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STOCHASTIC VOLTERRA–GAMMA INTEGRAL EQUATION WITH CONSTANT-DELAY FEEDBACK

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Abstract. This paper investigates a stochastic Volterra–Gamma delay integral equation. The existence and uniqueness of the solution are established by means of the Banach fixed-point theorem, while a generalized Grönwall inequality is employed to derive growth estimates and characterize the growth behaviour of the solution. The obtained results demonstrate that the parameter β has a significant influence on the solution behaviour. In particular, the numerical results and analysis reveal a pronounced parameter-dependent transition from the prescribed history period for $t < 0$ to a rapidly growing solution for $t > 0$.

Keywords. Volterra integral, Gamma–Volterra process, Burkholder-Davis-Gundy inequality, Growth moment, Mittag–Leffler.
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1. Introduction

The study of stochastic Volterra integral equations (SVIEs) can be traced back to the work of Berger and Mizel [1, 2]. SVIEs may be viewed as a generalization of stochastic differential equations (SDEs), incorporating memory effects through the dependence of the integrand on both the current time and the integration variable. In general, an SVIE can be expressed in the form

$$x(t) = \phi(t) + \int_0^t a(t, s, x(s))ds + \int_0^t b(t, s, x(s))dw_s, \ t \geq 0.$$

These equations have found wide applications in various fields, including mathematical finance, engineering, and biology. In mathematical finance, SVIEs are used to model rough volatility and financial assets exhibiting long-range dependence. In engineering, they arise in the modelling of turbulent fluid flows, heat transfer in materials with memory, and anomalous diffusion. In biology, SVIEs have been employed to describe population dynamics and the mathematical modelling of epidemic spread.

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Now, one briefly reviews some recent developments in the literature. The authors in [3] investigated optimal control problems for SVIEs with completely monotone kernels. The Hölder continuity of solutions and the convergence rate of a Milstein-type numerical scheme for SVIEs were studied in [8]. Moreover, the authors in [7] investigated the mean-square asymptotic stability of the exact solution and established a sharp strong convergence rate using a Mittag–Leffler Euler–Maruyama method for SVIEs. Further studies on SVIEs can be found in [6, 9, 10]. More recently, SVIEs involving delay arguments have also been investigated; see, for example, [12, 14] and the references therein.

This paper investigates a stochastic Gamma–Volterra integral equation driven by constant-delay feedback of the form:

$$x(t) = \begin{cases} \varphi(0) + \displaystyle\int_0^t (t-s)^{\beta} e^{-\lambda(t-s)} f(s, x(s), x(s-\tau)) dw_s, \; t \in (0, \mathfrak{T}], \\ \\ \varphi(t), \; t \in [-\tau, 0], \; \tau > 0, \end{cases} \tag{1.1}$$

where $\varphi \in C([-\tau, 0], \mathbb{R})$ is the prescribed history function, τ is a constant delay and $f : [0, \mathfrak{T}] \times \mathbb{R}^2 \to \mathbb{R}$ is a nonlinear term satisfying an appropriate Lipschitz continuity condition. To the best of the author's knowledge, and based on the available literature, the class of stochastic Volterra delay integral equations considered in this paper has not previously been studied. The stochastic Gamma–Volterra delay integral equation considered above incorporates several fundamental features, including memory effects induced by Volterra-type integral operators, Gamma-type or power-law kernels, time-delay mechanisms describing the dependence of the system on past states, and stochastic perturbations represented by stochastic processes. This combination provides a flexible framework for modelling complex dynamical systems in which the evolution of the state is influenced not only by its current and delayed values but also by accumulated historical effects and random fluctuations. In particular, the Volterra structure accounts for hereditary dependence, while the Gamma-type kernel provides a mechanism for characterizing the influence of past states over time. Consequently, such models have potential applications in a broad range of fields, including mathematical finance, physics, biology, engineering, and other areas involving memory-dependent and non-Markovian stochastic dynamics. The remainder of this paper is organized as follows. Section 2 introduces the necessary background notions and preliminary concepts. Section 3 presents the main results of the paper, together with their detailed proofs. Finally, Section 4 provides a brief summary of the principal results and concluding remarks.

2. Preliminaries

Some basic concepts on the study is presented in this section.

Definition 2.1 ([4, 5])**.** A Volterra process is a stochastic process given by

$$Y_t = \int_0^t g(t, s) dZ_s, \; t \in [0, \mathfrak{T}], \tag{2.1}$$

where $g(t,s)$ is a deterministic kernel, and Z is a Lévy process or a square integrable martingale process. For a fixed t, consider the characteristic function

$$\mathbf{E}[e^{i\xi Z_t}] = e^{t\psi(\xi)},\ \xi \in \mathbb{R},$$

where

$$\psi(\xi) = ia\xi - \frac{b\xi^2}{2} + \int_{\mathbb{R}} (e^{i\xi x} - 1 - i\xi\tau(x))\vartheta(dx),$$

$a \in \mathbb{R}, b \geq 0, \vartheta$ is a Lévy measure on $\mathbb{R}$, with $\int_0^t (x^2 \wedge 1)\vartheta(dx) < \infty, \vartheta(\{0\}) = 0$, and

$$\tau(x) = \begin{cases} x, & \text{for} \quad |x| \leq 1, \\ \frac{x}{|x|}, & \text{for} \quad |x| > 1. \end{cases}$$

The triplet (a, b, ϑ) is called the characteristic triplet of Z.

Remark 2.2. Depending on the regularity of the kernel g, the Volterra process can be expressed as $Y_t = \int_0^t g(t-s)dZ_s,\ t \in [0, \mathfrak{T}]$.

The Gaussian process $(Y_t)_{t\geq 0}$ is called Volterra, if there exists a standard Brownian motion $(w_t)_{t\geq 0}$ and a Volterra kernel $g \in L^2_{loc}([0,\infty))$ such that

$$Y_t = \int_0^t g(t,s)dw_s,\ a.s.,\ t \geq 0.$$

The label Volterra originates from the Volterra integral equation (of the first kind) which is of the type

$$x(t) = \int_0^t g(t,s)y(s)ds,\ t \in [0,\infty);$$

where the function x and the kernel g are known, and the function y is unknown.

Theorem 2.3 ([4], Lemma 3.4)**.** *The following properties hold for the Volterra process Y_t in Eq. (2.1).*

(i.) *Let $p \in [1,2)$. Assume that $g(t,.) \in L_p([0,t])$ and that the characteristic triplet of Z satisfies $b = 0$, ϑ is symmetric with $\int_{\mathbb{R}} |x|^p \vartheta(dx) < \infty$. Then*

$$\mathbf{E}\left| \int_0^t g(t,s)dZ_s \right|^p \leq C\left(|a|^p \|g(t,.)\|^p_{L_1([0,t])} + \|g(t,.)\|^p_{L_p([0,t])} \int_{\mathbb{R}} |x|^p \vartheta(dx) \right)$$
$$\leq \tilde{C}\|g(t,.)\|^p_{L_p([0,t])}.$$

(ii.) *Let $p \geq 2$. Assume that $g(t,.) \in L_p([0,t])$ and Z is such that ϑ is symmetric with $\int_{\mathbb{R}} |x|^p \vartheta(dx) < \infty$. Then*

$$\mathbf{E}\left| \int_0^t g(t,s)dZ_s \right|^p \leq C\Bigg(|a|^p \|g(t,.)\|^p_{L_1([0,t])} + |b|^{\frac{p}{2}} \|g(t,.)\|^p_{L_2([0,T])}$$
$$+ \|g(t,.)\|^p_{L_p([0,t])} \int_{\mathbb{R}} |x|^p \vartheta(dx) \Bigg).$$

Remark 2.4 ([11]). Alternatively, one can adopt a Burkholder-Davis-Gundy-type inequality for stochastic Volterra integrals as follows:

$$\mathbf{E}\left[\sup_{r\in[0,t]}\left|\int_0^r k(t-s)f(s)dw_s\right|^p\right] \leq C_p\mathbf{E}\left[\int_0^t |k(t-s)|^2|f(s)|^2ds\right]^{\frac{p}{2}}, \tag{2.2}$$

where $p \geq 2, t > 0, k \in L^2([0,t],\mathbb{R}^{d\times d})$, w is an m-dimensional Brownian motion, $C_p > 0$ and f is a previsible process in $\mathbb{R}^{d\times m}$.

Definition 2.5 ([4, 5])**.** A Gamma–Volterra Process is given by

$$Y_t = \int_0^t (t-s)^\beta e^{-\lambda(t-s)}dZ_s,$$

for $\beta \in (-\frac{1}{2}, \frac{1}{2}), \lambda \geq 0$, with $\int_{\mathbb{R}} |x|^p\vartheta(dx) < \infty$ for some p.

2.1. **Making sense of the solution.** For $t \in [0,\mathfrak{T}]$, denote $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ as a complete filtered probability space and define $L^p(\mathbf{P}) = L^p(\Omega, \mathcal{F}_t, \mathbf{P})$ to be a space of all $\mathcal{F}_t$- measurable and pth-moment integrable functions.

Lemma 2.6. *The function* $\{x(t), t \in [0,\mathfrak{T}]\}$ *is said to be a mild solution if Eq.*(1.1) *holds. Suppose further that* $\{x(t), t \in [0,\mathfrak{T}]\}$ *satisfies*

$$\sup_{t\in[0,\mathfrak{T}]} e^{p\lambda t}\mathbf{E}|x(t)|^p < \infty,$$

then $\{x(t), t \in [0,\mathfrak{T}]\}$ *is said to be a random solution to Eq.* (1.1).

3. Main results

In order to prove the main results of the paper, one imposes the following condition to facilitate the analysis and ensure the well-posedness of the problem:

Condition 3.1. The nonlinear function $f : [0,\mathfrak{T}] \times \mathbb{R}^2 \to \mathbb{R}$ is said to satisfy the following Lipschitz condition

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq \ell_f\big(|x_1-x_2| + |y_1-y_2|\big),\ \forall\, t \in [0,\mathfrak{T}],$$

and for convenience, $|f(t,x,y)| \leq \ell_f\big(|x| + |y|\big)$.

3.1. **Existence and Uniqueness Result.** Now, equip the space $L^2(\mathbf{P})$ with the norm

$$\|x\|^2_{2,\lambda} := \sup_{t\in[0,\mathfrak{T}]} e^{2\lambda t}\mathbf{E}|x(t)|^2 = \sup_{t\in[0,\mathfrak{T}]} e^{2\lambda(t-\tau)}\mathbf{E}|x(t-\tau)|^2, \tag{3.1}$$

almost surely, and let

$$\|\varphi\|^2_{2,\lambda} = \sup_{t\in[0,\mathfrak{T}]} e^{2\lambda t}|\varphi(t)|^2 = \sup_{t\in[0,\mathfrak{T}]} e^{2\lambda(t-\tau)}|\varphi(t-\tau)|^2 \leq \mathfrak{M} < \infty. \tag{3.2}$$

Remark 3.2. Note that: $\sup_{t\in[0,\mathfrak{T}]} e^{2\lambda(t-\tau)}\mathbf{E}|x(t-\tau)|^2 = \sup_{t\in[\tau,\mathfrak{T}+\tau]} e^{2\lambda t}\mathbf{E}|x(t)|^2$.

Define the operator $\mathcal{T} : L^2(\mathbf{P}) \to L^2(\mathbf{P})$ by

$$\mathcal{T}x(t) = \varphi(0) + \int_0^t (t-s)^\beta e^{-\lambda(t-s)}f(s,x(s),x(s-\tau))dw_s. \tag{3.3}$$

Lemma 3.3. *Let x be a random solution of Eq. (1.1) and let Condition 3.1 hold. Then, for positive constants c_1, c_2,*

$$\|\mathcal{T}x\|_{2,\lambda}^2 \leq c_1 + c_2\|x\|_{2,\lambda}^2,$$

where $c_1 = 2|\varphi(0)|^2 e^{2\lambda\mathfrak{T}} + 4C_2\ell_f^2\mathfrak{M}e^{2\lambda\tau}\frac{\mathfrak{T}^{2\beta+1}}{2\beta+1}$ and $c_2 = 4C_2\ell_f^2(1+e^{2\lambda\tau})\frac{\mathfrak{T}^{2\beta+1}}{2\beta+1}$.

Proof. Instead of the usual Itô Isometry, one applies the BDG inequality in Eq. (3.3) for $p = 2$ to obtain

$$\begin{aligned}
\mathbf{E}|\mathcal{T}x(t)|^2 \leq& 2|\varphi(0)|^2 + 2\mathbf{E}\left|\int_0^t (t-s)^\beta e^{-\lambda(t-s)} f(s,x(s),x(s-\tau))dw_s\right|^2 \\
\leq& 2|\varphi(0)|^2 + 2\mathbf{E}\left|\sup_{r\in[0,t]}\int_0^r (t-s)^\beta e^{-\lambda(t-s)} f(s,x(s),x(s-\tau))dw_s\right|^2 \\
\leq& 2|\varphi(0)|^2 + 2C_2\int_0^t (t-s)^{2\beta}e^{-2\lambda(t-s)}\mathbf{E}|f(s,x(s),x(s-\tau))|^2 ds.
\end{aligned}$$

By continuity of f, we obtain,

$$\begin{aligned}
\mathbf{E}|\mathcal{T}x(t)|^2 \leq& 2|\varphi(0)|^2 + 2C_2\ell_f^2\Bigg[\int_0^t (t-s)^{2\beta}e^{-2\lambda(t-s)}\mathbf{E}\big(|x(s)| + |x(s-\tau)|\big)^2 ds \\
\leq& 2|\varphi(0)|^2 + 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{-2\lambda(t-s)}\mathbf{E}|x(s)|^2 ds \\
&+ 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{-2\lambda(t-s)}\mathbf{E}|x(s-\tau)|^2 ds.
\end{aligned}$$

Let $t \in [0,\tau]$, that is, $t-\tau \in [-\tau, 0]$ to get $x(t-\tau) = \varphi(t-\tau)$ and thus,

$$\begin{aligned}
\mathbf{E}|\mathcal{T}x(t)|^2 \leq& 2|\varphi(0)|^2 + 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{-2\lambda(t-s)}\mathbf{E}|x(s)|^2 ds \\
&+ 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{-2\lambda(t-s)}|\varphi(s-\tau)|^2 ds.
\end{aligned}$$

Multiply through by $e^{2\lambda t}$ to obtain

$$\begin{aligned}
e^{2\lambda t}\mathbf{E}|\mathcal{T}x(t)|^2 \leq& 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{2\lambda s}\mathbf{E}|x(s)|^2 ds \\
&+ 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{2\lambda s}|\varphi(s-\tau)|^2 ds \\
=& 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{2\lambda s}\mathbf{E}|x(s)|^2 ds \\
&+ 4C_2\ell_f^2 e^{2\lambda\tau}\int_0^t (t-s)^{2\beta}e^{2\lambda(s-\tau)}|\varphi(s-\tau)|^2 ds.
\end{aligned}$$

Following Eq. (3.1) and Eq. (3.2), one obtains

$$e^{2\lambda t}\mathbf{E}|\mathcal{T}x(t)|^2 \leq 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2\|x\|_{2,\lambda}^2\frac{t^{2\beta+1}}{2\beta+1} + 4C_2\ell_f^2\|\varphi\|_{2,\lambda}^2 e^{2\lambda\tau}\frac{t^{2\beta+1}}{2\beta+1}.$$

Take the supremum over $t \in [0, \mathfrak{T}]$ to arrive at

$$\|\mathcal{T}x\|^2_{2,\lambda} \leq 2|\varphi(0)|^2 e^{2\lambda T} + 4C_2\ell_f^2\|x\|^2_{2,\lambda}\frac{\mathfrak{T}^{2\beta+1}}{2\beta+1} + 4C_2\ell_f^2\mathfrak{M}e^{2\lambda\tau}\frac{\mathfrak{T}^{1+2\beta}}{1+2\beta}.$$

Next, take $t \in [\tau, \mathfrak{T}]$ to obtain

$$\begin{aligned}
e^{2\lambda t}\mathbf{E}|\mathcal{T}x(t)|^2 \leq & 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{2\lambda s}\mathbf{E}|x(s)|^2 ds \\
& + 4C_2\ell_f^2\int_0^\tau (t-s)^{2\beta}e^{2\lambda s}\mathbf{E}|x(s-\tau)|^2 ds + 4C_2\ell_f^2\int_\tau^t (t-s)^{2\beta}e^{2\lambda s}\mathbf{E}|x(s-\tau)|^2 ds \\
= & 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{2\lambda s}\mathbf{E}|x(s)|^2 ds \\
& + 4C_2\ell_f^2 e^{2\lambda\tau}\int_0^\tau (t-s)^{2\beta}e^{2\lambda(s-\tau)}|\varphi(s-\tau)|^2 ds \\
& + 4C_2\ell_f^2 e^{2\lambda\tau}\int_\tau^t (t-s)^{2\beta}e^{2\lambda(s-\tau)}\mathbf{E}|x(s-\tau)|^2 ds,
\end{aligned}$$

since $s \in [0, \tau]$ implies that $x(s-\tau) = \varphi(s-\tau)$, and $s \in [\tau, t]$ gives $0 \leq s-\tau \leq t-\tau \leq t$. Hence,

$$\begin{aligned}
e^{2\lambda t}\mathbf{E}|\mathcal{T}x(t)|^2 \leq & 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{2\lambda s}\mathbf{E}|x(s)|^2 ds \\
& + 4C_2\ell_f^2 e^{2\lambda\tau}\int_0^t (t-s)^{2\beta}e^{2\lambda(s-\tau)}|\varphi(s-\tau)|^2 ds \\
& + 4C_2\ell_f^2 e^{2\lambda\tau}\int_0^t (t-s)^{2\beta}e^{2\lambda(s-\tau)}\mathbf{E}|x(s-\tau)|^2 ds.
\end{aligned}$$

From Eq. (3.1) and Eq. (3.2), one gets

$$e^{2\lambda t}\mathbf{E}|\mathcal{T}x(t)|^2 \leq 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2(1+e^{2\lambda\tau})\|x\|^2_{2,\lambda}\frac{t^{2\beta+1}}{2\beta+1} + 4C_2\ell_f^2 e^{2\lambda\tau}\|\varphi\|^2_{2,\lambda}\frac{t^{2\beta+1}}{2\beta+1}.$$

By taking the supremum over $t \in [0, \mathfrak{T}]$, one obtains

$$\|\mathcal{T}x\|^2_{2,\lambda} \leq 2|\varphi(0)|^2 e^{2\lambda\mathfrak{T}} + 4C_2\ell_f^2(1+e^{2\lambda\tau})\|x\|^2_{2,\lambda}\frac{\mathfrak{T}^{2\beta+1}}{2\beta+1} + 4C_2\ell_f^2 e^{2\lambda\tau}\mathfrak{M}\frac{\mathfrak{T}^{2\beta+1}}{2\beta+1}.$$

□

Lemma 3.4. *Let x and y be random solutions of Eq. (1.1), and let Condition 3.1 hold. Then,*

$$\|\mathcal{T}x - \mathcal{T}y\|^2_{2,\lambda} \leq c_3\|x-y\|^2_{2,\lambda},$$

where $c_3 = 2C_2\ell_f^2(1+e^{2\lambda\tau})\frac{\mathfrak{T}^{2\beta+1}}{2\beta+1} > 0$.

Proof. Taking the second moment of both sides and applying the BDG inequality for $p = 2$ yield

$$\begin{aligned}\mathbf{E}|\mathcal{T}x(t) - \mathcal{T}y(t)|^2 =& \mathbf{E}\left| \int_0^t (t-s)^{\beta} e^{-\lambda(t-s)}[f(s,x(s),x(s-\tau)) - f(s,y(s),y(s-\tau))]dw_s \right|^2 \\ \leq& 2C_2 \int_0^t (t-s)^{2\beta} e^{-2\lambda(t-s)} \mathbf{E}|f(s,x(s),x(s-\tau)) - f(s,y(s),y(s-\tau))|^2 ds \\ \leq& 2C_2\ell_f^2 \int_0^t (t-s)^{2\beta} e^{-2\lambda(t-s)} \mathbf{E}|x(s) - y(s)|^2 ds \\ &+ 2C_2\ell_f^2 \int_0^t (t-s)^{2\beta} e^{-2\lambda(t-s)} \mathbf{E}|x(s-\tau) - y(s-\tau)|^2 ds.\end{aligned}$$

Consider $t \in [0,\tau]$, which gives $x(t-\tau) - y(t-\tau) = \varphi(t-\tau)) - \varphi(t-\tau) = 0$, and thus,

$$e^{2\lambda t}\mathbf{E}|\mathcal{T}x(t) - \mathcal{T}y(t)|^2 \leq 2C_2\ell_f^2 \int_0^t (t-s)^{2\beta} e^{2\lambda s} \mathbf{E}|x(s) - y(s)|^2 ds \leq 2C_2\ell_f^2 \|x-y\|_{2,\lambda}^2 \frac{t^{2\beta+1}}{2\beta+1}.$$

Taking the supremum over $t \in [0,\mathfrak{T}]$, one obtains

$$\|\mathcal{T}x - \mathcal{T}y\|_{2,\lambda}^2 \leq C_2\ell_f^2 \|x-y\|_{2,\lambda}^2 \frac{\mathfrak{T}^{2\beta+1}}{2\beta+1}.$$

Next, let $t \in [\tau,\mathfrak{T}]$ to get

$$\begin{aligned}e^{2\lambda t}\mathbf{E}|\mathcal{T}x(t) - \mathcal{T}y(t)|^2 \leq& 2C_2\ell_f^2 \int_0^t (t-s)^{2\beta} e^{2\lambda s} \mathbf{E}|x(s) - y(s)|^2 ds \\ &+ C_2\ell_f^2 \int_0^\tau (t-s)^{2\beta} e^{2\lambda s} \mathbf{E}|x(s-\tau) - y(s-\tau)|^2 ds \\ &+ 2C_2\ell_f^2 \int_\tau^t (t-s)^{2\beta} e^{2\lambda s} \mathbf{E}|x(s-\tau) - y(s-\tau)|^2 ds \\ =& 2C_2\ell_f^2 \int_0^t (t-s)^{2\beta} e^{2\lambda s} \mathbf{E}|x(s) - y(s)|^2 ds \\ &+ 2C_2\ell_f^2 e^{2\lambda\tau} \int_\tau^t (t-s)^{2\beta} e^{2\lambda(s-\tau)} \mathbf{E}|x(s-\tau) - y(s-\tau)|^2 ds \\ \leq& 2C_2\ell_f^2 \int_0^t (t-s)^{2\beta} e^{2\lambda s} \mathbf{E}|x(s) - y(s)|^2 ds \\ &+ 2C_2\ell_f^2 e^{2\lambda\tau} \int_0^t (t-s)^{2\beta} e^{2\lambda(s-\tau)} \mathbf{E}|x(s-\tau) - y(s-\tau)|^2 ds \\ \leq& 2C_2\ell_f^2 (1 + e^{2\lambda\tau}) \|x-y\|_{2,\lambda}^2 \frac{t^{2\beta+1}}{2\beta+1}.\end{aligned}$$

Applying the supremum over $t \in [0,\mathfrak{T}]$ yields

$$\|\mathcal{T}x - \mathcal{T}y\|_{2,\lambda}^2 \leq 2C_2\ell_f^2 (1 + e^{2\lambda\tau}) \|x-y\|_{2,\lambda}^2 \frac{\mathfrak{T}^{2\beta+1}}{2\beta+1}.$$

□

Now apply the above lemmas to establish the existence and uniqueness of the solution to Eq. (1.1).

Theorem 3.5. *Let Condition 3.1 be satisfied, then Eq. (1.1) admits a unique solution if* $c_2 < 1$.

Proof. The proof follows by the Banach fixed-point theorem, so, one sets $\mathcal{T}x = x$. If one applies Lemma 3.3, then

$$\|x\|^2_{2,\lambda} \leq c_1 + c_2\|x\|^2_{2,\lambda}.$$

Thus, $\|x\|^2_{2,\lambda}(1-c_2) \leq c_1$, and $\|x\|^2_{2,\lambda} < \infty$ if and only if $c_2 < 1$. Similarly, let x and y be solutions of Eq. (1.1). Then, it follows from Lemma 3.4 that

$$\|\mathcal{T}x - \mathcal{T}y\|^2_{2,\lambda} = \|x-y\|^2_{2,\lambda} \leq c_3\|x-y\|^2_{2,\lambda}.$$

Therefore, $\|x-y\|^2_{2,\lambda}(1-c_3) \leq 0$. So, for $c_3 < 1$, one gets $x = y$. □

3.2. **Moment growth bound.** To obtain the growth bound, one applies the following generalized Gronwall inequality:

Lemma 3.6 ([13])**.** *Let* $\alpha > 0, b(t)$ *is a nonnegative function locally integrable on* $[0,\mathfrak{T}], \mathfrak{T} \leq \infty$, *and* $c(t)$ *is a nonnegative, nondecreasing continuous function defined on* $[0,\mathfrak{T}]$, $c(t) \leq M$, *and suppose* $a(t)$ *is nonnegative and locally integrable on* $[0,\mathfrak{T}]$ *with*

$$a(t) \leq b(t) + c(t)\int_0^t (t-s)^{\alpha-1}a(s)ds,$$

then

$$a(t) \leq b(t) + c(t)\int_0^t \sum_{k=1}^{\infty}\left[\frac{(c(t)\Gamma(\alpha))^k}{\Gamma(k\alpha)}(t-s)^{k\alpha-1}b(s)\right]ds,\ t \in [0,\mathfrak{T}].$$

Corollary 3.7. *Under the hypothesis of Lemma 3.6, let* b *be a nondecreasing function on* $[0,\mathfrak{T}]$, *then*

$$a(t) \leq b(t)E_\alpha[c(t)\Gamma(\alpha)t^\alpha],\ t \in [0,\mathfrak{T}].$$

The result is as follows.

Theorem 3.8. *Let Condition 3.1 hold, then for* $t \in [0,\mathfrak{T}]$, *Eq. (1.1) has a solution with the following upper moment bound*

$$\mathbf{E}|x(t)|^2 \leq \left[2|\varphi(0)|^2 + 4\mathfrak{M}C_2\ell_f^2 e^{-2\lambda(t-\tau)}\frac{t^{2\beta+1}}{2\beta+1}\right]E_{2\beta+1}\left[4C_2\ell_f^2(1+e^{2\lambda\tau})\Gamma(2\beta+1)t^{2\beta+1}\right].$$

Proof. From the estimate on the solution of Eq. (1.1),

$$\begin{aligned} e^{2\lambda t}\mathbf{E}|x(t)|^2 \leq & 2|\varphi(0)|^2e^{2\lambda t} + 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{2\lambda s}\mathbf{E}|x(s)|^2ds \\ & + 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}e^{2\lambda s}\mathbf{E}|x(s-\tau)|^2ds. \end{aligned}$$

If $t \in [0, \tau]$, one gets $x(t-\tau) = \varphi(t-\tau)$, then

$$\begin{aligned} e^{2\lambda t}\mathbf{E}|x(t)|^2 \leq & 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2 \int_0^t (t-s)^{2\beta} e^{2\lambda s}\mathbf{E}|x(s)|^2 ds \\ & + 4C_2\ell_f^2 e^{2\lambda\tau} \int_0^t (t-s)^{2\beta} e^{2\lambda(s-\tau)}|\varphi(s-\tau)|^2 ds. \end{aligned}$$

and by defining $g(t) = e^{2\lambda t}\mathbf{E}|x(t)|^2$,

$$\begin{aligned} g(t) &\leq 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2 \int_0^t (t-s)^{2\beta} g(s) ds + 4C_2\ell_f^2 e^{2\lambda\tau}\|\varphi\|_{2,\lambda}^2 \frac{t^{2\beta+1}}{2\beta+1} \\ &\leq 2|\varphi(0)|^2 e^{2\lambda t} + 4\mathfrak{M}C_2\ell_f^2 e^{2\lambda\tau}\frac{t^{2\beta+1}}{2\beta+1} + 4C_2\ell_f^2 \int_0^t (t-s)^{(2\beta+1)-1} g(s) ds. \end{aligned}$$

Applying Gronwall's inequality from Lemma 3.6 and Corollary 3.7 yields

$$g(t) \leq \left[2|\varphi(0)|^2 e^{2\lambda t} + 4\mathfrak{M}C_2\ell_f^2 e^{2\lambda\tau}\frac{t^{2\beta+1}}{2\beta+1}\right] E_{2\beta+1}\left[4C_2\ell_f^2\Gamma(2\beta+1)t^{2\beta+1}\right],$$

and thus,

$$\mathbf{E}|x(t)|^2 \leq \left[2|\varphi(0)|^2 + 4\mathfrak{M}C_2\ell_f^2 e^{-2\lambda(t-\tau)}\frac{t^{2\beta+1}}{2\beta+1}\right] E_{2\beta+1}\left[4C_2\ell_f^2\Gamma(2\beta+1)t^{2\beta+1}\right].$$

Similarly, for $t \in [\tau, \mathfrak{T}]$,

$$\begin{aligned} e^{2\lambda t}\mathbf{E}|x(t)|^2 \leq & 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2 \int_0^t (t-s)^{2\beta} e^{2\lambda s}\mathbf{E}|x(s)|^2 ds \\ & + 4C_2\ell_f^2 e^{2\lambda\tau} \int_0^\tau (t-s)^{2\beta} e^{2\lambda(s-\tau)}|\varphi(s-\tau)|^2 ds \\ & + 4C_2\ell_f^2 e^{2\lambda\tau} \int_\tau^t (t-s)^{2\beta} e^{2\lambda(s-\tau)}\mathbf{E}|x(s-\tau)|^2 ds. \end{aligned}$$

Thus, the above simplifies to

$$\begin{aligned} g(t) \leq & 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2 \int_0^t (t-s)^{2\beta} g(s) ds \\ & + 4C_2\ell_f^2 e^{2\lambda\tau} \int_0^t (t-s)^{2\beta} e^{2\lambda(s-\tau)}|\varphi(s-\tau)|^2 ds \\ & + 4C_2\ell_f^p e^{2\lambda\tau} \int_0^t (t-s)^{2\beta} g(s-\tau) ds \\ \leq & 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2\|\varphi\|_{2,\lambda}^2 e^{2\lambda\tau}\frac{t^{2\beta+1}}{2\beta+1} + 4C_2\ell_f^2 \int_0^t (t-s)^{2\beta} g(s) ds \\ & + 4C_2\ell_f^2 e^{2\lambda\tau} \int_0^t (t-s)^{2\beta} g(s-\tau) ds. \end{aligned}$$

Define $g^*(t) = \sup_{\vartheta\in[-\tau,0]} g(t+\vartheta)$ to obtain

$$\begin{aligned} g^*(t) \leq & 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2\|\varphi\|_{2,\lambda}^2 e^{2\lambda\tau}\frac{t^{2\beta+1}}{2\beta+1} + 4C_2\ell_f^2\int_0^t (t-s)^{2\beta}g^*(s)ds \\ & + 4C_2\ell_f^2 e^{2\lambda\tau}\int_0^t (t-s)^{2\beta}g^*(s)ds \\ \leq & 2|\varphi(0)|^2 e^{2\lambda t} + 4C_2\ell_f^2\mathfrak{M}e^{2\lambda\tau}\frac{t^{2\beta+1}}{2\beta+1} + 4C_2\ell_f^2(1+e^{2\lambda\tau})\int_0^t (t-s)^{2\beta}g^*(s)ds. \end{aligned}$$

Applying Gronwall's inequality of Lemma 3.6, one gets

$$g(t) \leq g^*(t) \leq \left[2|\varphi(0)|^2 e^{2\lambda t} + 4\mathfrak{M}C_2\ell_f^2 e^{2\lambda\tau}\frac{t^{2\beta+1}}{2\beta+1}\right] E_{2\beta+1}\left[4C_2\ell_f^2(1+e^{2\lambda\tau})\Gamma(2\beta+1)t^{2\beta+1}\right],$$

and consequently,

$$\mathbf{E}|x(t)|^2 \leq \left[2|\varphi(0)|^2 + 4\mathfrak{M}C_2\ell_f^2 e^{-2\lambda(t-\tau)}\frac{t^{2\beta+1}}{2\beta+1}\right] E_{2\beta+1}\left[4C_2\ell_f^2(1+e^{2\lambda\tau})\Gamma(2\beta+1)t^{2\beta+1}\right].$$

□

3.3. **Numerical Example.** Define $\alpha = 2\beta + 1$ for $\beta \in (-\frac{1}{2}, \frac{1}{2})$. Take $\beta = -\frac{1}{3}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{3}$ to obtain $\alpha = \frac{1}{3}, \frac{1}{2}, 1, \frac{3}{2}, \frac{5}{2}$. One considers the following example to illustrate the theoretical results.

Example 3.9. Let $\varphi(t) = |\cos(t)|$, and $f(t, x, y) = \sin(t + x + y)$ in Eq. (1.1). Thus, $\varphi(0) = 1$ and $\ell_f = 1$. For simplicity, take $C_2 = \frac{1}{2}, \mathfrak{M} = 2, \lambda = 1$ and $\tau = 1$. Then the solution growth estimate becomes

$$\mathbf{E}|x(t)|^2 \leq \left[2 + \frac{4}{\alpha} t^\alpha e^{-2(t-1)}\right] E_\alpha \left[4(1 + e^2)\Gamma(\alpha) t^\alpha\right].$$

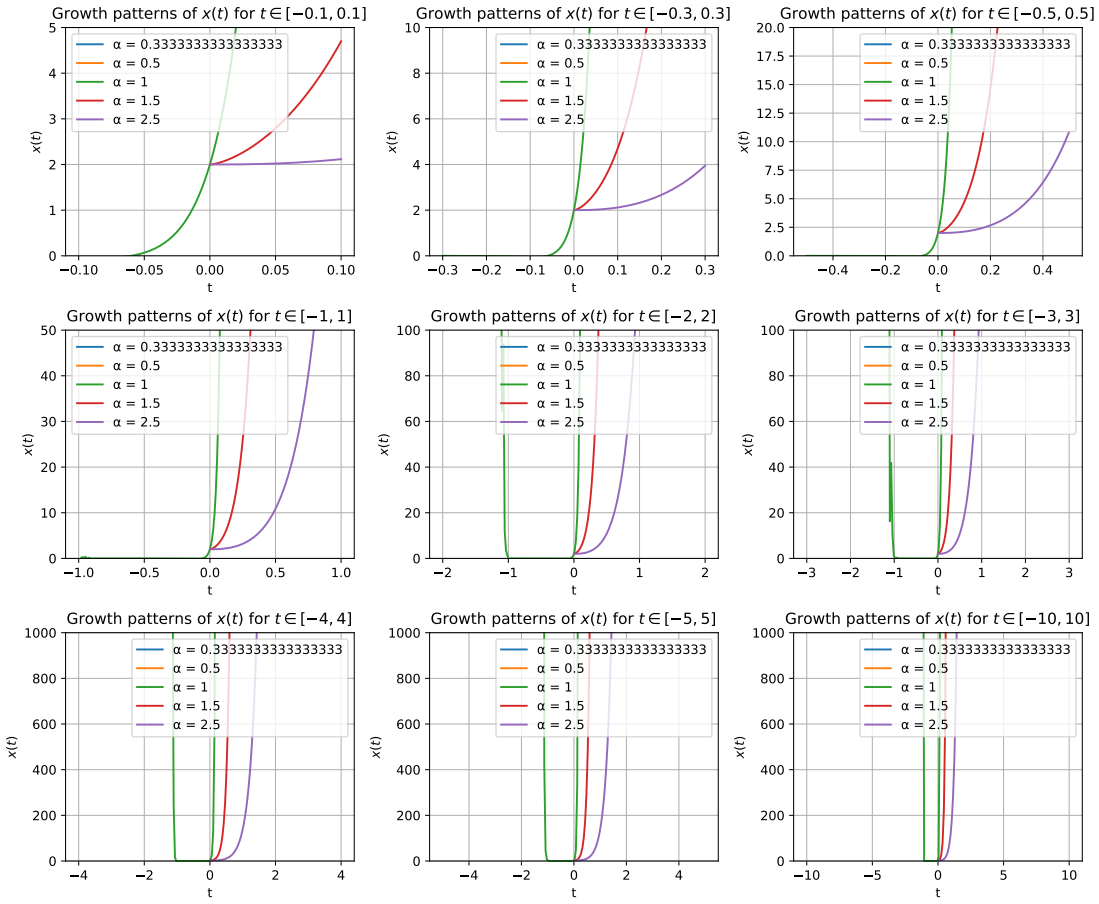


Fig. 1. Growth Behaviour of the solution for values of t and parameter $\alpha = 2\beta + 1$.

Next, one provides an interpretation of the plots presented in Figure 1 above and discuss their corresponding implications.

3.4. **Discussion of the result.** The numerical results reveal a distinct difference between the behaviour of the upper bound on the negative- and positive-time intervals. The plots illustrate the growth behavior of $x(t)$ for different values of the parameter α over increasingly large time intervals. For $-0.1 \leq t \leq 0$, the upper

bound exhibits a relatively smooth and moderate increase as t approaches zero. In contrast, for $0 \leq t \leq 0.1$, the results show a strong dependence of the growth rate on α. In particular, the curves corresponding to $\alpha = 1$ and $\alpha = \frac{3}{2}$ (that is, $\beta = 0$ and $\beta = \frac{1}{4}$ respectively) exhibit rapid growth near the origin, whereas the curve for $\alpha = \frac{5}{2}$ corresponding to $\beta = \frac{3}{4}$ initially increases more gradually before attaining larger values as time increases. As the observation interval expands, the solution values reach increasingly large magnitudes, highlighting the combined influence of the parameter α and the time interval in determining the observed growth patterns. This means that the parameter affects not only the magnitude of the solution but also how rapidly that magnitude changes with time. These observations highlight the influence of the parameter α on the solution. The widening intervals also reveal sharp transitions on the negative-time side that are not as apparent in the smaller intervals. Furthermore, for larger intervals, the results reveal markedly different behaviours on the negative- and positive-time intervals. For $t < 0$, corresponding to the history interval, the upper bound remains essentially negligible over a substantial portion of the interval, followed by pronounced localized spikes, after which the bound rapidly decays toward zero. In contrast, for $t > 0$, the upper bound exhibits a systematic and increasingly uniform rapid growth. The parameter α has a significant influence on this behaviour: smaller values of α, particularly $\alpha = 1$, lead to a much earlier onset of rapid growth, whereas larger values, such as $\alpha = \frac{3}{2}$ and $\alpha = \frac{5}{2}$, delay the rapid increase and result in comparatively slower growth over the same time interval. Thus, the graphs demonstrate a strong parameter-dependent transition from the prescribed history behaviour for $t < 0$ to the rapidly growing solution period for $t > 0$. In summary, the numerical results illustrate a transition from nonuniform behaviour on the history interval to pronounced growth on the positive-time domain. They also demonstrate that the parameter α influences both the local growth pattern and the magnitude attained over extended time intervals.

For a better visualization of the exponential growth behaviour of the solution, the graphs are plotted over the positive-time interval only, as shown in Figure 2 below.

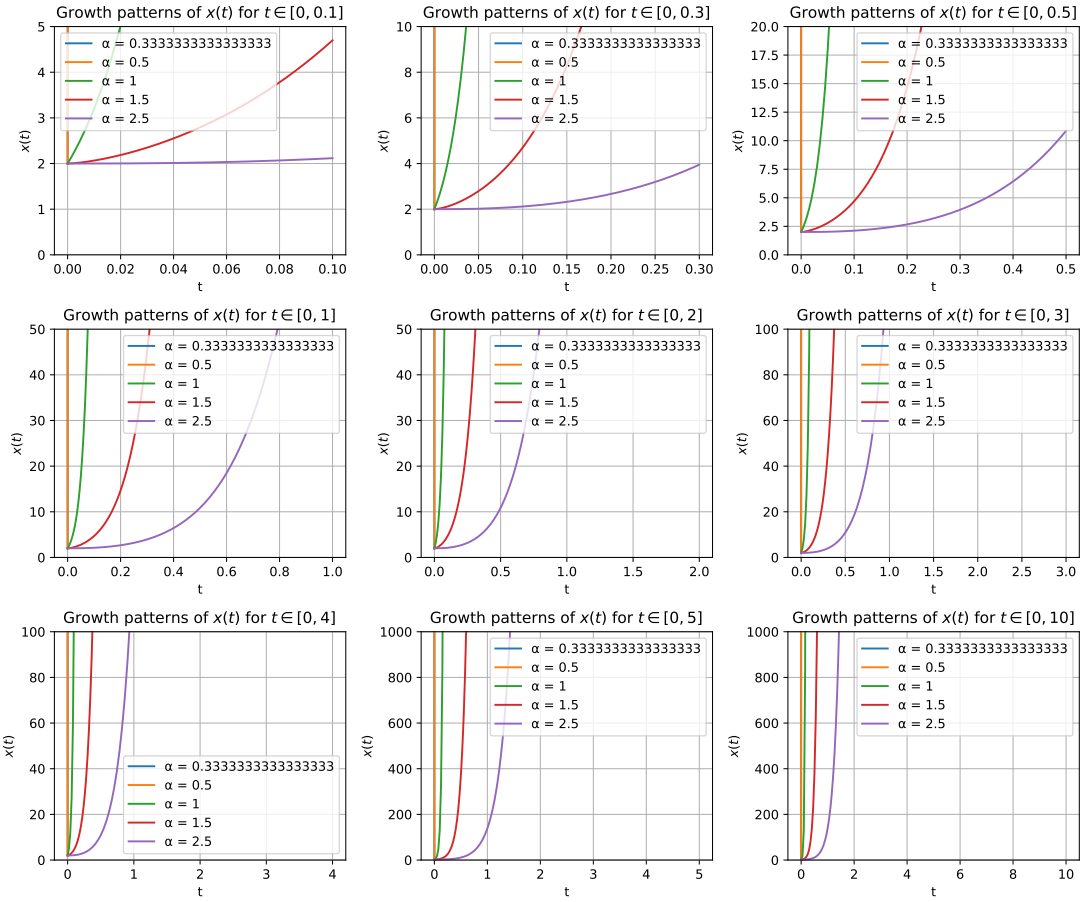


Fig. 2. Growth Behaviours of the solution for values of t and parameter $\alpha = 2\beta + 1$.

3.5. **Potential Applications.** The numerical results demonstrate the potential applications of the proposed fractional delay equation to dynamical systems exhibiting memory, delayed responses, and random fluctuations. Such characteristics arise naturally in a variety of applied real-world settings, including population dynamics, epidemiological models with incubation or reporting delays, viscoelastic materials with hereditary effects, biological and biochemical processes with maturation delays, financial systems with long-memory behaviour, and engineering systems involving delayed feedback and stochastic perturbations. In these applications, the history function may represent the pre-initial state of the system, while the delay accounts for the finite time required for past states to influence the current dynamics. The fractional parameter provides an additional mechanism for describing memory effects and can substantially influence the growth behaviour of the solution, as illustrated by the above numerical plots.

4. Conclusions

The author considered a stochastic Volterra delay equation driven by a Volterra–Gamma process. Such processes have important applications in the modelling of turbulent flows and environmental risk factors in energy finance, particularly in the context of wind energy. The main results establish the existence and uniqueness of a random solution and derive exponential growth bounds for the solution. Moreover, illustrative examples are provided to support and demonstrate the theoretical results. Future research may focus on establishing continuity properties, stability criteria, and long-term asymptotic behaviour of the solutions. One could also explore the equation for variable dependent delay.

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