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OPTIMAL ERROR ANALYSIS OF A SPECTRAL METHOD FOR A NONLINEAR TIME-FRACTIONAL ADVECTION-DIFFUSION EQUATION

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Abstract. This paper develops a Galerkin spectral element method (G-SEM) for a nonlinear time-fractional advection–diffusion equation involving the Caputo derivative. The fully discrete scheme combines the classical L1 formula in the temporal direction with Legendre spectral element discretization. The variational formulation and semi-discrete well-posedness are established. Rigorous analysis provides boundedness and stability results, while projection techniques and the discrete Gronwall inequality yield optimal error estimates without restrictive coupling between temporal and spatial discretization parameters. Numerical experiments confirm the theoretical convergence rates and demonstrate the accuracy and effectiveness of the proposed method for solving nonlinear time-fractional problems.

Keywords. Nonlinear advection-diffusion equation, Fractional derivative, Galerkin spectral element method (G-SEM), Legendre basis, Optimal error estimates
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1. Introduction

Fractional differential equations (FDEs) are widely used to model phenomena with memory and hereditary effects, with applications in physics, chemistry, biology, finance, economics, and engineering [21, 22, 29, 16, 23, 4, 24]. Meanwhile, nonlinear phenomena are widely encountered in applied sciences such as plasma physics, nonlinear optics, and fluid mechanics, and are often described by nonlinear partial differential equations (NPDEs). Since exact solutions are generally difficult to obtain, numerical methods play a crucial role in their analysis. The combination of fractional dynamics and nonlinearity has thus led to increasing interest in nonlinear fractional partial differential equations for modeling complex real-world systems. The spectral element method (SEM), originally developed for fluid dynamics, combines the geometric flexibility of the finite element method with the high accuracy of spectral techniques [25]. Owing to this

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hybrid nature, SEM has proven to be an effective tool for approximating solutions of complex models, including time-fractional partial differential equations, as well as for discretizing spatial operators of integer order [10, 28]. In recent years, several studies have extended SEM to fractional models. For instance, [13] considers a mobile–immobile fractional transport problem formulated with the Caputo–Fabrizio derivative, where a combination of a nonstandard finite difference scheme and SEM is employed. A hybrid SEM framework for solving fractional two-point boundary value problems involving both Caputo and Riemann–Liouville derivatives is proposed in [8]. In addition, [19] develops an SEM coupled with hierarchical matrix techniques to treat equations with two-sided second-order Riemann–Liouville operators. A high-order technique is presented in [35, 34] for two computational schemes of a Riesz fractional diffusion equation, where an SEM based on Legendre polynomials is employed for the spatial discretization, while the Crank–Nicolson scheme is used in the temporal direction. The proposed approach is supported by rigorous theoretical analysis and numerical experiments.

Motivated by these developments, the present work investigates an L1-formula/G-SEM method for the numerical solution of the following time-fractional nonlinear advection–diffusion equation [14, 15]:

$$\mathbf{v}_t - {}_0^c\mathrm{D}_t^{\mu}(\Delta \mathbf{v}) - \Delta \mathbf{v} + \nabla \mathbf{v} + G(\mathbf{v}) = g(\mathbf{x}, t), \quad \mathbf{x} \in \Xi \subset \mathbb{R}^d, \quad t \in (0, T]. \tag{1.1}$$

The initial condition is considered to be

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad \mathbf{x} \in \Xi, \tag{1.2}$$

together with the boundary condition

$$\mathbf{v}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Xi, \quad t \in (0, T], \tag{1.3}$$

where $G(v)$ denotes the nonlinear term, $g(\mathbf{x}, t)$ and $\mathbf{v}_0$ are given functions, and $d = 2$. Moreover, ∇ and Δ represent the gradient and Laplace operators, respectively. Moreover, many existing analyses assume that the nonlinear source term $G(v)$ satisfies a global Lipschitz condition with Lipschitz constant C_L. However, this assumption is restrictive and excludes many physically relevant nonlinearities, such as v^2, which is not globally Lipschitz continuous. The present work aims to overcome this limitation in the computational studies. Equation (1.1) is physically relevant in the following contexts:

- Diffusion of unidirectional long waves in nonlinear dispersive media [5].
- Long-wavelength surface waves in liquids.
- Acoustic (sound) waves in harmonic crystals and acoustic-gravity waves in compressible fluids.
- Predator–prey models of interacting populations [11].
- Molecular transport and heat and mass transfer in heterogeneous materials [2].

Following [26], the Caputo fractional derivative is given by

$$\mathrm{D}_t^{\mu}\mathbf{v} = \frac{1}{\Gamma(n-\mu)}\int_0^t \frac{\mathbf{v}^{(n)}(s)ds}{(t-s)^{\mu+1-n}}, \quad n-1<\mu<n, \quad n\in\mathbb{N},$$

with Γ denoting the gamma function.

A considerable amount of research has been devoted to the numerical treatment of fractional advection-diffusion equations. Among the existing contributions, [6] proposed a Legendre-Gauss-Lobatto spectral collocation approach for the numerical solution of space-fractional advection-diffusion equations with variable coefficients in both time and space under Dirichlet boundary conditions. In [20], the authors investigated a fractional advection-diffusion-reaction model involving a second-order RLC fractional derivative. Their methodology employs a shooting strategy combined with the secant iteration procedure, where the original boundary-value problem is first recast as an initial-value problem and subsequently transformed into an equivalent Volterra integral equation containing a weakly singular kernel. Furthermore, Abd-Elkawy and Alqahtani [1] introduced a space-time spectral collocation framework for variable-order Galilei-invariant advection-diffusion equations with nonlinear source terms. Their numerical formulation utilizes shifted Jacobi-Gauss-Lobatto and shifted Jacobi-Gauss-Radau collocation points to construct highly accurate approximations of the governing equation. In [27], Roul and Sundar developed numerical schemes based on uniform, graded, and adaptive meshes for the solution of a fractional advection–diffusion equation whose solution typically exhibits a weak singularity near the initial time. The time-fractional derivative, defined in the Caputo sense, is approximated using the L1 discretization on nonuniform temporal grids. For the spatial discretization, the authors employed a fourth-order compact finite difference method on a uniform mesh, resulting in a high-accuracy numerical framework capable of effectively capturing the singular behavior of the solution. Zhang and Jia [37] investigated the optimal error analysis of a numerical method for a two-dimensional nonlinear time-fractional advection–diffusion equation. Their approach combines a linearized L1 time discretization with a Legendre–Galerkin spectral method in space, while employing a temporal–spatial error splitting technique to establish unconditional optimal error estimates.

Obtaining optimal error estimates for nonlinear time-fractional problems often requires establishing the boundedness of the numerical solution in the L_∞norm, which typically restricts the temporal step size and the spatial mesh size. To overcome this limitation and derive unconditional optimal error estimates, Li [18] introduced the temporal–spatial error splitting technique, which has since been successfully applied to a wide range of integer-order partial differential equations [32, 31]. This paper develops a novel framework for the error analysis of fractional differential equations (FDEs), based on several nonstandard analytical techniques and newly established results. The proposed approach provides an effective methodology for the rigorous theoretical investigation of FDEs. In addition, we construct a G–SEM using Legendre polynomials as spatial basis functions. Particular attention is devoted to the implementation aspects of the method and the efficient numerical approximation of the underlying problem.

The remainder of this paper is organized as follows. In Section 2, we introduce the necessary notation, preliminary results, and theoretical background. Section 3 presents the fully discrete Galerkin spectral element scheme, where the Caputo fractional derivative is approximated by the standard L1 formula in time, while Legendre polynomial-based spectral elements are employed for the spatial discretization. A brief description of the computational implementation is also provided. In Section 4, several boundedness properties are established, together with the error analysis of the semi-discrete scheme. To derive the optimal error estimates of the G–SEM, a collection of auxiliary lemmas and fundamental analytical tools is developed in Section 5. Numerical experiments are then reported in Section 6 to verify the theoretical convergence results and demonstrate the accuracy and efficiency of the proposed method through a series of representative test problems and graphical illustrations. Finally, the main conclusions and perspectives for future research are presented in Section 7.

2. Preliminaries

This section presents properties [17, 9] that we use throughout the paper.

Let $m \geq 1$ be an integer. We denote by $H^m(\Xi)$ the Sobolev space of order m over Ξ, endowed with the norm $\|\cdot\|_m$, and by $H_0^1(\Xi)$ the closure of $C_0^\infty(\Xi)$ with respect to the norm $\|\cdot\|_1$. In addition, we use the notation

$$(u, v) = (u, v)_{L^2(\Xi)}$$

for the inner product in $L^2(\Xi)$ and

$$\|v\| = \|v\|_{L^2(\Xi)}, \qquad \|v\|_\infty = \|v\|_{L^\infty(\Xi)}.$$

Sobolev embedding inequality:

$$\|\mathbf{v}\|_\infty \leq m_1 \|\mathbf{v}\|_2, \quad \forall \mathbf{v} \in H_0^1(\Xi) \cap H^2(\Xi). \tag{2.1}$$

Elliptic regularity: Let $\Xi \subset \mathbb{R}^2$ be a bounded convex domain with either a polygonal or a smooth boundary. Then, there exists a positive constant m_2 such that

$$\|v\|_2 \leq m_2 \|\Delta v\|, \qquad \forall u \in H^2(\Xi) \cap H_0^1(\Xi). \tag{2.2}$$

If Ξ is convex, then $0 < m_2 \leq 1$. Otherwise, $m_2 > 1$.

Poincaré–Friedrichs inequality: Let $\Xi \subset \mathbb{R}^2$ be an open domain of finite width. Then, there exists a positive constant m_3 satisfying

$$\|\mathbf{v}\| \leq m_3 \|\nabla \mathbf{v}\|, \quad \forall \mathbf{v} \in H_0^1(\Xi). \tag{2.3}$$

Let $T > 0$ be the final time, and let $N \in \mathbb{N}^+$ denote the number of uniform time subintervals. We define the time step size by $\Delta t = T/N$, and introduce the uniform partition of the temporal interval $[0, T]$ where

$$t_n = n\Delta t, \quad \delta_t v^n = (\mathbf{v}^n - \mathbf{v}^{n-1})/\Delta t, \quad n = 0, 1, \ldots, N.$$

3. The L1-formula/G-SEM method

3.1. **Formulation of the fully discrete scheme.** For a function $\mathbf{v} \in C(\Xi \times [0,T])$, we denote its value at the discrete time level t_n by $\mathbf{v}^n(\cdot) = \mathbf{v}(\cdot, t_n)$. To approximate the Caputo fractional derivative, we employ the classical L1 scheme on a uniform temporal mesh. Accordingly, for $n = 1, \ldots, N$, the Caputo fractional derivative ${}^{c}\mathrm{D}_t^{\mu}\mathbf{v}(t)$ evaluated at $t = t_n$ is approximated by

$$\mathbb{D}_t^{\mu}\mathbf{v}^n = \frac{\Delta t^{-\mu}}{\Gamma(2-\mu)}\left[d_0^{\mu}\mathbf{v}^n + \sum_{i=1}^{n-1}\left(d_{n-i-1}^{\mu} - d_{n-i}^{\mu}\right)\mathbf{v}^{n-i} - d_{n-1}^{\mu}\mathbf{v}^0\right], \quad \text{where} \quad d_i^{\mu} = (i+1)^{1-\mu} - i^{1-\mu}, \quad i \geq 0. \tag{3.1}$$

We have the following property:

$$1 = d_0^{\mu} \geq d_i^{\mu} \geq d_{i+1}^{\mu}, \quad i \geq 0. \tag{3.2}$$

For $\mathbf{v} \in C^2([0,T], L^2(\Xi))$, the truncation error $r^n = {}^{c}\mathrm{D}_t^{\mu}\mathbf{v}^n - \mathbb{D}_t^{\mu}\mathbf{v}^n$ verifies [33]

$$\|r^n\| = O\left(\Delta t^{2-\mu}\right). \tag{3.3}$$

Evaluating (1.1) at the point $(\mathbf{x}, t_n)$ and substituting the L1 approximation for the Caputo fractional derivative, we obtain

$$\mathbf{v}_t^n - {}_0^{c}D_t^{\mu}\Delta\mathbf{v}^n - \Delta\mathbf{v}^n + \nabla\mathbf{v}^n + G\left(\mathbf{v}^n\right) = g^n, \quad n = 1, 2, \cdots, N. \tag{3.4}$$

Using the notation introduced above, Eq. (3.4) can be equivalently written in the following form:

$$\delta_t\mathbf{v}^n - \mathbb{D}_t^{\mu}\Delta\mathbf{v}^n - \Delta\mathbf{v}^n + \nabla\mathbf{v}^n + G\left(\mathbf{v}^{n-1}\right) = g^n + R_1^n, \quad n = 1, 2, \cdots, N, \tag{3.5}$$

Accordingly, the local truncation error associated with the L1 discretization can be expressed as

$$R_1^n = \delta_t\mathbf{v}^n - \mathbf{v}_t^n + \left({}_0^{C}D_t^{\mu}\Delta\mathbf{v}^n - \mathbb{D}_t^{\mu}\Delta\mathbf{v}^n\right) + G\left(\mathbf{v}^n\right) - G\left(\mathbf{v}^{n-1}\right). \tag{3.6}$$

Combining (3.3) with the assumption that $G(\cdot) \in C^1(\mathbb{R})$ and applying Taylor's expansion, we obtain

$$\|R_1^n\| \leq m_4\Delta t, \quad n = 1, 2, \cdots, N, \tag{3.7}$$

where $m_4 > 0$ is independent of Δt.

Neglecting the truncation term R_1^n in (3.6), we obtain the corresponding time-discrete elliptic system associated with the model (1.1)–(1.3), which is given by

$$\delta_t\mathbf{V}^n - \mathbb{D}_t^{\mu}\Delta\mathbf{V}^n - \Delta\mathbf{V}^n + \nabla\mathbf{V}^n + G\left(\mathbf{V}^{n-1}\right) = g^n, \quad n = 1, 2, \cdots, N, \tag{3.8}$$

under the following initial and boundary conditions

$$\mathbf{V}^0(\mathbf{x}) = \mathbf{v}^0(\mathbf{x}), \tag{3.9}$$

$$\mathbf{V}^n(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Xi, \quad n = 1, 2, \cdots, N, \tag{3.10}$$

where $\mathbf{V}^n$ is the numerical approximation.

We introduce the Ritz projection operator as follows:

$$P_M : H_0^1(\Xi) \to \mathbf{S}_M^0,$$

and P_M verifies:

$$\left(\nabla\left(\mathbf{v} - P_M\mathbf{v}\right), \nabla\mathbf{w}\right) = 0, \quad \mathbf{v} \in H_0^1(\Xi), \quad \forall \mathbf{w} \in \mathbf{S}_M^0,$$

and $\mathbf{S}_M^0$ is the spectral element function space

$$\mathbf{S}_M^0 = \left\{\mathbf{w} \in H_0^1(\Xi) : \mathbf{w}|_{\Xi_e} \in \mathbb{P}_M(\Xi), \quad e = 1, 2, \cdots, K_e\right\}.$$

where $\mathbb{P}_M$ is the space of polynomials of degree no greater than $M \in \mathbb{N}$.

Lemma 3.1. [36] *For any $\mathbf{v} \in \mathbf{S}_M^0$, there exists a positive constant C' independent of $n, \Delta t$ and M such that the following inverse inequality is obtained:*

$$\|\mathbf{v}\|_\infty \leq C' M \|\mathbf{v}\|_2.$$

Theorem 3.2. [7] *Let $\mathbf{v} \in H_1^0(\Xi) \cap H^\beta(\Xi)$ for $1 \leq \beta \leq M+1$. Then, there exists a positive constant m,*

$$\|\mathbf{v} - P_M\mathbf{v}\| \leq m \left[\sum_{e=1}^{K_e} h_e^{2(\min(M_e+1,\beta)-1)} M_e^{2(1-\beta)} \|\mathbf{v}\|_\beta^2\right]^{\frac{1}{2}}.$$

Specifically, if $M_e = M$ and $h \leq h_e \leq c'h$, we get

$$\|\mathbf{v} - P_M\mathbf{v}\| \leq m h^{(\min(M+1,\beta)-1)} M^{1-\beta} \|\mathbf{v}\|_\beta.$$

Testing (3.8) with an arbitrary function $w \in H_0^1(\Xi)$ and applying the integration-by-parts formula, we obtain the following variational formulation of (3.8)–(3.10): Find $\mathbf{V}^n \in H_0^1(\Xi)$, $n = 1, 2, \ldots, N$, such that for all $w \in H_0^1(\Xi)$,

$$\left(\delta_t \mathbf{V}^n, w\right) + \left(\mathbb{D}_t^\mu \nabla \mathbf{V}^n, \nabla w\right) + \left(\nabla \mathbf{V}^n, \nabla w\right) + \left(\nabla \mathbf{V}^n, w\right) + \left(G\left(\mathbf{V}^{n-1}\right), w\right) = \left(g^n, w\right). \tag{3.11}$$

The existence and uniqueness of the solution $\mathbf{V}^n$, for $n = 1, 2, \ldots, N$, follow directly from the Lax–Milgram theorem. Moreover, by the regularity theory for linear elliptic problems, we have

$$\mathbf{V}^n \in H_0^1(\Xi) \cap H^2(\Xi), \quad 1 \leq n \leq N. \tag{3.12}$$

Consequently, the fully discrete G-SEM scheme associated with the variational formulation (3.11) is defined as follows: Find $\mathbf{v}_M^n \in S_M^0$, $n = 1, 2, \ldots, N$, such that $\forall w_M \in \mathbf{S}_M^0$,

$$\begin{aligned}\left(\delta_t \mathbf{v}_M^n, w_M\right) + \left(\mathbb{D}_t^\mu \nabla \mathbf{v}_M^n, \nabla w_M\right) + \left(\nabla \mathbf{v}_M^n, \nabla w_M\right) \\ + \left(\nabla \mathbf{v}_M^n, w_M\right) + \left(G\left(\mathbf{v}_M^{n-1}\right), w_M\right) = \left(g^n, w_M\right),\end{aligned} \tag{3.13}$$

and $\mathbf{v}_M^0$ is taken as

$$\mathbf{v}_M^0 = P_M\mathbf{v}^0 = P_M\mathbf{V}^0.$$

3.2. **A brief computational implementation.** This subsection presents the implementation of the proposed method. The computational domain Ξ is partitioned into K_e non-overlapping elements, and the main steps of the spectral element method are briefly outlined.

Lemma 3.3. [30] *Let us introduce the following class of Legendre polynomials:*

$$\mathfrak{b}_k(x) = C_k \left(L_k(x) - L_{k+2}(x) \right), \tag{3.14}$$

in which $C_k = (4k+6)^{-\frac{1}{2}}$.

Let us define:

$$\begin{cases} W_{jk} = \int_{-1}^{1} \frac{\mathrm{d}\mathbf{b}_k(x)}{\mathrm{d}x} \frac{\mathrm{d}\mathbf{b}_j(x)}{\mathrm{d}x} \, \mathrm{d}x, \\ B_{jk} = \int_{-1}^{1} \mathbf{b}_k(x)\mathbf{b}_j(x)\mathrm{d}x, \\ S_{jk} = \int_{-1}^{1} \frac{\mathrm{d}\mathbf{b}_k(x)}{\mathrm{d}x}\mathbf{b}_j(x)\mathrm{d}x. \end{cases}$$

Then,

$$B_{jk} = B_{kj} = \begin{cases} C_k C_j \left(\frac{2}{2j+1} + \frac{2}{2j+5} \right), & k = j, \\ -C_k C_j \frac{2}{2k+1}, & k = j+2, \\ 0, & \text{otherwise} , \end{cases}$$

$$W_{jk} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \qquad S_{jk} = -S_{kj} = \begin{cases} 2C_k C_j, & k = j+1, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the following approximate solution in the one-dimensional setting [39, 38]:

$$V_M^{n(e)} = \sum_{i=0}^{M} V_{M,i}^n \mathfrak{b}_i, \quad 1 \leq e \leq K_e, \quad 1 \leq n \leq N.$$

Using the appropriate elemental mappings introduced in [12], the corresponding elemental mass matrices can be written in the following form:

$$W_{ij}^e = \int_{x_{e-1}}^{x_e} \frac{\mathrm{d}\mathbf{b}_i(x)}{\mathrm{d}x} \frac{\mathrm{d}\mathbf{b}_j(x)}{\mathrm{d}x} \, \mathrm{d}x = \frac{2}{h_e} \int_{-1}^{1} \frac{\mathrm{d}\mathbf{b}_i(\eta)}{\mathrm{d}\eta} \frac{\mathrm{d}\mathbf{b}_j(\eta)}{\mathrm{d}\eta} \, \mathrm{d}\eta$$

$$B_{ij}^e = \int_{x_{e-1}}^{x_e} \mathbf{b}_i(x)\mathbf{b}_j(x)\mathrm{d}x = \frac{h_e}{2} \int_{-1}^{1} \mathbf{b}_i(\eta)\mathbf{b}_j(\eta)\mathrm{d}\eta$$

For the two-dimensional problem, the approximate solution is represented in the following form:

$$V_M^{n(e)} = \sum_{i=0}^{M} \sum_{j=0}^{M} V_{M,i,j}^n \mathfrak{b}_i \mathfrak{b}_j, \quad 1 \leq e \leq K_e, \quad 1 \leq n \leq N.$$

The approximation space is constructed from polynomial basis functions of degree at most M. Accordingly, the corresponding elemental mass and stiffness matrices are defined as follows:

$$B^{Qe} = B_x^e \otimes B_y^e, \qquad W^{Qe} = W_x^e \otimes W_y^e.$$

4. Error analysis of the semi-discrete scheme

In this section, we investigate the error estimates associated with the semi-discrete scheme and establish several boundedness results.

To this end, subtracting (3.8) from (3.5) gives rise to the following error equation:

$$\delta_t \mathbf{e}^n - \mathbb{D}_t^{\mu} \Delta \mathbf{e}^n - \Delta \mathbf{e}^n + \nabla \mathbf{e}^n + G\left(\mathbf{v}^{n-1}\right) - G\left(\mathbf{V}^{n-1}\right) = R_1^n, \quad 1 \le n \le N, \tag{4.1}$$

where $\mathbf{e}_n$ is defined as

$$\mathbf{e}^n = \mathbf{v}^n - \mathbf{V}^n, \quad n = 1, 2, \cdots, N.$$

It follows directly from the definition of $\mathbf{V}^0$ that

$$\mathbf{e}^0 = \mathbf{v}^0 - \mathbf{V}^0 = 0. \tag{4.2}$$

Theorem 4.1. *Let $\mathbf{v}^n$ denote the solution of (1.1)–(1.3), and let $\mathbf{V}^n$ denote the solution of the time-discrete scheme (3.8)–(3.5). Then, there exists a positive constant $\Delta t_1 = \min((m_1 m_5)^{-2}, (2b_1)^{-1})$ such that, for $\Delta t \le \Delta t_1$, we have*

$$\|\mathbf{e}^n\|_2 \le m_5 \Delta t^{\frac{1}{2}}, \quad n = 1, 2, \cdots, N, \tag{4.3}$$

$$\|\mathbf{V}^n\|_\infty \le L_1, \quad n = 1, 2, \cdots, N, \tag{4.4}$$

where b_1, L_1 and m_5 are defined as

$$b_1 = 1 + 4(m_3 C_{L_1})^2$$

$$L_1 = \sup_{\Delta t} \left(\max_{0 \le n \le N} \|\mathbf{v}^n\|_\infty \right) + 1,$$

$$m_5 = m_2 m_4 \sqrt{8T \exp\left(2T(1 + 4\left(m_3 C_{L_1}\right)^2)\right)},$$

Proof. We first establish (4.3) by mathematical induction. According to (4.2), the result in (4.3) holds for $n = 0$ by direct computation. We then assume that (4.3) holds for all $n = 0, 1, \ldots, k-1$, where $k = 1, 2, \ldots, N$; that is,

$$\|\mathbf{e}^n\|_2 \le m_5 \Delta t^{\frac{1}{2}}, \quad 0 \le n \le k-1. \tag{4.5}$$

By the assumption $G(\cdot) \in C^1(\mathbb{R})$ and the Lagrange Mean Value Theorem,

$$\|G\left(\mathbf{v}^n\right) - G\left(\mathbf{V}^n\right)\| = \|G'\left(\rho_1\right)\mathbf{e}^n\| \le C_{L_1} \|\mathbf{e}^n\|, \quad n = 0, 1, \cdots, k-1, \tag{4.6}$$

ρ_1 is a constant, and C_{L_1} is given as

$$C_{L_1} = \max_{|x| \le L_1} |G'(x)|.$$

The above estimate remains valid for $n = k$. Taking the $L^2(\Xi)$ inner product of (4.1) with $-\Delta t \Delta \mathbf{e}^n$ and applying integration by parts, we obtain for $n = 1, 2 \cdots, k$,

$$\begin{aligned}\|\nabla \mathbf{e}^n\|^2 + \Delta t \|\Delta \mathbf{e}^n\|^2 = &\left(\nabla \mathbf{e}^{n-1}, \nabla \mathbf{e}^n\right) + \Delta t \left(\nabla \mathbf{e}^n, \Delta \mathbf{e}^n\right) \\ &+ \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)} \sum_{i=1}^{n-1} \left(d_{n-i-1}^{\mu} - d_{n-i}^{\mu}\right) \left(\Delta \mathbf{e}^i, \Delta \mathbf{e}^n\right) \\ &- \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)} \|\Delta \mathbf{e}^n\|^2 + \Delta t \left(G\left(\mathbf{v}^{n-1}\right) - G\left(\mathbf{V}^{n-1}\right), \Delta \mathbf{e}^n\right) - \Delta t \left(R_1^n, \Delta \mathbf{e}^n\right),\end{aligned}$$

By employing (3.2) together with the Cauchy-Schwarz and Young inequalities, we derive that

$$\begin{aligned}(1-\Delta t)\|\nabla \mathbf{e}^n\|^2 + \Delta t \|\Delta \mathbf{e}^n\|^2 \leq &\left\|\nabla \mathbf{e}^{n-1}\right\|^2 + \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)} \sum_{i=1}^{n-1} \left(d_{n-i-1}^{\mu} - d_{n-i}^{\mu}\right) \left\|\Delta \mathbf{e}^i\right\|^2 \\ &+ \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)} \left[\sum_{i=1}^{n-1} \left(d_{n-i-1}^{\mu} - d_{n-i}^{\mu}\right) - 2\right] \|\Delta \mathbf{e}^n\|^2 \\ &+ 2\Delta t \left(G\left(\mathbf{v}^{n-1}\right) - G\left(\mathbf{V}^{n-1}\right), \Delta \mathbf{e}^n\right) \\ &- 2\Delta t \left(R_1^n, \Delta \mathbf{e}^n\right), \quad n = 1, 2, \cdots, k.\end{aligned} \tag{4.7}$$

Observe that

$$\sum_{i=1}^{n-1} \left(d_{n-i-1}^{\mu} - d_{n-i}^{\mu}\right) - 2 = -1 - d_{n-1}^{\mu} \leq -1$$

From (4.7), we can immediately arrive at

$$\begin{aligned}\|\nabla \mathbf{e}^n\|^2 + \Delta t \|\Delta \mathbf{e}^n\|^2 \leq &\left\|\nabla \mathbf{e}^{n-1}\right\|^2 - \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)} \|\Delta \mathbf{e}^n\|^2 + \Delta t \|\nabla \mathbf{e}^n\|^2 \\ &+ \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)} \sum_{i=1}^{n-1} \left(d_{n-i-1}^{\mu} - d_{n-i}^{\mu}\right) \left\|\Delta \mathbf{e}^i\right\|^2 \\ &+ 2\Delta t \left(G\left(\mathbf{v}^{n-1}\right) - G\left(\mathbf{V}^{n-1}\right), \Delta \mathbf{e}^n\right) \\ &- 2\Delta t \left(R_1^n, \Delta \mathbf{e}^n\right), \quad n = 1, 2, \cdots, k.\end{aligned} \tag{4.8}$$

Furthermore, by invoking (2.3) and (4.6), together with the Cauchy–Schwarz and Young inequalities, we obtain for $1 \leq n \leq k$,

$$\begin{aligned}2\Delta t \left(G\left(\mathbf{v}^{n-1}\right) - G\left(\mathbf{V}^{n-1}\right), \Delta \mathbf{e}^n\right) \leq &2\Delta t \left\|G\left(\mathbf{v}^{n-1}\right) - G\left(\mathbf{V}^{n-1}\right)\right\| \|\Delta \mathbf{e}^n\| \\ \leq &2\Delta t C_{L_1} \left\|\mathbf{e}^{n-1}\right\| \|\Delta \mathbf{e}^n\| \\ \leq &\frac{\Delta t}{4} \|\Delta \mathbf{e}^n\|^2 + 4C_{L_1}^2 \Delta t \left\|\mathbf{e}^{n-1}\right\|^2 \\ \leq &\frac{\Delta t}{4} \|\Delta \mathbf{e}^n\|^2 + 4\left(m_3 C_{L_1}\right)^2 \Delta t \left\|\nabla \mathbf{e}^{n-1}\right\|^2.\end{aligned} \tag{4.9}$$

Following analogous arguments and applying (3.7), we can establish the following estimate:

$$\begin{aligned} -2\Delta t\left(R_1^n, \Delta \mathbf{e}^n\right) &\leq 2\Delta t \left\|R_1^n\right\| \left\|\Delta \mathbf{e}^n\right\| \\ &\leq \frac{\Delta t}{4}\left\|\Delta \mathbf{e}^n\right\|^2 + 4\Delta t \left\|R_1^n\right\|^2 \\ &\leq \frac{\Delta t}{4}\left\|\Delta \mathbf{e}^n\right\|^2 + 4m_4^2\Delta t^3, \quad 1 \leq n \leq k. \end{aligned} \tag{4.10}$$

Substituting (4.10) and (4.9) into (4.7), it holds that

$$\begin{aligned} \left\|\nabla \mathbf{e}^n\right\|^2 + \frac{\Delta t}{2}\left\|\Delta \mathbf{e}^n\right\|^2 \leq \left\|\nabla \mathbf{e}^{n-1}\right\|^2 &- \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)}\left\|\Delta \mathbf{e}^n\right\|^2 + \Delta t\left\|\nabla \mathbf{e}^n\right\|^2 \\ &+ \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)}\sum_{i=1}^{n-1}\left(d_{n-i-1}^{\mu} - d_{n-i}^{\mu}\right)\left\|\Delta \mathbf{e}^i\right\|^2 \\ &+ 4\left(m_3 C_L\right)^2 \Delta t\left\|\nabla \mathbf{e}^{n-1}\right\|^2 + 4m_4^2\Delta t^3, \quad 1 \leq n \leq k. \end{aligned}$$

Combining and rearranging the resulting terms, we obtain

$$\begin{aligned} \left\|\nabla \mathbf{e}^n\right\|^2 + \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)}\sum_{i=1}^{n} d_{n-i}^{\mu}\left\|\Delta \mathbf{e}^i\right\|^2 + \frac{\Delta t}{2}\left\|\Delta \mathbf{e}^n\right\|^2 &\leq \left\|\nabla \mathbf{e}^{n-1}\right\|^2 \\ + \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)}\sum_{i=1}^{n-1} d_{n-i-1}^{\mu}\left\|\Delta \mathbf{e}^i\right\|^2 + 4\left(m_3 C_{L_1}\right)^2 \Delta t\left\|\nabla \mathbf{e}^{n-1}\right\|^2 & \\ + \Delta t\left\|\nabla \mathbf{e}^n\right\|^2 + 4m_4^2\Delta t^3, \quad n = 1, 2, \cdots, k. & \end{aligned} \tag{4.11}$$

We introduce the following notation:

$$S^n = \left\|\nabla \mathbf{e}^n\right\|^2 + \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)}\sum_{i=1}^{n} d_{n-i}^{\mu}\left\|\Delta \mathbf{e}^i\right\|^2, \quad 0 \leq n \leq N. \tag{4.12}$$

Using this notation, (4.11) can be written for $n = 1, 2, \cdots, k$ as:

$$S^n + \frac{\Delta t}{2}\left\|\Delta \mathbf{e}^n\right\|^2 \leq S^{n-1} + \Delta t\left\|\nabla e^n\right\|^2 + 4\left(m_3 C_{L_1}\right)^2 \Delta t\left\|\nabla e^{n-1}\right\|^2 + 4m_4^2\Delta t^3. \tag{4.13}$$

Replacing n by m in (4.13), summing the resulting inequality over $m = 1, \ldots, n$, and using the relation $\left|\nabla e^n\right|^2 \leq S^n$, we deduce for $1 \leq n \leq k$,

$$\left\|\nabla e^n\right\|^2 + \frac{\Delta t}{2}\sum_{m=1}^{n}\left\|\Delta e^m\right\|^2 \leq \Delta t\left(1 + 4\left(m_3 C_{L_1}\right)^2\right)\sum_{m=1}^{n}\left\|\nabla e^m\right\|^2 + 4Tm_4^2\Delta t^2.$$

Since $\Delta t_1 \leq \Delta t$, an application of the discrete Gronwall inequality yields

$$\left\|\nabla \mathbf{e}^k\right\|^2 + \frac{\Delta t}{2}\sum_{m=1}^{k}\left\|\Delta \mathbf{e}^m\right\|^2 \leq 4Tm_4^2\Delta t^2 \exp\left(2T(1 + 4\left(m_3 C_{L_1}\right)^2)\right).$$

Consequently, we arrive at for $1 \leq k \leq N$,

$$\sum_{m=1}^{k} \|\mathbf{e}^m\|_2^2 \leq m_2^2 \sum_{m=1}^{k} \|\Delta \mathbf{e}^m\|^2 \leq 8T(m_2 m_4)^2 \exp\left(2T(1+4\left(m_3 C_{L_1}\right)^2)\right) \Delta t = m_5^2 \Delta t,$$

and

$$\left\|\mathbf{e}^k\right\|_2 \leq \sqrt{\sum_{m=1}^{k} \|\mathbf{e}^m\|_2^2} = m_5 \Delta t^{\frac{1}{2}}, \quad k = 1, 2, \cdots, N. \tag{4.14}$$

where the elliptic regularity estimate (2.2) has been used. In view of (4.14), it follows that (4.3) remains valid for $n = k$.

By combining (2.1), (4.5), and the triangle inequality, we have

$$\begin{aligned} \|\mathbf{V}^n\|_\infty &= \|\mathbf{v}^n - \mathbf{e}^n\|_\infty \\ &\leq \|\mathbf{v}^n\|_\infty + \|\mathbf{e}^n\|_\infty \\ &\leq \|\mathbf{v}^n\|_\infty + m_1 \|\mathbf{e}^n\|_2 \\ &\leq \|\mathbf{v}^n\|_\infty + m_1 m_5 \Delta t^{\frac{1}{2}} \leq L_1. \end{aligned}$$

This completes the proof.

Theorem 4.2. *Let $\mathbf{v}^n \in H_1^0(\Xi) \cap H^\beta(\Xi)$ for $1 \leq \beta$, and $\mathbf{V}^n$ denote the solutions of (1.1)–(1.3) and (3.8)–(3.5), respectively. Such that, for $\Delta t \leq \Delta t_1$,*

$$\sum_{m=1}^{n} \Delta t \left\|\delta_t \left(\mathbf{V}^m - P_M \mathbf{V}^m\right)\right\|^2 \leq m_6 M^{2-2^\beta}, \quad 1 \leq n \leq N, \tag{4.15}$$

$$\|\mathbf{V}^n\|_2 \leq m_7, \quad 0 \leq n \leq N, \tag{4.16}$$

where m_6 and m_7 are defined as

$$\begin{aligned} m_6 &= 2m^2 \left(\|\mathbf{v}_t\|_{L^2(0,T;H^2(\Xi))}^2 + 4m_5^2\right), \\ m_7 &= \|\mathbf{v}\|_{L^\infty(0,T;H^2(\Xi))} + m_5. \end{aligned}$$

Proof. In fact, by the Ritz projection operator and applying Theorem 3.2, we can easily arrive at

$$\begin{aligned} &\sum_{m=1}^{n} \Delta t \left\|\delta_t \left(\mathbf{V}^m - P_M \mathbf{V}^m\right)\right\|^2 = \sum_{m=1}^{n} \Delta t \left\|\delta_t \mathbf{V}^m - P_M \left(\delta_t \mathbf{V}^m\right)\right\|^2 \\ &\quad \leq m^2 \sum_{m=1}^{n} \Delta t M^{2-2\beta} \left\|\delta_t \mathbf{V}^m\right\|_2^2 \\ &\quad \leq 2m^2 \left[\sum_{m=1}^{n} \Delta t \left\|\delta_t \mathbf{v}^m\right\|_2^2 + \sum_{m=1}^{n} \Delta t \left\|\delta_t e^m\right\|_2^2\right] M^{2-2\beta}, \quad n = 1, 2, \cdots, N. \end{aligned} \tag{4.17}$$

It is readily verified that

$$\begin{aligned}\sum_{m=1}^{n}\Delta t\,\|\delta_t\mathbf{v}^m\|_2^2 &= \sum_{m=1}^{n}\Delta t\left\|\Delta t^{-1}\int_{t_{m-1}}^{t_m}\mathbf{v}_t\;\mathrm{d}t\right\|_2^2\\ &\le \sum_{m=1}^{n}\Delta t^{-1}\left(\int_{t_{m-1}}^{t_m}1\cdot\|\mathbf{v}_t\|_2\;\mathrm{d}t\right)^2\\ &\le \sum_{m=1}^{n}\left(\int_{t_{m-1}}^{t_m}\|\mathbf{v}_t\|_2^2\;\mathrm{d}t\right)\le\|\mathbf{v}\|_{L^2(0,T;H^2(\Xi))}.\end{aligned}\tag{4.18}$$

And Theorem 4.1 lead to,

$$\begin{aligned}\sum_{m=1}^{n}\Delta t\,\|\delta_t e^m\|_2^2 &= \Delta t^{-2}\sum_{m=1}^{n}\Delta t\left\|e^m-e^{m-1}\right\|_2^2\\ &\le 4\Delta t^{-2}\sum_{m=1}^{n}\Delta t\,\|e^m\|_2^2\\ &\le 4m_5^2,\quad 1\le n\le N\end{aligned}\tag{4.19}$$

Substituting (4.18) and (4.19) into (4.17), we obtain

$$\begin{aligned}\sum_{m=1}^{n}\Delta t\,\|\delta_t\left(\mathbf{V}^m-P_M\mathbf{V}^m\right)\|^2 &\le 2m^2\left(\|\mathbf{v}_t\|^2_{L^2(0,T;H^2(\Xi))}+4m_5^2\right)M^{2-2\beta}\\ &= m_6M^{2-2\beta},\quad 1\le n\le N.\end{aligned}\tag{4.20}$$

Also, we can conclude that

$$\|\mathbf{V}^n\|_2\le\|\mathbf{v}^n\|_2+\|e^n\|_2\le\|\mathbf{v}\|_{L^\infty(0,T;H^2(\Xi))}+m_5\Delta t^{\frac{1}{2}}\le m_7,\quad 0\le n\le N.\tag{4.21}$$

5. Optimal error estimates

To establish the optimal error estimates for the G–SEM, we first derive several boundedness results that will be used in the subsequent analysis.

Noting that $\mathbf{S}_M^0\subset H_0^1(\Xi)$ and using (3.8), we have $\forall w_M\in\mathbf{S}_M^0$,

$$\left(\delta_t\mathbf{V}^n,w_M\right)+\left(\mathbb{D}_t^{\mu}\nabla\mathbf{V}^n,\nabla w_M\right)+\left(\nabla\mathbf{V}^n,\nabla w_M\right)+\left(\mathbf{G}\left(\mathbf{V}^{n-1}\right),w_M\right)=\left(\mathbf{g}^n,w_M\right).\tag{5.1}$$

Subtracting (3.8) from (5.1) and using the properties of the Ritz projection operator, we obtain the following error equation: $\forall w_M\in\mathbf{S}_M^0$

$$\begin{aligned}\left(\delta_t\xi_M^n,w_M\right)+\left(\mathbb{D}_t^{\mu}\nabla\xi_M^n,\nabla w_M\right)+\left(\nabla\xi_M^n,\nabla w_M\right) &= \left(\mathbf{G}\left(\mathbf{v}_M^{n-1}\right)-\mathbf{G}\left(\mathbf{V}^{n-1}\right),w_M\right)\\ &\quad-\left(R_2^n,w_M\right).\end{aligned}$$

where ξ_M^n and R_2^n are defined as

$$\xi_M^n=P_M\mathbf{V}^n-\mathbf{v}_M^n,\quad R_2^n=\delta_t\left(\mathbf{V}^n-P_M\mathbf{V}^n\right),\quad 0\le n\le N.$$

By the definition of $\mathbf{v}_M^0$, it is evident that we have

$$\xi_M^0 = P_M\mathbf{V}^0 - \mathbf{v}_M^0 = P_M\mathbf{V}^0 - P_M\mathbf{v}^0 = 0.$$

In view of (4.4) in Theorem 4.2, it follows that

$$\|P_M\mathbf{V}^n\|_\infty < \infty, \quad 0 \le n \le N. \tag{5.2}$$

Theorem 5.1. *Suppose $\mathbf{v}_M^n$ and $\mathbf{V}^n$ are the solutions of (3.11) and (3.13), respectively. Then, there exists a positive constant Δt_2 such that when $\Delta t \le \Delta t_2$,*

$$\|\xi_M^n\| \le C_\star M^{-1}, \quad 0 \le n \le N, \tag{5.3}$$

$$\|\mathbf{v}_M^n\|_\infty \le L_2, \quad 0 \le n \le N. \tag{5.4}$$

$C_\star > 0$ denotes a constant arising in the proof. In view of (5.2), we define

$$L_2 = \sup_{\Delta t}\left(\max_{0\le n\le N}\|P_M\mathbf{V}^n\|_\infty\right) + C'C_\star.$$

Proof. The estimate (5.3) can be established analogously by following the same arguments used to derive (4.3).

Using Lemma 3.1, the inequality (5.3) and the triangle inequality, we get that

$$\begin{aligned}\|\mathbf{v}_M^n\|_\infty &= \|P_M\mathbf{V}^n - \xi_M^n\|_\infty\\ &\le \|P_M\mathbf{V}^n\|_\infty + \|\xi_M^n\|_\infty\\ &\le \|P_M\mathbf{V}^n\|_\infty + C'M\,\|\xi_M^n\|\\ &\le \|P_M\mathbf{V}^n\|_\infty + C'C_\star\\ &\le L_2, \quad 0 \le n \le k-1.\end{aligned}$$

Theorem 5.2. *Let $\mathbf{v}$ be the exact solution of (1.1)–(1.3), and let $\mathbf{v}_M^n$ denote the spectral element approximation obtained from the fully discrete scheme (3.13). Then, there exists a positive constant $\Delta t_3 = \min\left(\Delta t_2, (m_1m_5)^{-2}, (2b_2)^{-1}\right)$ such that, whenever $\Delta t \le \Delta t_3$,*

$$\|\mathbf{v}^n - \mathbf{v}_M^n\| \le m_8\Delta t + m_{10}M^{1-\beta}, \quad 0 \le n \le N, \tag{5.5}$$

where b_2, m_8, m_9 and m_{10} are defined as

$$\begin{aligned}b_2 =& 1 + 6(m_3C_{L_1L_2})^2\\ m_8 =& \left\{3T\left(m_3m_4\right)^2\exp\left(2T\left(1+6\left(m_3m_L\right)^2\right)\right)\right\}^{\frac{1}{2}}\\ m_{10} =& m\|\mathbf{v}\|_{L^\infty\left(0,T,H^\beta(\Xi)\right)} + m_9\\ m_9 =& \left\{\left(6T\left(mm_3m_L\right)^2\|\mathbf{v}\|^2_{L^\infty\left(0,T;H^\beta(\Xi)\right)}\right.\right.\\ &\left.\left.+3\left(mm_3\right)^2\|\mathbf{v}_t\|^2_{L^2\left(0,T;H^\beta(\Xi)\right)}\right)\exp\left(2T\left(1+6\left(m_3m_L\right)^2\right)\right)\right\}^{\frac{1}{2}}\end{aligned}$$

Proof. The variational formulation associated with (3.5) is stated as follows: the exact solution verifies

$$\begin{aligned}(\delta_t \mathbf{v}^n, w) + (\mathbb{D}_t^\mu \nabla \mathbf{v}^n, \nabla w) + (\nabla \mathbf{v}^n, \nabla w) + (\nabla \mathbf{v}^n, w) + (G(\mathbf{v}^{n-1}), w) \\ = (f^n, w) + (R_1^n, w), \quad \forall w \in H_0^1(\Xi), \quad 1 \le n \le N.\end{aligned} \tag{5.6}$$

Taking $w = w_M$ in (5.6), we arrive at

$$\begin{aligned}(\delta_t \mathbf{v}^n, w_M) + (\mathbb{D}_t^\mu \nabla \mathbf{v}^n, \nabla w_M) + (\nabla \mathbf{v}^n, \nabla w_M) + (\nabla \mathbf{v}^n, w_M) + \left(G\left(\mathbf{v}^{n-1}\right), w_M\right) \\ = (f^n, w_M) + (R_1^n, w_M), \quad \forall w_M \in \mathbf{S}_M^0, \quad 1 \le n \le N.\end{aligned} \tag{5.7}$$

Let

$$\mathbf{v}^n - \mathbf{v}_M^n = (\mathbf{v}^n - P_M \mathbf{v}^n) + (P_M \mathbf{v}^n - \mathbf{v}_M^n) = \psi^n + \rho_M^n, \quad n = 0, 1, \cdots, N.$$

Subtracting (3.13) from (5.7) and exploiting the properties of the projection operator P_M, we derive the following error equation for ρ_M^n:

$$\begin{aligned}(\delta_t \rho_M^n, w_M) + (\mathbb{D}_t^\mu \nabla \rho_M^n, \nabla w_M) + (\nabla \rho_M^n, \nabla w_M) + (\nabla \rho_M^n, w_M) \\ + \left(G\left(\mathbf{v}^{n-1}\right) - G\left(\mathbf{v}_M^{n-1}\right), w_M\right) \\ = -(\delta_t \psi^n, w_M) + (R_1^n, w_M), \quad \forall w_M \in \mathbf{S}_M^0, \quad n = 1, 2, \cdots, N.\end{aligned} \tag{5.8}$$

Let $w_M = \Delta t\, \rho_M^n$ as the test function in (5.8) yields

$$\begin{aligned}\|\rho_M^n\|^2 + \Delta t \|\nabla \rho_M^n\|^2 = \left(\rho_M^{n-1}, \rho_M^n\right) + \Delta t \left(\nabla \rho_M^n, \rho_M^n\right) \\ + \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)} \sum_{i=1}^{n-1} \left(d_{n-i-1}^\mu - d_{n-i}^\mu\right) \left(\nabla \rho_M^i, \nabla \rho_M^n\right) \\ - \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)} \|\nabla \rho_M^n\|^2 + \Delta t \left(G\left(\mathbf{v}_M^{n-1}\right) - G\left(\mathbf{v}^{n-1}\right), \rho_M^n\right) \\ - \Delta t \left(\delta_t \psi^n, \rho_M^n\right) + \Delta t \left(R_1^n, \rho_M^n\right), \quad n = 1, 2, \cdots, N.\end{aligned} \tag{5.9}$$

Following arguments analogous to those employed in the derivation of (4.8), we obtain

$$\begin{aligned}\|\rho_M^n\|^2 + \Delta t \|\nabla \rho_M^n\|^2 \le \left\|\rho_M^{n-1}\right\|^2 + \Delta t \|\rho_M^n\|^2 - \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)} \|\nabla \rho_M^n\|^2 \\ + \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)} \sum_{i=1}^{n-1} \left(d_{n-i-1}^\mu - d_{n-i}^\mu\right) \left\|\nabla \rho_M^i\right\|^2 \\ + 2\Delta t \left(G\left(\mathbf{v}_M^{n-1}\right) - G\left(\mathbf{v}^{n-1}\right), \rho_M^n\right) \\ - 2\Delta t \left(\delta_t \psi^n, \rho_M^n\right) + 2\Delta t \left(R_1^n, \rho_M^n\right), \quad n = 1, 2, \cdots, N.\end{aligned} \tag{5.10}$$

Using (5.4) and similar to the derivation of (4.6), we obtain

$$\|\mathbf{G}\left(\mathbf{v}_M^n\right)-\mathbf{G}\left(\mathbf{V}^n\right)\| \leq C_{L_1L_2}\left\|\mathbf{v}_M^n-\mathbf{V}^n\right\|, \tag{5.11}$$

where $C_{L_1L_2}=\max_{|x|\leq\max\{L_1,L_2\}}\left|\mathbf{G}'(x)\right|$.

Moreover, by invoking the boundedness established in Theorem 3.2, we have

$$\begin{aligned}
2\Delta t\left(G\left(\mathbf{v}_M^{n-1}\right)-G\left(\mathbf{v}^{n-1}\right),\rho_M^n\right) \leq & 2\Delta t\left\|G\left(\mathbf{v}_M^{n-1}\right)-G\left(\mathbf{v}^{n-1}\right)\right\|\left\|\rho_M^n\right\| \\
\leq & 2\Delta t C_{L_1L_2}\left\|\psi^{n-1}+\rho_M^{n-1}\right\|\left\|\rho_M^n\right\| \\
\leq & \frac{\Delta t}{3m_3^2}\left\|\rho_M^n\right\|^2+3\left(m_3C_{L_1L_2}\right)^2\Delta t\left\|\psi^{n-1}+\rho_M^{n-1}\right\|^2 \\
\leq & \frac{\Delta t}{3}\left\|\nabla\rho_M^n\right\|^2+6\left(m_3C_{L_1L_2}\right)^2\Delta t\left\|\psi^{n-1}\right\|^2+6\left(m_3C_{L_1}\right)^2\Delta t\left\|\rho_M^{n-1}\right\|^2 \\
\leq & \frac{\Delta t}{3}\left\|\nabla\rho_M^n\right\|^2+6\left(mm_3C_{L_1L_2}\right)^2\Delta t\left\|\mathbf{v}^{n-1}\right\|_\beta^2M^{2-2\beta}+6\left(m_3C_{L_1}\right)^2\Delta t\left\|\rho_M^{n-1}\right\|^2,
\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}
-2\Delta t\left(\delta_t\psi^n,\rho_M^n\right) \leq & \frac{\Delta t}{3}\left\|\nabla\rho_M^n\right\|^2+3\left(mm_3\right)^2\Delta t\left\|\Delta t^{-1}\int_{t_{n-1}}^{t_n}\mathbf{v}_t\,\mathrm{d}t\right\|_\beta^2M^{2-2\beta} \\
\leq & \frac{\Delta t}{3}\left\|\nabla\rho_M^n\right\|^2+3\left(mm_3\right)^2\Delta t^{-1}\left(\int_{t_{n-1}}^{t_n}1\cdot\left\|\mathbf{v}_t\right\|_\beta\,\mathrm{d}t\right)^2M^{2-2\beta} \\
\leq & \frac{\Delta t}{3}\left\|\nabla\rho_M^n\right\|^2+3\left(mm_3\right)^2M^{2-2\beta}\int_{t_{n-1}}^{t_n}\left\|\mathbf{v}_t\right\|_\beta^2\,\mathrm{d}t.
\end{aligned} \tag{5.13}$$

For the last term on the right-hand side of (5.10), using (3.7), we obtain

$$\begin{aligned}
2\Delta t\left(R_1^n,\rho_M^n\right) \leq & 2\Delta t\left\|R_1^n\right\|\left\|\rho_M^n\right\| \\
\leq & \frac{\Delta t}{3m_3^2}\left\|\rho_M^n\right\|^2+3m_3^2\Delta t\left\|R_1^n\right\|^2 \\
\leq & \frac{\Delta t}{3}\left\|\nabla\rho_M^n\right\|^2+3\left(m_3m_4\right)^2\Delta t^3
\end{aligned} \tag{5.14}$$

Substituting the estimates (5.12)-(5.14) into (5.10) yields

$$\begin{aligned}
\left\|\rho_M^n\right\|^2 \leq & \left\|\rho_M^{n-1}\right\|^2+\Delta t\left\|\rho_M^n\right\|^2-\frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)}\left\|\nabla\rho_M^n\right\|^2 \\
& +\frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)}\sum_{i=1}^{n-1}\left(d_{n-i-1}^\mu-d_{n-i}^\mu\right)\left\|\nabla\rho_M^i\right\|^2 \\
& +6\left(m_3C_{L_1L_2}\right)^2\Delta t\left\|\rho_M^{n-1}\right\|^2+6\left(mm_3C_{L_1L_2}\right)^2\Delta t\left\|\mathbf{v}^{n-1}\right\|_\beta^2M^{2-2\beta} \\
& +3\left(mm_3\right)^2M^{2-2\beta}\int_{t_{n-1}}^{t_n}\left\|\mathbf{v}_t\right\|_\beta^2\,\mathrm{d}t \\
& +3\left(m_3m_4\right)^2\Delta t^3, \quad n=1,2,\cdots,N.
\end{aligned} \tag{5.15}$$

Let

$$T^n = \|\rho_M^n\|^2 + \frac{\Delta t^{1-\mu}}{\Gamma(2-\mu)} \sum_{i=1}^{n} d_{n-i}^{\mu} \left\|\nabla \rho_M^i\right\|^2, \quad n = 1, 2, \cdots, N. \tag{5.16}$$

Consequently, (5.15) can be expressed in the form

$$\begin{aligned} T^n \leq & T^{n-1} + \Delta t \|\rho_M^n\|^2 \\ & + 6\left(m_3 C_{L_1 L_2}\right)^2 \Delta t \left\|\rho_M^{n-1}\right\|^2 + 6\left(m m_3 C_{L_1 L_2}\right)^2 \Delta t \left\|\mathbf{v}^{n-1}\right\|_{\beta}^2 M^{2-2\beta} \\ & + 3\left(m m_3\right)^2 M^{2-2\beta} \int_{t_{n-1}}^{t_n} \|\mathbf{v}_t\|_{\beta}^2 \, \mathrm{d}t \\ & + 3\left(m_3 m_4\right)^2 \Delta t^3, \quad n = 1, 2, \cdots, N. \end{aligned} \tag{5.17}$$

Replacing n by m in (5.17), summing the resulting inequalities over $m = 1, \ldots, n$, and using the inequality $|\rho_M^n|^2 \leq T^n$, we arrive at

$$\begin{aligned} \|\rho_M^n\|^2 \leq & (1 + 6\left(m_3 C_{L_1 L_2}\right)^2) \Delta t \sum_{m=1}^{n} \|\rho_M^m\|^2 + \Big(6T\left(m m_3 C_{L_1 L_2}\right)^2 \|\mathbf{v}\|_{L^{\infty}\left(0,T;H^{\beta}(\Xi)\right)}^2 \\ & + 3\left(m m_3\right)^2 \|\mathbf{v}_t\|_{L^2\left(0,T;H^{\beta}(\Xi)\right)}^2\Big) M^{2-2\beta} \\ & + 3T\left(m_3 m_4\right)^2 \Delta t^2, \quad 1 \leq n \leq N. \end{aligned}$$

Since $\Delta t \leq \Delta t_2$, applying the discrete Gronwall's inequality then

$$\begin{aligned} \|\rho_M^n\| \leq & \Big\{\Big(6T\left(m m_3 C_{L_1 L_2}\right)^2 \|\mathbf{v}\|_{L^{\infty}\left(0,T;H^{\beta}(\Xi)\right)}^2 \\ & + 3\left(m m_3\right)^2 \|u_t\|_{L^2\left(0,T;H^{\beta}(\Xi)\right)}^2\Big) \exp\left(2T\left(1 + 6\left(m_3 C_{L_1 L_2}\right)^2\right)\right)\Big\}^{\frac{1}{2}} M^{1-\beta} \\ & + \left\{3T\left(m_3 m_4\right)^2 \exp\left(2T\left(1 + 6\left(m_3 C_{L_1 L_2}\right)^2\right)\right)\right\}^{\frac{1}{2}} \Delta t \\ = & m_8 \Delta t + m_9 M^{1-\beta}, \quad 0 \leq n \leq N \end{aligned}$$

Moreover, by the triangle inequality, we have

$$\begin{aligned} \|\mathbf{v}^n - \mathbf{v}_M^n\| &= \|\psi^n + \rho_M^n\| \\ &\leq \|\psi^n\| + \|\rho_M^n\| \\ &\leq \left(m\|\mathbf{v}\|_{L^{\infty}\left(0,T;H^{\beta}(\Xi)\right)} + m_9\right) M^{1-\beta} + m_8 \Delta t \\ &= m_8 \Delta t + m_{10} M^{1-\beta}, \quad n = 0, 1, \cdots, N. \end{aligned}$$

This completes the proof.

6. Computational experiments

This section aims to numerically validate the theoretical results established in the preceding sections through a series of carefully designed experiments. The proposed method is assessed in terms of the accuracy of the computed solutions,

the associated numerical errors, and the observed convergence rates. These numerical investigations provide a comprehensive assessment of the reliability, accuracy, and convergence properties of the proposed scheme, and demonstrate its effectiveness in approximating the underlying problem.

$$Rate = \frac{\log\left(\| \text{ error }_1\| / \| \text{ error }_2\|\right)}{\log\left(k_1/k_2\right)}, \tag{6.1}$$

where $= \|\mathbf{v}_M^n - \mathbf{v}^n\|$ is the error, and $k_1 \neq k_2$.

Consider the time-fractional nonlinear advection–diffusion equation posed on the square domain $\Xi = (0,1)^2$. Let $\mathbf{G}(\mathbf{v}) = \mathbf{v}^2$ and let the exact solution be $\mathbf{v}(x,y,t) = e^{x+y}t^{3-\beta}$ and $T = 1$. Then, we can obtain the source function $\mathbf{v}(x,y,t) = e^{x+y}\left[(3-\beta)t^{2-\beta} - \frac{2\Gamma(4-\beta)}{\Gamma(4-2\beta)}t^{3-2\beta}\right] + e^{2(x+y)}t^{6-2\beta}$.

- Figure 1 illustrates the accuracy and convergence behavior of the proposed numerical method at the final time $T = 1$ for two sets of fractional derivative orders, namely $\beta = 0.3, 0.6, 0.8$ and $\beta = 0.2, 0.5, 0.7$. The numerical errors in the L_2-norm are plotted against the polynomial degree M, with $\Delta t = 2e-3$ and $K_e = 10$ fixed (left panel), and against the number of elements K_e, with $\Delta t = 2e-3$ and $M = 4$ fixed (right panel). These results demonstrate the convergence of the proposed method with respect to both the temporal and spatial discretizations.
- Tables 1 and 2 provide further evidence of the accuracy and computational efficiency of the proposed G-SEM. In Table 1, the L_2-norm errors and corresponding convergence orders obtained by the G-SEM, FEM, and FDM are reported for several time step sizes Δt, with $K_e = 4$ and $\beta = 0.5$. The results demonstrate that the G-SEM consistently achieves higher accuracy and exhibits more favorable convergence behavior than the conventional FEM and FDM. Table 2 reports the L_2-norm errors, convergence orders, and CPU times of the G-SEM for different values of K_e, with $\Delta t = 10^{-2}$ and $\beta = 0.5$ and 0.7. These results confirm the accuracy of the proposed method while also demonstrating its computational efficiency. Moreover, the observed temporal convergence rates are in excellent agreement with the theoretical estimates established in Theorem 5.2, providing strong numerical evidence for the validity of the proposed scheme. The rapid reduction in the L_2-norm error confirms the high-order approximation capability of the G-SEM and highlights its effectiveness in achieving highly accurate numerical solutions with relatively modest spatial discretization.

7. Conclusion

In this work, we developed an efficient Galerkin spectral element method for a time-fractional nonlinear advection–diffusion equation. For the resulting fully discrete scheme, we derived boundedness properties and optimal error estimates, achieving a convergence rate of $\mathcal{O}\left(\Delta t + M^{1-\beta}\right)$. A series of numerical experiments demonstrated the accuracy, stability, and computational efficiency of the proposed approach, while also confirming the theoretically predicted convergence rates. It should be noted that, due to the weakly singular nature of the Caputo

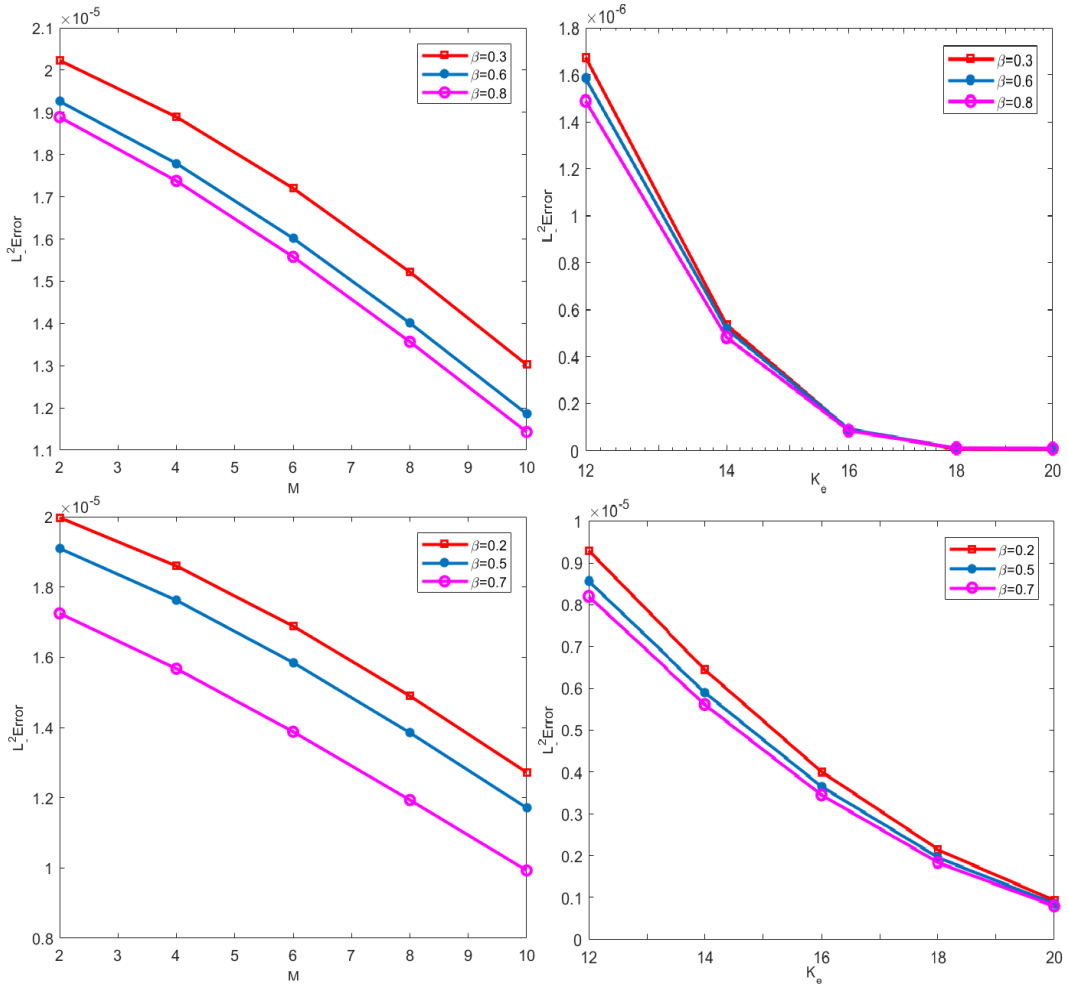


Fig. 1. L_2 Errors for different values of M, and K_e. at $T = 1$.

Table 1. Comparison of the G-SEM, FEM, and FDM errors and orders based on L_2 norm.

$K_e = 4$	G-SEM		FEM		FDM	
Δt	$L_{2_}error$	rate	$L_{2_}error$	rate	$L_{2_}error$	rate
$1/4$	$2.5238e-05$		$3.9823e-03$		$4.4456e-02$	
$1/8$	$1.3155e-05$	0.94	$2.1489e-03$	0.89	$2.4156e-02$	0.88
$1/16$	$6.7157e-06$	0.97	$1.1516e-03$	0.90	$1.3126e-02$	0.88
$1/32$	$3.4047e-06$	0.98	$6.1286e-04$	0.91	$6.9855e-03$	0.91
$1/64$	$1.7142e-06$	0.99	$3.1944e-04$	0.94	$3.6919e-03$	0.92
$1/128$	$8.5710e-07$	1.00	$1.6421e-04$	0.97	$1.9110e-03$	0.95

kernel, second-order temporal convergence cannot generally be attained at $t = t_n$. As a natural extension of the present work, higher-order temporal discretization techniques, particularly the $L_{2-1\sigma}$ scheme on non-uniform meshes [3], will be considered in future investigations.

Table 2. G-SEM errors, rates based on L_2 norm and CPU time(s) for $\beta = 0.5$ and 0.7.

$\Delta t = 10^{-2}$	$\beta = 0.5$		$\beta = 0.7$		
K_e	L_2_error	rate	L_2_error	rate	CPU Time
4	4.3458e-04		2.5679e-03		19.5654
6	1.0865e-04	2.00	6.4198e-04	2.00	35.4463
8	2.3160e-05	2.23	1.4667e-04	2.13	52.4649
10	3.7936e-06	2.61	2.3694e-05	2.63	80.4546
12	5.4850e-07	2.79	3.5466e-06	2.74	99.3564
14	6.9520e-08	2.98	4.4641e-07	2.99	203.353

Declarations

Conflicts of interest: The authors declare that they have no competing interests regarding the publication of this paper.

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