1,x}L_\Upsilon(Q_T)$, and since the embedding $W_0^{1,x}L_\Upsilon(Q_T) \hookrightarrow\hookrightarrow L^1(Q_T)$ is compact, then by using the Lebesgue's theorem and Banach–Alaoglu–Bourbaki's theorem, there exists a subsequence still denoted $(T_k(\vartheta_n))_n$ and a measurable function $\xi_k \in W_0^{1,x}L_\Upsilon(Q_T)$ such that

$$\begin{cases} T_k(\vartheta_n) \rightharpoonup \xi_k & \text{weakly in } W_0^{1,x}L_\Upsilon(Q_T) \text{ for } \sigma(\Pi L_\Upsilon, \Pi E_{\overline{\Upsilon}}), \\ T_k(\vartheta_n) \longrightarrow \xi_k & \text{strongly in } L^1(Q_T) \text{ and a.e. in } Q_T. \end{cases} \tag{4.23}$$

By using the same argument as in [17], we have $(\vartheta_n)_n$ is a Cauchy sequence in measure, then there exists a subsequence still denoted by $(\vartheta_n)_n$ and a measurable function ϑ such that

$$\vartheta_n \longrightarrow \vartheta \quad \text{a.e in } Q_T \quad \text{as } n \to \infty. \tag{4.24}$$

By combining (4.23) and (4.24), we obtain the convergent result (4.15).
Moreover, from (4.22), (1.3), and following the same arguments in [10], we deduce (4.16).
Returning to (4.22) and using Fatou's lemma, we get

$$\frac{1}{k}\int_\Omega \widetilde{T}_k(\vartheta(\tau))\, dx \le C_1 \quad \text{for almost any } \tau \in (0,T).$$

Since $\dfrac{1}{k}\displaystyle\int_\Omega \widetilde{T}_k(\vartheta(\tau))\,dx$ converges pointwise to $\|\vartheta(\tau)\|_{L^1(\Omega)}$ as $k\to 0$. Then, we obtain (4.17).
Finally, thanks to (4.13) we have $|\phi_n(T_k(\vartheta_n))|\le \sup_{|s|\le k}(|\phi(s)|)\in (L_{\overline{\Upsilon}}(Q_T))^N$, it implies from (4.24) and Vitali's theorem that

$$\phi_n(T_k(\vartheta_n))\longrightarrow \phi(T_k(\vartheta)) \quad \text{strongly in } (L_{\overline{\Upsilon}}(Q_T))^N \text{ as } n\to\infty. \tag{4.25}$$

Now, we will prove the convergence results

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{h\le|\vartheta_n|\le h+1\}} a_n(x,t,\vartheta_n,\nabla\vartheta_n)\cdot\nabla\vartheta_n\,dx\,dt=0, \tag{4.26}$$

and

$$\lim_{h\to+\infty}\limsup_{n\to+\infty}\int_{\{|\vartheta_n|>h+1\}} \Upsilon(x,|\vartheta_n|)\lambda(\varepsilon|\vartheta_n|)|\,dx\,dt=0. \tag{4.27}$$

Choosing $T_1(\vartheta_n-T_h(\vartheta_n))\exp(B(|\vartheta_n|))$ as a test function in (4.6) for any $h>0$, and by using (1.6) we obtain

$$\begin{aligned}
&\int_\Omega Z_h(\vartheta_n(T))\,dx+\int_{\{h\le|\vartheta_n|\le h+1\}} a_n(x,t,\vartheta_n,\nabla\vartheta_n)\cdot\nabla\vartheta_n\exp(B(|\vartheta_n|))\,dx\,dt\\
&\quad+\int_{Q_T} a_n(x,t,\vartheta_n,\nabla\vartheta_n)\cdot\nabla\vartheta_n\lambda(\varepsilon|\vartheta_n|)|T_1(\vartheta_n-T_h(\vartheta_n))|\exp(B(|\vartheta_n|))\,dx\,dt\\
&\quad+\int_{Q_T}\phi_n(\vartheta_n)\cdot(\nabla T_{h+1}(\vartheta_n)-\nabla T_h(\vartheta_n))\exp(B(|\vartheta_n|))\,dx\,dt\\
&\quad+\int_{Q_T}\phi_n(\vartheta_n)\cdot\nabla\vartheta_n\lambda(\varepsilon|\vartheta_n|)|T_1(\vartheta_n-T_h(\vartheta_n))|\exp(B(|\vartheta_n|))\,dx\,dt\\
&\quad\le\int_{Q_T} fT_1(\vartheta_n-T_h(\vartheta_n))\exp(B(|\vartheta_n|))\,dx\,dt\\
&\quad+\int_{Q_T}\overline{\Upsilon}_x^{-1}(\psi(x,|\vartheta_n|))T_1(\vartheta_n-T_h(\vartheta_n))\exp(B(|\vartheta_n|))\,dx\,dt+\int_\Omega Z_h(\vartheta_n(0))\,dx,
\end{aligned} \tag{4.28}$$

where $Z_h(r)=\displaystyle\int_0^r T_1(s-T_h(s))\exp(B(|s|))\,ds$. The definition of $Z_h(r)$ implies that

$$\int_\Omega Z_h(\vartheta_n(T))\,dx\ge 0 \quad\text{and}\quad \int_\Omega Z_h(\vartheta_n(0))\,dx\le C_0\int_{\{|\vartheta_{0,n}|>h\}}|\vartheta_{0,n}|\,dx.$$

By applying the same argument as in step of a priori estimates, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\{h\le|\vartheta_n|\le h+1\}} a_n(x,t,\vartheta_n,\nabla\vartheta_n)\cdot\nabla\vartheta_n\,dx\,dt+(\alpha-\varepsilon C_0)\int_{\{|\vartheta_n|>h+1\}}\Upsilon(x,|\vartheta_n|)\lambda(\varepsilon|\vartheta_n|)|\,dx\,dt\\
&\le\int_{Q_T} fT_1(u-T_h(\vartheta))\exp(B(|\vartheta|))\,dx\,dt+\int_{Q_T}\Upsilon(x,1)T_1(u-T_h(\vartheta))\exp(B(|\vartheta|)))\,dx\,dt\\
&+\int_{Q_T} g_\varepsilon(x)T_1(u-T_h(\vartheta))\exp(B(|\vartheta|))\,dx\,dt+C_0\int_{\{|\vartheta_0|>h\}}|\vartheta_0|\,dx.
\end{aligned}$$

By the Lebesgue Dominated Convergence Theorem, passing to the limit as $h \to +\infty$, we conclude that (4.26) and (4.27).
Let E be a measurable subset of Q_T and $h > 0$. By using (1.6), (2.5) and Young inequality, we obtain

$$\begin{aligned}\int_E |F_n(x,t,\vartheta_n)|\,dx\,dt \leq \int_E |f(x,t)|\,dx\,dt + \int_E \Upsilon(x,\varepsilon|\vartheta_n|)\lambda(\varepsilon|\vartheta_n|)|\,dx\,dt \\ + \int_E \Upsilon(x,1)\,dx\,dt + \int_E g_\varepsilon(x)\,dx\,dt.\end{aligned} \tag{4.29}$$

Moreover, we have

$$\begin{aligned}\int_E \Upsilon(x,\varepsilon|\vartheta_n|)\lambda(\varepsilon|\vartheta_n|)\,dx\,dt &= \int_{E\cap\{|\vartheta_n|\leq h\}} \Upsilon(x,\varepsilon|T_h(\vartheta_n|)\lambda(\varepsilon|T_h(\vartheta_n|)|\,dx\,dt \\ &\quad + \int_{E\cap\{|\vartheta_n|>h\}} \Upsilon(x,\varepsilon|\vartheta_n|)\lambda(\varepsilon|\vartheta_n|)|\,dx\,dt \\ &\leq \varepsilon\int_E \Upsilon(x,h)\,dx\,dt + \int_{\{|\vartheta_n|>h\}} \Upsilon(x,\varepsilon|\vartheta_n|)\lambda(\epsilon|\vartheta_n|)|\,dx\,dt.\end{aligned} \tag{4.30}$$

By combining (4.27)-(4.30), we conclude that

$$\lim_{|E|\to 0}\limsup_{n\to\infty}\int_E |F_n(x,t,\vartheta_n)|\,dx\,dt = 0.$$

Thus the sequences $(F_n(x,t,\vartheta_n))_n$ is equi-integrable. According to $F_n(x,t,\vartheta_n) \longrightarrow F(x,t,\vartheta)$ a.e. in Q_T and by applying Vitali's theorem, we deduce that

$$|F(x,t,\vartheta)| \in L^1(Q_T), \tag{4.31}$$

and

$$F_n(x,t,\vartheta_n) \longrightarrow F(x,t,\vartheta) \quad \text{strongly in } L^1(Q_T) \quad \text{as } n\to\infty. \tag{4.32}$$

Now we verify that the assumptions of Lemma 3.4 are satisfied.
Firstly, thanks to (4.16) and Banach–Alaoglu–Bourbaki's theorem there exists a measurable function $\varpi_k \in (L_{\overline{\Upsilon}}(Q_T))^N$ such that

$$a_n(x,t,T_k(\vartheta_n),\nabla T_k(\vartheta_n)) \rightharpoonup \varpi_k \quad \text{weakly in } (L_{\overline{\Upsilon}}(Q_T))^N \text{ as } n\to\infty. \tag{4.33}$$

Moreover, we have $\mathcal{D}(Q_T)$ dense modularly in $W_0^{1,x}L_\Upsilon(Q_T)$ and $T_k(\vartheta) \in W_0^{1,x}L_\Upsilon(Q_T)$ for any $k > 0$. Then, there exists a sequence $(v_j)_j \subset \mathcal{D}(Q_T)$ such that

$$T_k(v_j) \longrightarrow T_k(\vartheta) \quad \text{modularly in } W_0^{1,x}L_\Upsilon(Q_T) \quad \text{and a.e. in } Q_T \quad \text{as } j\to\infty. \tag{4.34}$$

Let $(T_k(v_j))_\mu \in L^\infty(Q_T)\cap W_0^{1,x}L_\Upsilon(Q_T)$ represent the unique solution of the problem for any $\mu > 0$ (see Section 3 of [3]):

$$\begin{cases}\dfrac{\partial(T_k(v_j))_\mu}{\partial t} + \mu((T_k(v_j))_\mu - T_k(v_j)) = 0 \quad \text{in } \mathcal{D}'(Q_T),\\ (T_k(v_j))_\mu(t=0) = \alpha_0^\mu \quad \text{in } \Omega,\end{cases}$$

where $(\alpha_0^\mu)_\mu$ is a sequence of functions defined on Ω satisfying

$$\alpha_0^\mu \in L^\infty(\Omega)\cap W_0^1L_\Upsilon(\Omega) \text{ and } \alpha_0^\mu \longrightarrow T_k(\vartheta_0) \text{ a.e. in } \Omega \quad \text{as } \mu\to\infty.$$

Moreover, by using (4.34) we obtain

$$(T_k(v_j))_\mu \longrightarrow (T_k(\vartheta))_\mu \quad \text{modularly in } W_0^{1,x}L_\Upsilon(Q_T) \quad \text{as } j \to \infty, \tag{4.35}$$

and

$$(T_k(\vartheta))_\mu \longrightarrow T_k(\vartheta) \quad \text{modularly in } W_0^{1,x}L_\Upsilon(Q_T) \quad \text{as } \mu \to \infty. \tag{4.36}$$

Fixed $k > 0$ and $\tau \in (0, T)$. By taking $S_h(\vartheta_n)(T_k(\vartheta_n) - (T_k(v_j))_\mu)\chi_{(0,\tau)}$ as a test function in (4.6), we get

$$I^1_{n,j,\mu,h} + I^2_{n,j,\mu,h} + I^3_{n,j,\mu,h} + I^4_{n,j,\mu,h} + I^5_{n,j,\mu,h} = I^6_{n,j,\mu,h}, \tag{4.37}$$

where

$$\begin{aligned}
I^1_{n,j,\mu,h} &= \int_{Q_\tau} \frac{\partial \vartheta_n}{\partial t} S_h(\vartheta_n)(T_k(\vartheta_n) - (T_k(v_j))_\mu)\, dx\, dt,\\
I^2_{n,j,\mu,h} &= \int_{Q_\tau} a_n(x,t,\vartheta_n,\nabla\vartheta_n)\cdot\nabla(T_k(\vartheta_n) - (T_k(v_j))_\mu)S_h(\vartheta_n)\, dx\, dt,\\
I^3_{n,j,\mu,h} &= -\int_{\{h\le|\vartheta_n|\le h+1\}} a_n(x,t,\vartheta_n,\nabla\vartheta_n)\cdot\nabla\vartheta_n\ \mathrm{sign}(\vartheta_n)(T_k(\vartheta_n) - (T_k(v_j))_\mu)\, dx\, dt,\\
I^4_{n,j,\mu,h} &= \int_{Q_\tau} \phi_n(\vartheta_n)\cdot\nabla(T_k(\vartheta_n) - (T_k(v_j))_\mu)S_h(\vartheta_n)\, dx\, dt,\\
I^5_{n,j,\mu,h} &= -\int_{\{h\le|\vartheta_n|\le h+1\}} \phi_n(\vartheta_n)\cdot\nabla\vartheta_n\ \mathrm{sign}(\vartheta_n)(T_k(\vartheta_n) - (T_k(v_j))_\mu)\, dx\, dt,\\
I^6_{n,j,\mu,h} &= \int_{Q_\tau} F_n(x,t,\vartheta_n)S_h(\vartheta_n)(T_k(\vartheta_n) - (T_k(v_j))_\mu)\, dx\, dt.
\end{aligned}$$

Concerning the first term $I^1_{n,j,\mu,h}$, we have

$$\begin{aligned}
I^1_{n,j,\mu,h} &= \int_{Q_\tau} \frac{\partial \widetilde{S}_h(\vartheta_n)}{\partial t}(T_k(\vartheta_n) - (T_k(v_j))_\mu)\, dx\, dt\\
&= \left[\int_\Omega (\widetilde{S}_h(\vartheta_n) - T_k(\vartheta_n))(T_k(\vartheta_n) - (T_k(v_j))_\mu)\, dx\right]_0^\tau\\
&\quad - \int_{Q_\tau} (\widetilde{S}_h(\vartheta_n) - T_k(\vartheta_n))\left(\frac{\partial T_k(\vartheta_n)}{\partial t} - \frac{\partial (T_k(v_j))_\mu}{\partial t}\right) dx\, dt\\
&\quad + \int_{Q_\tau} \frac{\partial T_k(\vartheta_n)}{\partial t}(T_k(\vartheta_n) - (T_k(v_j))_\mu)\, dx\, dt\\
&= J_1 + J_2 + J_3,
\end{aligned} \tag{4.38}$$

where $\widetilde{S}_h(r) = \displaystyle\int_0^r S_h(s)\, ds$.

For the first term J_1, it is clear that $\widetilde{S}_h(\vartheta_n)$ and $T_k(\vartheta_n)$ have the same sign as ϑ_n. Moreover, we have $\widetilde{S}_h(\vartheta_n) = T_k(\vartheta_n)$ on the set $\{|\vartheta_n| \le k\}$, and since

$|\widetilde{S}_h(\vartheta_n)| \geq |T_k(\vartheta_n)|$ on the set $\{|\vartheta_n| > k\}$. Hence

$$\begin{aligned} J_1 &= \left[\int_{\{|\vartheta_n|>k\}} (\widetilde{S}_h(\vartheta_n) - T_k(\vartheta_n))(T_k(\vartheta_n) - (T_k(v_j))_\mu)\, dx\right]_0^\tau \\ &\geq -\int_{\{|\vartheta_{0,n}|>k\}} (\widetilde{S}_h(\vartheta_{0,n}) - T_k(\vartheta_{0,n}))(T_k(\vartheta_{0,n}) - \alpha_0^\mu)\, dx \\ &\geq \varepsilon(n,\mu). \end{aligned} \tag{4.39}$$

For the second term J_2, we have $(\widetilde{S}_h(\vartheta_n) - T_k(\vartheta_n))\dfrac{\partial T_k(\vartheta_n)}{\partial t} = 0$. Then

$$\begin{aligned} J_2 &= \int_{\{|\vartheta_n|>k\}} (\widetilde{S}_h(\vartheta_n) - T_k(\vartheta_n))\frac{\partial (T_k(v_j))_\mu}{\partial t}\, dx\, dt \\ &= \mu \int_{\{|\vartheta_n|>k\}} (\widetilde{S}_h(\vartheta_n) - T_k(\vartheta_n))(T_k(v_j) - T_k(\vartheta_n))\, dx\, dt \\ &\quad + \mu \int_{\{|\vartheta_n|>k\}} (\widetilde{S}_h(\vartheta_n) - T_k(\vartheta_n))(T_k(\vartheta_n) - (T_k(v_j))_\mu)\, dx\, dt \\ &\geq \mu \int_{\{|\vartheta_n|>k\}} (\widetilde{S}_h(\vartheta_n) - T_k(\vartheta_n))(T_k(v_j) - T_k(\vartheta_n))\, dx\, dt \\ &\geq \varepsilon(n,j). \end{aligned} \tag{4.40}$$

For the last term J_3, we have

$$\begin{aligned} J_3 &= \frac{1}{2}\left[\int_\Omega (T_k(\vartheta_n) - (T_k(v_j))_\mu))^2\, dx\right]_0^\tau + \mu \int_{Q_\tau} (T_k(v_j) - (T_k(v_j))_\mu)(T_k(\vartheta_n) - (T_k(v_j))_\mu)\, dx\, dt \\ &\geq -\frac{1}{2}\int_\Omega (T_k(\vartheta_0) - \alpha_0^\mu)^2\, dx + \mu \int_{Q_\tau} (T_k(v_j) - T_k(\vartheta))(T_k(\vartheta) - (T_k(v_j))_\mu)\, dx\, dt \\ &\quad + \mu \int_{Q_\tau} (T_k(\vartheta) - (T_k(v_j))_\mu)^2\, dx\, dt + \varepsilon(n) \\ &\geq \varepsilon(n,j). \end{aligned} \tag{4.41}$$

By combining (4.38)-(4.41), we deduce that

$$I^1_{n,j,\mu,h} \geq \varepsilon(n,j,\mu). \tag{4.42}$$

Concerning the second term $I^2_{n,j,\mu,h}$, notice that $S_h(\vartheta_n) = 1$ on $\{|\vartheta_n| \leq k\}$ and $|S_h(\vartheta_n)| \leq 1$, then

$$\begin{aligned} I^2_{n,j,\mu,h} &= \int_{Q_\tau} a_n(x,t,T_k(\vartheta_n), \nabla T_k(\vartheta_n)) \cdot (\nabla T_k(\vartheta_n) - \nabla (T_k(v_j))_\mu)\, dx\, dt \\ &\quad - \int_{\{k<|\vartheta_n|\leq h+1\}} a_n(x,t,T_{h+1}(\vartheta_n), \nabla T_{h+1}(\vartheta_n)) \cdot \nabla (T_k(v_j))_\mu S_h(\vartheta_n)\, dx\, dt. \end{aligned}$$

Since $\nabla (T_k(v_j))_\mu \in (E_\Upsilon(Q_T))^N$, it follows from (4.24) and Lemma 4.6 in [17] that

$$\nabla (T_k(v_j))_\mu S_h(\vartheta_n)\chi_{\{k<|\vartheta_n|\leq h+1\}} \longrightarrow \nabla (T_k(v_j))_\mu S_h(\vartheta)\chi_{\{k<|\vartheta|\leq h+1\}}$$

strongly in $(E_\Upsilon(Q_T))^N$ as $n \to \infty$.
By using (4.33), we obtain

$$\int_{\{k<|\vartheta_n|\le h+1\}} a_n(x,t,T_{h+1}(\vartheta_n),\nabla T_{h+1}(\vartheta_n))\cdot\nabla(T_k(v_j))_\mu S_h(\vartheta_n)\,dx\,dt$$
$$\longrightarrow \int_{\{k<|\vartheta|\le h+1\}} \varpi_{h+1}\cdot\nabla(T_k(v_j))_\mu S_h(\vartheta)\,dx\,dt \quad \text{as } n\to\infty.$$

Then

$$\int_{\{k<|\vartheta|\le h+1\}} \varpi_{h+1}\cdot\nabla(T_k(v_j))_\mu S_h(\vartheta)\,dx\,dt \longrightarrow 0 \quad \text{as } j,\mu\to\infty.$$

It follows that

$$I^2_{n,j,\mu,h} = \int_{Q_\tau} a_n(x,t,T_k(\vartheta_n),\nabla T_k(\vartheta_n))\cdot(\nabla T_k(\vartheta_n)-\nabla(T_k(v_j))_\mu)\,dx\,dt+\varepsilon(n,j,\mu). \tag{4.43}$$

For the term $I^3_{n,j,\mu,h}$, since $|(T_k(\vartheta_n)-(T_k(v_j))_\mu)|\le 2k$, we have

$$|I^3_{n,j,\mu,h}|\le 2k\int_{\{h\le|\vartheta_n|\le h+1\}} a_n(x,t,\vartheta_n,\nabla\vartheta_n)\cdot\nabla\vartheta_n\,dx\,dt,$$

thanks to (4.26), we conclude that

$$I^3_{n,j,\mu,h}\le\varepsilon(n,h). \tag{4.44}$$

For the term $I^4_{n,j,\mu,h}$, since $\mathrm{supp}(S_h)\subset[-h-1,h+1]$, then

$$|I^4_{n,j,\mu,h}|\le\int_{Q_\tau}|\phi_n(T_{h+1}(\vartheta_n))\cdot\nabla(T_k(\vartheta_n)-(T_k(v_j))_\mu)||S_h(\vartheta_n)|\,dx\,dt,$$
$$\le\int_{Q_\tau}|\phi_n(T_{h+1}(\vartheta_n))||\nabla(T_k(\vartheta_n)-(T_k(v_j))_\mu)|\,dx\,dt.$$

By using (4.15), we obtain $|\nabla(T_k(\vartheta_n)-(T_k(v_j))_\mu)|\rightharpoonup 0$ weakly in $L_\Upsilon(Q_T)$ as $n,j,\mu\to\infty$, and thanks to (4.25) we conclude that

$$I^4_{n,j,\mu,h}\le\varepsilon(n,j,\mu). \tag{4.45}$$

Concerning the term $I^5_{n,\mu,j,h}$, we set $\Phi_n(t)=\int_0^t\phi_n(s)\,ds\in\mathcal{C}^1(\mathbb{R},\mathbb{R}^N)$. Then

$$I^5_{n,\mu,j,h}=-\int_{\{h\le|\vartheta_n|\le h+1\}}\phi_n(\vartheta_n)\cdot\nabla\vartheta_n\ (T_k(\vartheta_n)-(T_k(v_j))_\mu)\mathrm{sign}(\vartheta_n)\,dx\,dt,$$
$$=-\int_{\{h\le|\vartheta_n|\le h+1\}}\Phi_n(T_{h+1}(\vartheta_n))\cdot\nabla(T_k(v_j))_\mu\mathrm{sign}(\vartheta_n)\,dx\,dt,$$

Since $\Phi_n(T_{h+1}(\vartheta_n))\longrightarrow\Phi(T_{h+1}(\vartheta))$ strongly in $(E_{\overline{\Upsilon}}(Q_T))^N$ as $n\to\infty$, we obtain

$$\lim_{n\to+\infty}\lim_{j\to+\infty}\lim_{\mu\to+\infty} I^5_{n,j,\mu,h}=-\int_{\{h\le|\vartheta|\le h+1\}}\Phi(T_{h+1}(\vartheta))\cdot\nabla(T_k(\vartheta))\mathrm{sign}(\vartheta)\,dx\,dt=0. \tag{4.46}$$

Finally, for the last term $I^6_{n,j,\mu,h}$, we have

$$|S_h(\vartheta_n)(T_k(\vartheta_n)-(T_k(v_j))_\mu)|\le 2k,$$

and

$$\lim_{n\to+\infty}\lim_{\mu\to+\infty}\lim_{j\to+\infty} S_h(\vartheta_n)(T_k(\vartheta_n)-(T_k(v_j))_\mu)=0 \quad \text{a.e in } Q_T \text{ for a subsequence.}$$

Then

$$S_h(\vartheta_n)(T_k(\vartheta_n)-(T_k(v_j))_\mu)\rightharpoonup 0 \quad \text{weak-* in } \ L^\infty(Q_T),$$

by using (4.32), we conclude that

$$I^6_{n,j,\mu,h}=\varepsilon(n,j,\mu). \tag{4.47}$$

By combining (4.42)-(4.47) we conclude that

$$\int_{Q_\tau} a_n(x,t,T_k(\vartheta_n),\nabla T_k(\vartheta_n))\cdot(\nabla T_k(\vartheta_n)-\nabla(T_k(v_j))_\mu)\,dx\,dt\le\varepsilon(n,j,\mu,h). \tag{4.48}$$

Now, we will establish that

$$\begin{aligned}\lim_{\tau\to\infty}\lim_{n\to\infty}\int_{Q_T}(a_n(x,t,T_k(\vartheta_n),&\nabla T_k(\vartheta_n))-a_n(x,t,T_k(\vartheta_n),\nabla T_k(\vartheta)\chi^\tau))\\ &(\nabla T_k(\vartheta_n)-\nabla T_k(\vartheta)\chi^\tau)\,dx\,dt=0,\end{aligned} \tag{4.49}$$

where χ^τ is the characteristic function of the set $\{(x,t)\in Q_T : |\nabla T_k(\vartheta)|\le\tau\}$. In view of (4.49), we have

$$\begin{aligned}&\int_{Q_T}(a_n(x,t,T_k(\vartheta_n),\nabla T_k(\vartheta_n))-a_n(x,t,T_k(\vartheta_n),\nabla T_k(\vartheta)\chi^\tau))\cdot(\nabla T_k(\vartheta_n)-\nabla T_k(\vartheta)\chi^\tau)\,dx\,dt\\ &\le\varepsilon(n,j,\mu,h)+\int_{Q_T}a_n(x,t,T_k(\vartheta_n),\nabla T_k(\vartheta_n))\cdot(\nabla(T_k(v_j))_\mu-\nabla T_k(\vartheta)\chi^\tau)\,dx\,dt\\ &-\int_{Q_T}a_n(x,t,T_k(\vartheta_n),\nabla T_k(\vartheta)\chi^\tau)\cdot(\nabla T_k(\vartheta_n)-\nabla T_k(\vartheta)\chi^\tau)\,dx\,dt\\ &=\varepsilon(n,j,\mu,h)+I_1+I_2,\end{aligned} \tag{4.50}$$

For the term I_1, we have $(\nabla(T_k(v_j))_\mu-\nabla T_k(\vartheta)\chi^\tau)\in(E_\Upsilon(Q_T))^N$, and by using (4.33) we obtain

$$\begin{aligned}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} I_1&=\lim_{j\to\infty}\lim_{\mu\to\infty}\int_{Q_T}\varpi_k\cdot(\nabla(T_k(v_j))_\mu-\nabla T_k(\vartheta)\chi^\tau)\,dx\,dt\\ &=\int_{Q_T}\varpi_k\cdot\nabla T_k(\vartheta)\chi_{\{|\nabla T_k(\vartheta)|>\tau\}}\,dx\,dt.\end{aligned}$$

Since $|\varpi_k\cdot\nabla T_k(\vartheta)|\in L^1(Q_T)$, by using Lebesgue's dominated convergence theorem, we obtain

$$\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} I_1=0. \tag{4.51}$$

For the term I_2, since $\overline{\Upsilon}$ satisfies the Δ_2-condition and in view of (1.2), (4.26) and Lebesgue dominated convergence theorem, we deduce that

$$a_n(x,\Gamma,T_k(\vartheta_n),\nabla T_k(\vartheta)\chi^\tau)\longrightarrow a(x,\Gamma,T_k(\vartheta),\nabla T_k(\vartheta)\chi^\tau)$$

strongly in $(E_{\overline{\Upsilon}}(Q_T))^N$ as $n \to \infty$.
Then,

$$\begin{aligned}\lim_{n\to\infty} I_2 &= -\lim_{n\to\infty}\int_{Q_T} a_n(x,t,T_k(\vartheta_n),\nabla T_k(\vartheta)\chi^\tau)\cdot(\nabla T_k(\vartheta_n)-\nabla T_k(\vartheta)\chi^\tau)\,dx\,dt\\ &= -\int_{Q_T} a(x,t,T_k(\vartheta),\nabla T_k(\vartheta)\chi^\tau)\cdot\nabla T_k(\vartheta)\chi_{\{|\nabla T_k(\vartheta)|>\tau\}}\,dx\,dt\\ &= -\int_{Q_T} a(x,t,T_k(\vartheta),0)\cdot\nabla T_k(\vartheta)\chi_{\{|\nabla T_k(\vartheta)|>\tau\}}\,dx\,dt\\ &= 0.\end{aligned} \tag{4.52}$$

By combining (4.50)-(4.51) , we obtain (4.52).
In view of Lemma 3.4, we conclude that

$$\nabla\vartheta_n \longrightarrow \nabla\vartheta \quad \text{a.e. in } Q_T, \tag{4.53}$$

$$T_k(\vartheta_n) \longrightarrow T_k(\vartheta) \quad \text{modularly in } W_0^{1,x}L_\Upsilon(Q_T), \tag{4.54}$$

and

$$a_n(x,t,T_k(\vartheta_n),\nabla T_k(\vartheta_n))\cdot\nabla T_k(\vartheta_n) \longrightarrow a(x,t,T_k(\vartheta),\nabla T_k(\vartheta))\cdot\nabla T_k(\vartheta) \quad \text{strongly in } L^1(Q_T). \tag{4.55}$$

Moreover, thanks to (4.24), (4.53) and Lemma 1 in [9], we conclude that

$$a_n(x,t,T_k(\vartheta_n),\nabla T_k(\vartheta_n)) \rightharpoonup a(x,t,T_k(\vartheta),\nabla T_k(\vartheta)) \quad \text{weakly in } (L_{\overline{\Upsilon}}(Q_T))^N. \tag{4.56}$$

Let $v \in W_0^{1,x}L_\Upsilon(Q_T)\cap L^\infty(Q_T)$ such that $\dfrac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{\Upsilon}}(Q_T)+L^1(Q_T)$. By using Theorem 2.14 in [16], there exists a prolongation $\overline{v} = v$ on Q_T, $\overline{v} \in W_0^{1,x}L_\Upsilon(\Omega\times\mathbb{R})\cap L^1(\Omega\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R})$ and $\dfrac{\partial\overline{v}}{\partial t} \in W^{-1,x}L_{\overline{\Upsilon}}(\Omega\times\mathbb{R})+L^1(\Omega\times\mathbb{R})$. Then, there exists a sequence $(v_j)_j$ in $\mathcal{D}(\Omega\times\mathbb{R})$ such that

$$v_j \longrightarrow \overline{v} \quad \text{in } W_0^{1,x}L_\Upsilon(\Omega\times\mathbb{R}),$$

$$\frac{\partial v_j}{\partial t} \longrightarrow \frac{\partial\overline{v}}{\partial t} \quad \text{in } W^{-1,x}L_{\overline{\Upsilon}}(\Omega\times\mathbb{R})+L^1(\Omega\times\mathbb{R}),$$

for the modular convergence and $\|v_j\|_{\infty,Q_T} \le (N+2)\|v\|_{\infty,Q_T}$.
For every $\tau\in(0,T)$, by using $T_k(\vartheta_n - v_j)\chi_{(0,\tau)}$ as a test function in (4.6), we get

$$\begin{aligned}&\int_{Q_\tau}\frac{\partial\vartheta_n}{\partial t}T_k(\vartheta_n-v_j)\,dx\,dt + \int_{Q_\tau} a_n(x,t,\vartheta_n,\nabla\vartheta_n)\cdot\nabla T_k(\vartheta_n-v_j)\,dx\,dt\\ &\le \int_{Q_\tau} F_n(x,t,\vartheta_n)T_k(\vartheta_n-v_j)\,dx\,dt - \int_{Q_\tau}\phi_n(\vartheta_n)\cdot\nabla T_k(\vartheta_n-v_j)\,dx\,dt.\end{aligned} \tag{4.57}$$

Concerning the first term of (4.57), we obtain

$$\begin{aligned}\int_{Q_\tau} \frac{\partial \vartheta_n}{\partial t} T_k(\vartheta_n - v_j)\,dx\,dt &= \left[\int_\Omega T_k(\vartheta_n - v_j)\,dx\right]_0^\tau + \int_{Q_\tau} \frac{\partial v_j}{\partial t} T_k(\vartheta_n - v_j)\,dx\,dt \\ &= \left[\int_\Omega T_k(\vartheta - v_j)\,dx\right]_0^\tau + \int_{Q_\tau} \frac{\partial v_j}{\partial t} T_k(\vartheta - v_j)\,dx\,dt + \varepsilon(n) \\ &= \int_{Q_\tau} \frac{\partial \vartheta}{\partial t} T_k(\vartheta - v)\,dx\,dt + \varepsilon(n,j).\end{aligned}$$

By using the same method as in [16], we establish that

$$\begin{aligned}\liminf_{j\to\infty} \liminf_{n\to\infty} \int_{Q_\tau} a_n(x,t,\vartheta_n,\nabla\vartheta_n)\cdot\nabla T_k(\vartheta_n - v_j)\,dx\,dt \\ \geq \int_{Q_\tau} a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla T_k(\vartheta - v)dx\,dt.\end{aligned}$$

Since $\nabla T_k(\vartheta_n - v_j) \rightharpoonup \nabla T_k(\vartheta - v_j)$ in $L_\Upsilon(Q_T)$ as $n \to \infty$, we have

$$\int_{Q_\tau} \phi_n(\vartheta_n)\cdot\nabla T_k(\vartheta_n - v_j)\,dx\,dt \longrightarrow \int_{Q_\tau} \phi(\vartheta)\cdot\nabla T_k(\vartheta - v)\,dx\,dt \quad \text{as } n, j \to \infty.$$

Consequently, since $T_k(\vartheta_n - v_j) \rightharpoonup T_k(\vartheta - v)$ weak-* in $L^\infty(Q_T)$ as $n, j \longrightarrow \infty$ and by using (4.32), we conclude that

$$\begin{aligned}&\int_{Q_T} \frac{\partial \vartheta}{\partial t} T_k(\vartheta - v)\,dx\,dt + \int_{Q_T} a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla T_k(\vartheta - v)\,dx\,dt \\ &\leq \int_{Q_T} F(x,t,\vartheta) T_k(\vartheta - v)\,dx\,dt - \int_{Q_T} \phi(\vartheta)\cdot\nabla T_k(\vartheta - v)\,dx\,dt.\end{aligned} \tag{4.58}$$

This completes the proof of Theorem 4.1. □

5. Uniqueness result

In this final section, we prove that every entropy solution of problem (1.1) is also a renormalized solution. Moreover, we conclude that the entropy and renormalized solutions for our problem are equivalent. Finally we prove the uniqueness of the renormalized and entropy solutions to our problem.

Theorem 5.1. *Under the same assumptions of theorem 4.1, every entropy solution ϑ of problem (1.1) is also renormalized solution in the following sense:*

$$T_k(\vartheta) \in W_0^{1,x} L_\Upsilon(Q_T) \ \text{ for any } k > 0, \tag{5.1}$$

$$\vartheta \in L^\infty(0,T;L^1(\Omega)), \tag{5.2}$$

$$F(x,t,\vartheta) \in L^1(Q_T), \tag{5.3}$$

$$\lim_{h\longrightarrow+\infty} \int_{\{(x,t)\in Q_T \ :\ h\leq|\vartheta|\leq h+1\}} a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla\vartheta\,dx\,dt = 0, \tag{5.4}$$

$$\begin{aligned}&\int_{Q_T}\frac{\partial\vartheta}{\partial t}S'(\vartheta)\nu\,dx\,dt+\int_{Q_T}a(x,t,\vartheta,\nabla\vartheta)\cdot(\nabla\vartheta S''(\vartheta)\nu+\nabla\nu S'(\vartheta))\,dx\,dt\\&+\int_{Q_T}\phi(\vartheta)\cdot(\nabla\vartheta S''(\vartheta)\nu+\nabla\nu S'(\vartheta))\,dx\,dt=\int_{Q_T}F(x,t,\vartheta)\nu S'(\vartheta)\,dx\,dt,\end{aligned}\tag{5.5}$$

$$S(\vartheta)(t=0)=S(\vartheta_0)\quad in\ \Omega,\tag{5.6}$$

for any $\nu\in W_0^{1,x}L_\Upsilon(Q_T)\cap L^\infty(Q_T)$ *and for any* $S(\cdot)\in W^{2,\infty}(\mathbb{R})$ *with* S' *has a compact support.*

Proof of theorem 5.1
In view of theorem 4.2, the results (5.1)–(5.3) holds true.
Let $h>0$, by taking $k=1$ and $v=T_h(\vartheta)$ in (4.5) we have

$$\begin{aligned}&\int_{Q_T}\frac{\partial\vartheta}{\partial t}T_1(\vartheta-T_h(\vartheta))\,dx\,dt+\int_{Q_T}a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla T_1(\vartheta-T_h(\vartheta))\,dx\,dt\\&+\int_{Q_T}\phi(\vartheta)\cdot\nabla T_1(\vartheta-T_h(\vartheta))\,dx\,dt\le\int_{Q_T}F(x,t,\vartheta)T_1(\vartheta-T_h(\vartheta))\,dx\,dt.\end{aligned}$$

By applying the same argument as in the proof of Theorem 4.1, we conclude that

$$\begin{aligned}\int_{\{h\le|\vartheta|\le h+1\}}a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla\vartheta\,dx\,dt&\le\int_{Q_T}F(x,t,\vartheta)T_1(\vartheta-T_h(\vartheta))\,dx\,dt\\&+\int_{\{|\vartheta_0|>h\}}|\vartheta_0|\,dx.\end{aligned}$$

Applying Lebesgue's Theorem and passing $h\to+\infty$ we obtain (5.4).
Let $h>m>0$. Returning to (4.5) and taking $v=T_h(\vartheta)-S'(\vartheta)\nu$, with $supp(S')\subset[-m,m]$ and $k>\|S'(\cdot)\|_\infty\times\|\nu\|_\infty+1$, we get

$$\mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3\le\mathcal{K}_4,\tag{5.7}$$

where

$$\begin{aligned}\mathcal{K}_1&=\int_{Q_T}\frac{\partial\vartheta}{\partial t}T_k(\vartheta-T_h(\vartheta)+S'(\vartheta)\nu)\,dx\,dt,\\\mathcal{K}_2&=\int_{Q_T}a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla T_k(\vartheta-T_h(\vartheta)+S'(\vartheta)\nu)\,dx\,dt,\\\mathcal{K}_3&=\int_{Q_T}\phi(\vartheta)\cdot\nabla T_k(\vartheta-T_h(\vartheta)+S'(\vartheta)\nu)\,dx\,dt,\\\mathcal{K}_4&=\int_{Q_T}F(x,t,\vartheta)T_k(\vartheta-T_h(\vartheta)+S'(\vartheta)\nu)\,dx\,dt.\end{aligned}$$

For the first term $\mathcal{K}_1$, since $k > \|S'(\cdot)\|_\infty \times \|\nu\|_\infty + 1$, we have $T_k(S'(\vartheta)\nu) = S'(\vartheta)\nu$. Then

$$\begin{aligned}\mathcal{K}_1 &= \int_{Q_T} \frac{\partial \vartheta}{\partial t} T_k(\vartheta - T_h(\vartheta) + S'(\vartheta)\nu)\, dx\, dt \\ &= \int_{Q_T} \frac{\partial \vartheta}{\partial t} T_k(S'(\vartheta)\nu)\, dx\, dt + \int_{\{|\vartheta|>h\}} \frac{\partial \vartheta}{\partial t} T_k(\vartheta - T_h(\vartheta))\, dx\, dt \\ &\geq \int_{Q_T} \frac{\partial \vartheta}{\partial t} S'(\vartheta)\nu\, dx\, dt.\end{aligned} \tag{5.8}$$

For the term $\mathcal{K}_2$, we have

$$\begin{aligned}\mathcal{K}_2 &= \int_{Q_T} a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla T_k(\vartheta - T_h(\vartheta) + S'(\vartheta)\nu)\, dx\, dt \\ &= \int_{Q_T} a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla(S'(\vartheta)\nu)\, dx\, dt + \int_{\{h\leq|\vartheta|\leq h+k\}} a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla\vartheta\, dx\, dt.\end{aligned}$$

By using (5.4) , we obtain

$$\lim_{h\to\infty} \mathcal{K}_2 = \int_{Q_T} a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla(S'(\vartheta)\nu)\, dx\, dt. \tag{5.9}$$

For the term $\mathcal{K}_3$, we have

$$\begin{aligned}\mathcal{K}_3 &= \int_{Q_T} \phi(\vartheta)\cdot\nabla T_k(\vartheta - T_h(\vartheta) + S'(\vartheta)\nu)\, dx\, dt \\ &= \int_{Q_T} \phi(\vartheta)\cdot\nabla(S'(\vartheta)\nu)\, dx\, dt + \int_{Q_T} \phi(\vartheta)\cdot\nabla T_k(\vartheta - T_h(\vartheta))\, dx\, dt \\ &= \int_{Q_T} \phi(\vartheta)\cdot\nabla(S'(\vartheta)\nu)\, dx\, dt.\end{aligned} \tag{5.10}$$

For the first term $\mathcal{K}_4$, since $F(x,t,\vartheta) \in L^1(Q_T)$ and $T_k(\vartheta - T_h(\vartheta) + S'(\vartheta)\nu) \rightharpoonup S'(\vartheta)\nu$ weak-* in $L^\infty(Q_T)$ as $h \longrightarrow \infty$ we conclude that

$$\lim_{h\longrightarrow\infty} \mathcal{K}_4 = \int_{Q_T} F(x,t,\vartheta)S'(\vartheta)\nu\, dx\, dt. \tag{5.11}$$

By combining (5.7)-(5.11) and arguing for both ν and $-\nu$, we obtain (5.7). □

Remark 5.2. By taking $S'(\vartheta) = S_h(\vartheta)$ and $\nu = T_k(\vartheta - v)$ in (5.5), and then letting $h \to \infty$, we obtain (4.5). Then, the renormalized solutions of problem (1.1) are also entropy solutions. Therefore, we conclude that the entropy and renormalized solutions of problem (1.1) are equivalents.

Now, we present the following essential lemma.

Lemma 5.3. *For any $0 < k < \ell$, we define*

$$H(\vartheta, v, \ell, k) = \int_{\{\ell-k<|\vartheta|<\ell+k\}} \Big(a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla\vartheta + |\phi(\vartheta)||\nabla\vartheta|\Big)\,dx\,dt + \int_{\{\ell-k<|v|<\ell+k\}} \Big(a(x,t,v,\nabla v)\cdot\nabla v + |\phi(v)||\nabla v|\Big)\,dx\,dt.$$

where ϑ and v are two renormalized solutions to problem (1.1). Assume that (1.2)-(1.8) hold, then

$$\liminf_{\ell\to+\infty}\limsup_{k\to 0}\frac{1}{k}H(\vartheta, v, \ell, k) = 0. \tag{5.12}$$

The proof of Lemma 5.3 follows the same argument as in Lemma 4.1 of [2].

Theorem 5.4. *Under the assumptions (1.2)-(1.8) and for any compact set $K \subset \mathbb{R}$, there exists $L_K \in E_{\overline{\Upsilon}}(Q_T)$ and $\rho_K > 0$ such that : for any $r, \overline{r} \in K$,*

$$\big|a(x,t,r,\xi) - a(x,t,\overline{r},\xi)\big| \le \Big(L_K(x,t) + \rho_K \overline{\Upsilon}_x^{-1}\Upsilon(x,|\xi|)\Big)|r-\overline{r}|, \tag{5.13}$$

$$\big|\phi(r) - \phi(\overline{r})\big| \le L_K(x,t)|r-\overline{r}|, \tag{5.14}$$

for almost every $(x,t) \in Q_T$ and for every $\xi \in \mathbb{R}^N$. Then the problem (1.1) has a unique renormalized solution.

Proof of theorem 5.4

For all $n > 0$ and $\sigma > 0$, we introduce a smooth approximation $\widetilde{S}_n^\sigma$ of the truncation function T_n such that $\widetilde{S}_n^\sigma(0) = 0$ and

$$\big(\widetilde{S}_n^\sigma\big)'(r) = \begin{cases} 0 & \text{for} \quad |r| \ge n+\sigma, \\ \frac{n+\sigma-|r|}{\sigma} & \text{for} \quad n \le |r| \le n+\sigma, \\ 1 & \text{for} \quad |r| \le n. \end{cases} \tag{5.15}$$

Fixed $n > 0$ and for almost every $(x,t) \in Q_T$, we have

$$\lim_{\sigma\to 0}\big(\widetilde{S}_n^\sigma\big)'(r) = \chi_{\{|r|\le n\}} \quad \text{and} \quad \lim_{\sigma\to 0}\widetilde{S}_n^\sigma(r) = T_n(r). \tag{5.16}$$

Since $\widetilde{S}_n^\sigma \in W^{2,\infty}(\mathbb{R})$ and $\operatorname{supp}((\widetilde{S}_n^\sigma)') \subset [-n-\sigma, n+\sigma]$. By taking $S = \widetilde{S}_n^\sigma$ and $\nu = \frac{1}{k}T_k\big(\widetilde{S}_n^\sigma(\vartheta) - \widetilde{S}_n^\sigma(v)\big)\chi_{(0,t)}$ in the difference of equation (5.5), with ϑ and v being two renormalized solutions of problem (1.1), we obtain

$$I_{1,n}^\sigma + I_{2,n}^\sigma + I_{3,n}^\sigma + I_{4,n,}^\sigma + I_{5,n}^\sigma = I_{6,n}^\sigma, \tag{5.17}$$

where

$$I^{\sigma}_{1,n} = \frac{1}{k}\int_0^T\int_0^t \Big\langle \frac{\partial\big(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v)\big)}{\partial t}; T_k\big(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v)\big)\Big\rangle\, ds\, dt,$$

$$I^{\sigma}_{2,n} = \frac{1}{k}\int_0^T\int_0^t\int_\Omega \Big[(\widetilde{S}^{\sigma}_n)'(\vartheta)a(x,\Gamma,\vartheta,\nabla\vartheta) - (\widetilde{S}^{\sigma}_n)'(v)a(x,t,v,\nabla v)\Big]\cdot\nabla T_k\big(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v)\big)\, dx\, ds\, dt,$$

$$I^{\sigma}_{3,n} = \frac{1}{k}\int_0^T\int_0^t\int_\Omega \Big[(\widetilde{S}^{\sigma}_n)''(\vartheta)a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla\vartheta - (\widetilde{S}^{\sigma}_n)''(v)a(x,t,v,\nabla v)\cdot\nabla v\Big] T_k\big(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v)\big)\, dx\, ds\, dt,$$

$$I^{\sigma}_{4,n} = \frac{1}{k}\int_0^T\int_0^t\int_\Omega \Big[(\widetilde{S}^{\sigma}_n)'(\vartheta)\phi(\vartheta) - (\widetilde{S}^{\sigma}_n)'(v)\phi(v)\Big]\cdot\nabla T_k\big(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v)\big)\, dx\, ds\, dt,$$

$$I^{\sigma}_{5,n} = \frac{1}{k}\int_0^T\int_0^t\int_\Omega \Big[(\widetilde{S}^{\sigma}_n)''(\vartheta)\phi(\vartheta)\cdot\nabla\vartheta - (\widetilde{S}^{\sigma}_n)''(v)\phi(v)\cdot\nabla v\Big] T_k\big(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v)\big)\, dx\, ds\, dt,$$

$$I^{\sigma}_{6,n} = \frac{1}{k}\int_0^T\int_0^t\int_\Omega \Big[F(x,t,\vartheta)(\widetilde{S}^{\sigma}_n)'(\vartheta) - F(x,t,v)(\widetilde{S}^{\sigma}_n)'(v)\Big] T_k\big(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v)\big)\, dx\, ds\, dt.$$

Firstly, for the term $I^{\sigma}_{1,n}$, notice that

$$\widetilde{S}^{\sigma}_n(\vartheta)(t=0) = \widetilde{S}^{\sigma}_n(v)(t=0) = \widetilde{S}^{\sigma}_n(\vartheta_0) \quad a.e.\text{ in } \Omega.$$

Then,

$$\int_0^t \Big\langle \frac{\partial(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v))}{\partial t}; T_k(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v))\Big\rangle\, ds = \int_\Omega \widetilde{T}_k(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v))(\Gamma)\, dx,$$

by using (5.16), we have

$$\begin{aligned}\lim_{k\to0}\lim_{\sigma\to0}\frac{1}{k}\int_0^T\int_0^t \Big\langle \frac{\partial(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v))}{\partial t}; T_k(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v))\Big\rangle\, ds\, dt&\\ = \lim_{k\to0}\lim_{\sigma\to0}\frac{1}{k}\int_{Q_T}\widetilde{T}_k(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v))(t)\, dx\, dt&\\ = \int_{Q_T}|T_n(\vartheta)-T_n(v)|\, dx\, dt.&\end{aligned}$$

Passing to the limit as $n\to+\infty$ in this last equality, we deduce that

$$\lim_{n\to\infty}\lim_{k\to0}\lim_{\sigma\to0} I^{\sigma}_{1,n} = \int_{Q_T}|\vartheta - v|\, dx\, dt. \tag{5.18}$$

For the term $I^{\sigma}_{2,n}$, we set

$$B^{\sigma}_n = \Big((\widetilde{S}^{\sigma}_n)'(\vartheta)a(x,t,\vartheta,\nabla\vartheta) - \widetilde{S}^{\sigma}_n)'(v)a(x,t,v,\nabla v)\Big)\cdot\nabla T_k(\widetilde{S}^{\sigma}_n(\vartheta)-\widetilde{S}^{\sigma}_n(v)).$$

Then,

$$I^{\sigma}_{2,n} = \frac{1}{k}\int_0^T\int_0^t\int_\Omega B^{\sigma}_n\, dx\, ds\, dt = \frac{1}{k}\int_0^T (T-t)B^{\sigma}_n\, dx\, dt.$$

Moreover, we have

$$(\widetilde{S}^{\sigma}_n)'(\vartheta)a(x,t,\vartheta,\nabla\vartheta) = (\widetilde{S}^{\sigma}_n)'(\vartheta)a(x,t,T_{n+1}(\vartheta),\nabla T_{n+1}(\vartheta)),$$

and

$$(\widetilde{S}_n^\sigma)'(v)a(x,t,v,\nabla v) = (\widetilde{S}_n^\sigma)'(v)a(x,t,T_{n+1}(v),\nabla T_{n+1}(v)).$$

In view of (5.15) and (5.16), we have B_n^σ converges to $\big(\chi_{\{|\vartheta|\le n\}}a(x,t,\vartheta,\nabla\vartheta) - \chi_{\{|v|\le n\}}a(x,t,v,\nabla v)\big)\cdot\nabla T_k(T_n(\vartheta)-T_n(v))$ and $|B_n^\sigma|\le R_n\in L^1(Q_T)$, where

$$R_n = \big(|a(x,t,T_{n+1}(\vartheta),\nabla T_{n+1}(\vartheta))| + |a(x,t,T_{n+1}(v),\nabla T_{n+1}(v))|\big) \big(|\nabla T_{n+1}(\vartheta)| + |\nabla T_{n+1}(v)|\big)\chi_{\{|T_n(\vartheta)-T_n(v)|\le k\}}.$$

By using Lebesgue's theorem, we get

$$\begin{aligned}\lim_{\sigma\to 0} I_{2,n}^\sigma &= \lim_{\sigma\to 0}\frac{1}{k}\int_{Q_T}(T-t)B_n^\sigma\,dx\,dt\\ &= \frac{1}{k}\int_{Q_T}(T-t)(\chi_{\{|\vartheta|\le n\}}a(x,t,\vartheta,\nabla\vartheta) - \chi_{\{|v|\le n\}}a(x,t,v,\nabla v))\cdot\nabla T_k(T_n(\vartheta)-T_n(v))\,dx\,dt\\ &= J_1+J_2+J_3+J_4,\end{aligned} \tag{5.19}$$

where

$$\begin{aligned}J_1 &= \frac{1}{k}\int_{\{|\vartheta-v|\le k,|\vartheta|\le n,|v|\le n\}}(T-t)(a(x,t,\vartheta,\nabla\vartheta)-a(x,t,\vartheta,\nabla v))\cdot(\nabla\vartheta-\nabla v)\,dx\,dt,\\ J_2 &= \frac{1}{k}\int_{\{|\vartheta-v|\le k,|\vartheta|\le n,|v|\le n\}}(T-t)(a(x,t,\vartheta,\nabla v)-a(x,t,v,\nabla v))\cdot(\nabla\vartheta-\nabla v)\,dx\,dt,\\ J_3 &= \frac{1}{k}\int_{\{|T_n(\vartheta)-T_n(v)|\le k,|\vartheta|>n,|v|\le n\}}(T-t)a(x,t,v,\nabla v)\cdot\nabla v\,dx\,dt,\\ J_4 &= \frac{1}{k}\int_{\{|T_n(\vartheta)-T_n(v)|\le k,|\vartheta|\le n,|v|>n\}}(T-t)a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla\vartheta\,dx\,dt.\end{aligned}$$

Since $a(x,t,\vartheta,\nabla\vartheta)$ satisfies the condition (1.3), we deduce that

$$J_1\ge 0. \tag{5.20}$$

Also, by (5.13) we obtain

$$|J_2|\le T\int_{\{|\vartheta-v|\le k\}}\Big(L_K(x,t)+\rho_K\overline{\Upsilon}_x^{-1}\Upsilon(x,|v|)\Big)\,(|\nabla\vartheta|+|\nabla v|)\,dx\,dt.$$

Since $\Big(L_K(x,t)+\rho_K\overline{\Upsilon}_x^{-1}\Upsilon(x,|v|)\Big)\,(|\nabla\vartheta|+|\nabla v|)\in L^1(Q_T)$ and by using the Lebesgue dominated convergence theorem, we conclude that

$$\limsup_{k\to 0} J_2 = 0. \tag{5.21}$$

From the definition of T_n, it follows that

$$J_3 = \frac{1}{k}\int_{\{n-k\le v\le n\}\cup\{-n\le v\le -n+k\}}(T-t)a(x,t,v,\nabla v)\cdot\nabla v\,dx\,dt.$$

By using (1.3), we get

$$\liminf_{n\to+\infty}\limsup_{k\to 0} J_3\ge 0. \tag{5.22}$$

Similarly we deduce that

$$\liminf_{n\to+\infty}\limsup_{k\to 0} J_4 \geq 0. \tag{5.23}$$

By combining (5.19)-(5.23), we obtain

$$\liminf_{n\to+\infty}\limsup_{k\to 0}\lim_{\sigma\to 0} I^{\sigma}_{2,n} \geq 0. \tag{5.24}$$

For the terms $I^{\sigma}_{3,n}$ and $I^{\sigma}_{5,n}$, for any $\sigma>0$ and $k>0$ we have

$$\begin{aligned} |I^{\sigma}_{3,n}| \leq \frac{T}{\sigma}\int_{\{n-\sigma\leq\vartheta\leq n\}\cup\{-n\leq v\leq -n+\sigma\}} a(x,t,\vartheta,\nabla\vartheta)\cdot\nabla\vartheta\,dx\,dt \\ + \frac{T}{\sigma}\int_{\{n-\sigma\leq v\leq n\}\cup\{-n\leq v\leq -n+\sigma\}} a(x,t,v,\nabla v)\cdot\nabla v\,dx\,dt, \end{aligned} \tag{5.25}$$

and

$$\begin{aligned} |I^{\sigma}_{5,n}| \leq \frac{T}{\sigma}\int_{\{n-\sigma\leq\vartheta\leq n\}\cup\{-n\leq v\leq -n+\sigma\}} \phi(\vartheta)\cdot\nabla\vartheta\,dx\,dt \\ + \frac{T}{\sigma}\int_{\{n-\sigma\leq v\leq n\}\cup\{-n\leq v\leq -n+\sigma\}} \phi(v)\cdot\nabla v\,dx\,dt. \end{aligned} \tag{5.26}$$

By combining (5.25)-(5.26), we deduce that

$$|I^{\sigma}_{3,n}| + |I^{\sigma}_{5,n}| \leq \frac{T}{\sigma}H(\vartheta,v,n,\sigma). \tag{5.27}$$

For the term $I^{\sigma}_{4,n}$, for any $n\geq 0$ we have

$$\begin{aligned} \limsup_{\sigma\to 0}|I^{\sigma}_{4,n}| &= \Big|\frac{1}{k}\int_{Q_T}(T-t)(\chi_{\{|\vartheta|\leq n\}}\phi(\vartheta)-\chi_{\{|v|\leq n\}}\phi(v))\cdot\nabla T_k(T_n(\vartheta)-T_n(v))\,dx\,dt\Big| \\ &\leq I_1+I_2+I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{T}{k}\int_{Q_T}\chi_{\{|\vartheta|\leq n,|v|>n\}}|\phi(\vartheta)||\nabla T_k(T_n(\vartheta)-n.\mathrm{sgn}(v))|\,dx\,dt, \\ I_2 &= \frac{T}{k}\int_{Q_T}\chi_{\{|\vartheta|>n,|v|\leq n\}}|\phi(v)||\nabla T_k(T_n(v)-n.\mathrm{sgn}(\vartheta))|\,dx\,dt, \\ I_3 &= \frac{T}{k}\int_{Q_T}\chi_{\{|\vartheta|\leq n,|v|\leq n\}}|\phi(\vartheta)-\phi(v)||\nabla T_k(T_n(\vartheta)-\nabla T_k(T_n(v)|\,dx\,dt. \end{aligned}$$

For the terms I_1 and I_2, we have

$$\begin{aligned} I_1 &\leq \frac{T}{k}\int_{Q_T}\chi_{\{|\vartheta|\leq n,|v|>n\}}\chi_{\{|\vartheta-n.\mathrm{sgn}(v)|\leq k\}}|\phi(\vartheta)||\nabla(\vartheta)|\,dx\,dt \\ &\leq \frac{T}{k}\int_{\{n-k\leq\vartheta\leq n\}\cup\{-n\leq\vartheta\leq -n+k\}}|\phi(\vartheta)||\nabla(\vartheta)|\,dx\,dt, \end{aligned} \tag{5.28}$$

and similarly

$$I_2 \leq \frac{T}{k}\int_{\{n-k\leq v\leq n\}\cup\{-n\leq v\leq -n+k\}}|\phi(v)||\nabla(v)|\,dx\,dt. \tag{5.29}$$

Moreover, by (5.14) we get

$$\begin{aligned} I_3 &\leq \frac{T}{k}\int_{\{|\vartheta|\leq n,|v|\leq n\}\cap\{|T_n(\vartheta)-T_n(v)|\leq k\}} L_K(x,t)|\vartheta - v||\nabla T_k(T_n(\vartheta)-T_n(v))|\,dx\,dt \\ &= \frac{T}{k}\int_{\{|\vartheta|\leq n,|v|\leq n\}\cap\{|T_n(\vartheta)-T_n(v)|\leq k\}} L_K(x,t)|T_n(\vartheta)-T_n(v)||\nabla T_k(T_n(\vartheta)-T_n(v))|\,dx\,dt \\ &\leq T\int_{\{|\vartheta|\leq n,|v|\leq n\}\cap\{|T_n(\vartheta)-T_n(v)|\leq k\}} L_K(x,t)(|\nabla T_n(\vartheta)|+|\nabla T_n(v)|)\,dx\,dt. \end{aligned}$$

Since $L_K(x,t)(|\nabla T_n(\vartheta)|+|\nabla T_n(v)|)\in L^1(Q_T)$. By Lebesgue's Dominated convergence theorem, we obtain

$$\lim_{k\to 0} I_3 = 0. \tag{5.30}$$

By combining (5.28)-(5.30), we conclude that

$$\limsup_{\sigma\to 0}|I^\sigma_{4,n}| \leq \frac{T}{k}H(\vartheta,v,n,k)+\varepsilon(k), \tag{5.31}$$

where $\varepsilon(k)$ is a positive function such that $\varepsilon(k)\longrightarrow 0$ as $k\to 0$.
For the term $I^\sigma_{6,n}$, by using Lebesgue's theorem and (5.16) we obtain

$$\lim_{\sigma\to 0}|I^\sigma_{6,n}| \leq \frac{T}{k}\int_{Q_T}|F(x,t,\vartheta)\chi_{\{|\vartheta|\leq n\}}-F(x,t,v)\chi_{\{|v|\leq n\}}|\times|T_k(T_n(\vartheta)-T_n(v))|\,dx\,dt.$$

Since $\dfrac{|T_k(T_n(\vartheta)-T_n(v))|}{k}\leq 1$ and by Lebesgue theorem, we have

$$\lim_{\sigma\to 0}|I^\sigma_{6,n}| \leq T\int_{Q_T}|F(x,t,\vartheta)\chi_{\{|\vartheta|\leq n\}}-F(x,t,v)\chi_{\{|v|\leq n\}}|\,dx\,dt\longrightarrow 0 \quad \text{as } n\longrightarrow+\infty.$$

Then,

$$\lim_{k\to 0}\liminf_{n\to+\infty}\lim_{\sigma\to 0}|I^\sigma_{6,n}| = 0. \tag{5.32}$$

Finally, returning to (5.17) and using (5.18), we deduce that $\vartheta = v$ a.e. in Q_T. Therefore, in view of remark 5.2, our problem (1.1) has an unique entropy and renormalized solution. □

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