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DIAMETRAL METRIC DIMENSION OF GENERALIZED CORONA GRAPHS

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Abstract. This study investigates the diametral metric dimension of generalized corona graphs. The diametral metric dimension requires the resolving set to contain all diametral vertices. To determine the diametral metric dimension of a generalized corona graph, one must first determine the relevant vertex distances and identify the diametral set in the graph and the graph's diametral set. The results show that the diametral metric dimension of the generalized corona graph depends on the diameter of the main graph and the metric dimensions of the graphs in the sequence. These concepts provide theoretical insights into resolving structures in the planning infrastructure.

Keywords. Metric dimension, diametral metric dimension, diametral set, corona graph, planning infrastructure
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1. Introduction

Graph theory is a discipline in discrete mathematics that studies mathematical structures consisting of vertices and edges, where edges represent relationships between pairs of vertices. Since the work of Leonhard Euler on the Seven Bridges of Königsberg problem, graph theory has developed into an important area of mathematical research with applications in transportation, communication networks, optimization, and network design [1].

Among the fundamental distance-based concepts in graph theory are distance, eccentricity, diameter, and radius. For two vertices $v, w \in V(G)$, the distance between v and w, denoted by $d_G(v, w)$ or simply by $d(v, w)$, is defined as the length of a shortest path between v and w in G. Based on the concept of distance, the eccentricity of a vertex, as well as the diameter and radius of a graph, can be defined. These concepts have led to the development of various graph parameters, one of which is the metric dimension. The metric dimension was introduced independently by Slater in 1975 and by Harary and Melter in 1976 [9, 5]. Since

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then, metric dimension has been extensively studied for various graph classes and graph operations.

The significance of this study lies in the development of a concept in graph theory that not only enriches the study of metric dimension but also contributes to network structure analysis. The development of the diametral metric dimension is expected to broaden the theoretical foundation for analyzing network structures, particularly by considering the distance structure and diameter of a graph. Furthermore, this study may provide a basis for further research on distance-based graph parameters and their extensions to various graph classes and graph operations. These developments may also support future investigations into the potential applications of such parameters in network monitoring and the analysis of critical locations within networks.

A related direction in the study of the metric dimension is obtained by requiring a resolving set that contains the central set. This is referred to as the central metric dimension. Susilowati et al. have investigated several variants of metric dimension, including rooted product graphs and edge coronation graphs [10, 11]. Similarly, the diametral metric dimension requires a resolving set to contain all diametral vertices, namely, the vertices whose eccentricity is equal to the diameter of the graph.

Graph operations provide an object for investigating how structural modifications affect distance-based parameters. One such operation is the corona product, introduced by Frucht and Harary [3]. The generalized corona extends this operation by allowing different graphs to be attached to different vertices of the main graph [8]. Consequently, the resulting distance structure, eccentricities, diameter, and diametral vertices may depend on both the main graph and the attached graphs. This makes the generalized corona an interesting graph operation for studying the diametral metric dimension.

Motivated by these observations, this paper investigates the diametral metric dimension of generalized corona graphs $G \odot (H^1, H^2, \ldots, H^n)$. The main graph G is considered from four fundamental graph families, namely the path, cycle, complete, and star graphs. For each graph family, we characterize the diametral vertices of the resulting generalized corona graph and determine its diametral metric dimension. The results provide further insight into the relationship between graph operations, the position of diametral vertices, and resolving sets.

2. Preliminaries

In this paper, all graphs are assumed to be connected. Let $G = (V(G), E(G))$ be a graph. The distance between two vertices $u, v \in V(G)$, denoted by $d_G(u, v)$, is the length of a shortest u–v path in G.

For a vertex $v \in V(G)$, the eccentricity of v is defined by

$$\mathrm{e}_G(v) = \max_{u \in V(G)} d_G(u, v). \tag{2.1}$$

The diameter and radius of G are respectively defined by

$$\mathrm{diam}(G) = \max_{v \in V(G)} \mathrm{e}_G(v) \tag{2.2}$$

and

$$\mathrm{rad}(G) = \min_{v \in V(G)} \mathrm{e}_G(v). \tag{2.3}$$

The diameters of several fundamental graph families are well known.

Lemma 2.1. [2, 4] *Let G be a connected graph of order n. The following properties hold:*

(1) $\mathrm{diam}(P_n) = n - 1$ *for* $n \geq 1$;
(2) $\mathrm{diam}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 3$;
(3) $\mathrm{diam}(K_n) = 1$ *for* $n \geq 2$;
(4) Define $S_n \cong K_{1,n-1}$, $diam(S_n) = 2$ *for* $n \geq 3$.

Definition 2.2. [7] Diametral vertices are vertices whose eccentricity is equal to the diameter of the graph.

The definition of metric dimension of the graph G can be seen in [2].
The following result gives the metric dimension of $K_1 + C_n$ and $K_1 + P_n$.

Lemma 2.3. [6]

(1) *For* $n > 3$ *and* $n \notin \{3, 6\}$, $dim(K_1 + C_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$;
(2) *For* $n > 1$ *and* $n \notin \{1, 2, 3, 6\}$, $dim(K_1 + P_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$.

We next recall the generalized corona operation.

Definition 2.4. [8] Let G be a graph of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ be a sequence of graphs. The generalized corona graph, denoted by

$$G \odot \mathcal{H} = G \odot (H^1, H^2, \ldots, H^n), \tag{2.4}$$

is the graph obtained by taking one copy of G and one copy of each graph H^i, and joining every vertex v_i of G to every vertex of H^i for $i = 1, 2, \ldots, n$.

For the generalized corona graph, the vertex labeling is given by $V(G \odot \mathcal{H}) = V(G^0) \cup \bigcup_{i=1}^{n} V(H_i^i)$, where $V(G^0) = \{v_i^0 \in V(G \odot H) \mid v_i \in V(G)\}$ and $V(H_i^i) = \{u_{ij}^i \mid u_{ij} \in V(H^i),\ i = 1, 2, 3, \ldots, n\}$. Here, G^0 is referred to as the main graph, while each H^i is referred to as a branch graph.

3. Main results

In this section, we determine the diametral metric dimension of the generalized corona graph $G \odot \mathcal{H}$ where G is a connected graph of order n and $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ is a sequence of nontrivial connected graphs. In this section, G^0 denotes the copy of the main graph G in $G \odot \mathcal{H}$, while H_i^i denotes the copy of H^i attached to $v_i \in V(G)$.

The first step in this study is to define the concept of the diametral metric dimension.

Definition 3.1. A diametral set is a set whose elements are all diametral vertices. The diametral set of a graph G is denoted by $D(G)$.

Definition 3.2. Let $W \subseteq V(G)$ be a resolving set. W is called the diametral resolving set if W contains the diametral set $D(G)$. A diametral resolving set of minimum cardinality is called a diametral basis. The cardinality of a diametral basis in G is called the diametral metric dimension, denoted by $\dim_{\text{diam}}(G)$.

Lemma 3.3. *If every set with cardinality k is not a diametral resolving set, then any set with a cardinality less than k is also not a diametral resolving set.*

Proof. By definition, if every set with cardinality k is not a diametral resolving set of G. Therefore, no set with cardinality less than k can be a diametral resolving set. □

The next lemma presents the diametral set of the corona operation $G \odot \mathcal{H}$.

Lemma 3.4. *The diametral set of $G \odot \mathcal{H}$ is*

$$D(G \odot \mathcal{H}) = \{u_{ij}^i \in V(H_i^i) \mid v_i \in D(G)\}. \tag{3.1}$$

Proof. Let $V(G) = \{v_i \mid i = 1, 2, 3, \ldots, n\}$ and $\mathcal{H}$ be a sequence of n graphs, namely $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ with $V(H^i) = \{u_{ij} \mid j = 1, 2, 3, \ldots, |V(H^i)|\}$, $i = 1, 2, 3, \ldots, n$. The vertex set of the graph $G \odot \mathcal{H}$ is $V(G \odot \mathcal{H}) = V(G^0) \cup_{i=1}^{n} V(H_i^i)$ with $V(G^0) = \{v_i^0 \in V(G \odot \mathcal{H}) \mid v_i \in V(G)\}$ and $V(H_i^i) = \{u_{ij}^i \mid u_{ij} \in V(H^i);\ i = 1, 2, 3, \ldots, n\}$. For any chosen $v_i \in D(G)$, then $e_G(v_i) = \max\{e_G(v) \mid v \in V(G)\}$. For any chosen $v_i^0 \in V(G \odot \mathcal{H})$ for $v_i \in D(G)$, then for every $u_{ij}^i \in V(H_i^i)$, it follows that $e_{G\odot\mathcal{H}}(u_{ij}^i) = e_{G\odot\mathcal{H}}(v_i^0) + 2$. It is known that $e_{G\odot\mathcal{H}}(u) \leq e_{G\odot\mathcal{H}}(u_{ij}^i)$, for every $u \in V(G \odot \mathcal{H})$. Therefore, $e_{G\odot\mathcal{H}}(u_{ij}^i) = \max\{e_{G\odot\mathcal{H}}(u) \mid u \in V(G \odot \mathcal{H})\}$. Thus, $D(G \odot \mathcal{H}) = \{u_{ij}^i \in V(H_i^i) \mid v_i \in D(G)\}$. □

Lemma 3.5. *Let $W \subseteq V(G)$. If $x \in W$ or $y \in W$ then $r(x \mid W) \neq r(y \mid W)$.*

Proof. Let W be a diametral resolving set of graph G. Taking any two distinct vertices $x, y \in W$, there are two possibilities, namely (1) $x, y \in W$; (2) $x \in W$ and $y \notin W$.

(1) $x, y \in W$, based on [12] it follows that $r(x \mid W) \neq r(y \mid W)$.
(2) Without loss of generality, take any $x \in W$ and $y \notin W$. Then, in $r(x \mid W)$ there exists an element 0 corresponding to x, whereas in $r(y \mid W)$ there is no element 0 since $y \notin W$. Thus, $r(x \mid W) \neq r(y \mid W)$.

Therefore, it is proved that for $x \in W$ or $y \in W$, $r(x \mid W) \neq r(y \mid W)$. □

We first consider the case where the main graph is a path.

3.1. G is a Path Graph.

In this section, we determine the diametral metric dimension of $G \odot \mathcal{H}$ where G is a path of order at least 3.

Theorem 3.6. *The diametral metric dimension of $P_n \odot \mathcal{H}$ is*

$$\dim_{\text{diam}}(P_n \odot \mathcal{H}) = |D(P_n \odot \mathcal{H})| + \sum_{i=2}^{n-1} \dim(H^i). \tag{3.2}$$

for $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ is a sequence of star graphs or complete graphs.

Proof. Let $V(P_n) = \{v_i \mid i = 1, 2, 3, \ldots, n\}$ and $\mathcal{H}$ be a sequence of n star graphs or complete graphs, namely $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ with $V(H^i) = \{u_{ij} \mid j = 1, 2, 3, \ldots, |V(H^i)|\}, \quad i = 1, 2, 3, \ldots, n$. For H^i is a star graph, there is $u_{i1} \in H^i$ such that $deg(u_{i1}) = m-1$. The vertex set of the graph $P_n \odot \mathcal{H}$ is given by $V(P_n \odot \mathcal{H}) = V(P_n^0) \cup \bigcup_{i=1}^{n} V(H_i^i)$, where $V(P_n^0) = \{v_i^0 \in V(P_n \odot \mathcal{H}) \mid v_i \in V(P_n)\}$ and $V(H_i^i) = \{u_{ij}^i \mid u_{ij} \in V(H^i)\}, \quad i = 1, 2, 3, \ldots, n$. Let B^i be a basis of graph H^i, so that for $i = 1, 2, 3, \ldots, n$,

$$B^i = \begin{cases} \{u_{i2}, u_{i3}, \ldots, u_{i(m-2)}, u_{i(m-1)}\}, & H^i \cong S_m, \\ \{u_{i1}, u_{i2}, \ldots, u_{i(q-2)}, u_{i(q-1)}\}, & H^i \cong K_q. \end{cases}$$

Let $B_i^i = \{u_{ij}^i \in V(P_n \odot \mathcal{H}) \mid u_{ij} \in B^i\}, \quad i = 2, 3, \ldots, n-1$. We define $W = \{u_{ij}^i \in V(H_i^i) \mid v_i \in D(P_n)\} \cup \bigcup_{i=2}^{n-1} \{u_{ij}^i \in V(P_n \odot \mathcal{H}) \mid u_{ij} \in B^i\}$. Thus, $|W| = |D(P_n \odot \mathcal{H})| + \sum_{i=2}^{n-1} \dim(H^i)$. For any distinct vertices $x, y \in V(P_n \odot \mathcal{H})$ where $x \neq y$, there are three possible cases: (i) $x, y \in W$; (ii) $x \in W$ and $y \in V(P_n \odot \mathcal{H}) \setminus W$; (iii) $x, y \in V(P_n \odot \mathcal{H}) \setminus W$. For cases (1) and (2), by Lemma 3.5, it is proven that $r(x \mid W) \neq r(y \mid W)$. For case (3), there are four subcases.

Subcase 3.1: $u_{ij}^i, u_{ik}^i \in V(H_i^i)$.

Since $r(u_{ij}^i \mid B_i^i) \neq r(u_{ik}^i \mid B_i^i)$ and $B_i^i \subseteq W$, then $r(u_{ij}^i \mid W) \neq r(u_{ik}^i \mid W)$.

Subcase 3.2: $v_x^0, v_y^0 \in V(P_n^0)$ where $x \neq y$.

Since $v_x^0, v_y^0 \in V(P_n^0)$ is a path, $d(v_x^0, v_y^0) = s, \quad 1 \leq s \leq n-1$. For any chosen $u_{xj}^x \in W$. We know that $d(v_x^0, u_{xj}^x) = 1$, for $j = 1, 2, \ldots, |V(H^i)|$, so $d(v_y^0, u_{xj}^x) = d(v_x^0, v_y^0) + d(v_x^0, u_{xj}^x) = s+1$. Because $d(v_y^0, u_{xj}^x) > d(v_x^0, u_{xj}^x)$, it holds that $r(v_x^0 \mid W) \neq r(v_y^0 \mid W)$.

Subcase 3.3: $v_x^0 \in V(P_n^0)$ and $u_{ij}^i \in V(H_i^i)$.

There are two possibilities, namely, $x = i$ or $x \neq i$.

If $x = i$, let $u_{pj}^p \in W$. Then $d(v_p^0, u_{pj}^p) = 1 = d(v_i^0, u_{ij}^i)$. Since $d(v_p^0, v_i^0) = s$, where $1 \leq s \leq n-1$, we have $d(v_i^0, u_{pj}^p) = d(v_p^0, u_{pj}^p) + d(v_p^0, v_i^0) = 1 + s$. Furthermore, $d(u_{ij}^i, u_{pj}^p) = d(v_p^0, u_{pj}^p) + d(v_i^0, u_{ij}^i) = s + 1 + 1 = s + 2$. Thus, $d(v_i^0, u_{pj}^p) < d(u_{ij}^i, u_{pj}^p)$.

If $x \neq i$, let $u_{ik}^i \in W$. Then $d(u_{ij}^i, v_i^0) = d(u_{ik}^i, v_i^0) = 1$ and $d(u_{ij}^i, u_{ik}^i) \leq 2$. Since $d(v_x^0, v_i^0) = s$, where $1 \leq s \leq n-1$, we have $d(v_x^0, u_{ik}^i) = d(v_x^0, v_i^0) + d(u_{ik}^i, v_i^0) = s + 1$. Thus, $d(u_{ij}^i, u_{ik}^i) < d(v_x^0, u_{ik}^i)$.

Therefore, for $v_x^0, u_{ij}^i \in V(P_n \odot \mathcal{H}) \setminus W$, we have $r(v_x^0 \mid W) \neq r(u_{ij}^i \mid W)$.

Subcase 3.4: $u_{xj}^x \in V(H_x^x)$ and $u_{yj}^y \in V(H_y^y)$ for $x \neq y$.

Since $d(u_{xj}^x, v_x^0) = 1, \quad d(v_x^0, v_y^0) = s, \quad 1 \leq s \leq n-1$. For any chosen $u_{yk}^y \in W$, we have $d(u_{yj}^y, u_{yk}^y) \leq 2$ and $d(v_y^0, u_{yj}^y) = d(v_y^0, u_{yk}^y) = 1$. We know that $d(u_{xj}^x, v_y^0) = d(u_{xj}^x, v_x^0) + d(v_x^0, v_y^0) = 1 + s$ and $d(u_{xj}^x, u_{yk}^y) = d(u_{xj}^x, v_y^0) + d(v_y^0, u_{yk}^y) = (1 + s) + 1 = 2 + s$. Thus, $d(u_{yj}^y, u_{yk}^y) < d(u_{xj}^x, u_{yk}^y)$. Hence, for $u_{xj}^x, u_{yj}^y \in V(P_n \odot \mathcal{H}) \setminus W$, it follows that $r(u_{xj}^x \mid W) \neq r(u_{yj}^y \mid W)$.

Next, to prove that W is a diametral resolving set with minimum cardinality, let $T \subseteq V(P_n \odot \mathcal{H})$ be a set containing the diametral set such that $|T| < |W|$. Let $|T| = |W| - 1$. Then there is an index i such that at most

$|B_i^i| - 1$ vertices in H_i^i are elements of T. Consequently, there are two vertices $u_{ij}^i, u_{ik}^i \in V(H_i^i), \quad u_{ij}^i, u_{ik}^i \notin T$, such that $r(u_{ij}^i \mid B_i^i) = r(u_{ik}^i \mid B_i^i)$. Moreover, for every vertex $z \in V(P_n \odot \mathcal{H}) \setminus V(H_i^i)$, every path from z to a vertex of H_i^i must pass through v_i^0. Since both u_{ij}^i and u_{ik}^i are adjacent to v_i^0, we have $d(z, u_{ij}^i) = d(z, v_i^0) + 1 = d(z, u_{ik}^i)$. Therefore, no vertex outside H_i^i can distinguish u_{ij}^i and u_{ik}^i. Consequently, T is not a diametral resolving set. By Lemma 3.3, W is the diametral resolving set with minimum cardinality. Thus, $\dim_{\text{diam}}(P_n \odot \mathcal{H}) = |D(P_n \odot \mathcal{H})| + \sum_{i=2}^{n-1} \dim(H^i)$. □

Theorem 3.7. *The diametral metric dimension of $P_n \odot \mathcal{H}$ is*

$$\dim_{\text{diam}}(P_n \odot \mathcal{H}) = |D(P_n \odot \mathcal{H})| + \sum_{i=2}^{n-1} \dim(K_1 + H^i). \tag{3.3}$$

for $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ is a sequence of path graphs or cycle graphs for $|V(H^i)| > 6$, then

Proof. Let $V(P_n) = \{v_i \mid i = 1, 2, 3, \ldots, n\}$ and $\mathcal{H}$ be a sequence of n path graphs or cycle graphs, namely $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ with $V(H^i) = \{u_{ij} \mid j = 1, 2, 3, \ldots, |V(H^i)|\}, \quad i = 1, 2, 3, \ldots, n$. The vertex set of the graph $P_n \odot \mathcal{H}$ is given by $V(P_n \odot \mathcal{H}) = V(P_n^0) \cup \bigcup_{i=1}^{n} V(H_i^i)$, where $V(P_n^0) = \{v_i^0 \in V(P_n \odot \mathcal{H}) \mid v_i \in V(P_n)\}$ and $V(H_i^i) = \{u_{ij}^i \mid u_{ij} \in V(H^i)\}, \quad i = 1, 2, 3, \ldots, n$. Since each H^i is either a path graph or a cycle graph, by Lemma 2.3, a metric basis of $K_1 + H^i$ can be chosen with cardinality $\dim(K_1 + H^i)$, so that for $i = 1, 2, 3, \ldots, n$,

$$B^i = \begin{cases} \{u_{im_1}, u_{im_2}, \ldots, u_{im_r}\}, & H^i \cong P_m, \\ \{u_{iq_1}, u_{iq_2}, \ldots, u_{iq_s}\}, & H^i \cong C_q, \end{cases}$$

such that $|B^i| = \left\lfloor \frac{2m_r+2}{5} \right\rfloor$ or $|B^i| = \left\lfloor \frac{2q_s+2}{5} \right\rfloor$. Let $B_i^i = \{u_{ij}^i \in V(P_n \odot \mathcal{H}) \mid u_{ij} \in B^i\}, \quad i = 2, 3, \ldots, n-1$. Define $W = \{u_{ij}^i \in V(H_i^i) \mid v_i \in D(P_n)\} \cup \bigcup_{i=2}^{n-1} \{u_{ij}^i \in V(P_n \odot \mathcal{H}) \mid u_{ij} \in B^i\}$, so that $|W| = |D(P_n \odot \mathcal{H})| + \sum_{i=2}^{n-1} \dim(K_1 + H^i)$. The next step, we can prove that any two vertices in $P_n \odot \mathcal{H}$ have the distinct representation to W. We have three cases for two possibilities of vertices using the same logic with the proof of Theorem 3.6. Next, by Lemma 3.3, we can prove $\dim_{\text{diam}}(P_n \odot \mathcal{H}) = |D(P_n \odot \mathcal{H})| + \sum_{i=2}^{n-1} \dim(K_1 + H^i)$. □

In the next subsection, we discuss the case where the main graph is a star graph below.

3.2. G is a Star Graph.

In this part, we determine the diametral metric dimension of $G \odot \mathcal{H}$ where G is a star of order at least 3. Let $v_1 \in S_n$ and $\deg(v_1) = n - 1$.

Theorem 3.8. *The diametral metric dimension of $S_n \odot \mathcal{H}$ is*

$$\dim_{\text{diam}}(S_n \odot \mathcal{H}) = |D(S_n \odot \mathcal{H})| + \dim(H^1). \tag{3.4}$$

for $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ is a sequence of star graphs or complete graphs.

Proof. Let $V(S_n) = \{v_i \mid i = 1, 2, 3, \ldots, n\}$ and $\mathcal{H}$ be a sequence of n star graphs or complete graphs, namely $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ with $V(H^i) = \{u_{ij} \mid j = 1, 2, 3, \ldots, |V(H^i)|\}, \quad i = 1, 2, 3, \ldots, n$. For H^i is a star graph, there is $u_{i1} \in H^i$ such that $deg(u_{i1}) = m-1$. The vertex set of the graph $S_n \odot \mathcal{H}$ is given by $V(S_n \odot \mathcal{H}) = V(S_n^0) \cup \bigcup_{i=1}^{n} V(H_i^i)$, where $V(S_n^0) = \{v_i^0 \in V(S_n \odot \mathcal{H}) \mid v_i \in V(S_n)\}$ and $V(H_i^i) = \{u_{ij}^i \mid u_{ij} \in V(H^i)\}, \quad i = 1, 2, 3, \ldots, n$. Let B^1 be a metric basis of H^1, so

$$B^1 = \begin{cases} \{u_{12}, u_{13}, \ldots, u_{1(m-2)}, u_{1(m-1)}\}, & H^1 \cong S_m, \\ \{u_{11}, u_{12}, \ldots, u_{1(q-2)}, u_{1(q-1)}\}, & H^1 \cong K_q. \end{cases}$$

Let $B_1^1 = \{u_{1j}^1 \in V(S_n \odot \mathcal{H}) \mid u_{1j} \in B^1\}$. Define $W = \{u_{ij}^i \in V(H_i^i) \mid v_i \in D(S_n)\} \cup \{u_{1j}^1 \in V(S_n \odot \mathcal{H}) \mid u_{1j} \in B^1\}$, so that $|W| = |D(S_n \odot \mathcal{H})| + \dim(H^1)$. For any distinct vertices $x, y \in V(S_n \odot \mathcal{H})$ where $x \neq y$, there are three possible cases: (i) $x, y \in W$; (ii) $x \in W$ and $y \in V(S_n \odot \mathcal{H}) \setminus W$; (iii) $x, y \in V(S_n \odot \mathcal{H}) \setminus W$. For cases (1) and (2), by Lemma 3.5, it is proven that $r(x \mid W) \neq r(y \mid W)$. For case (3), there are three subcases.

Subcase 3.1: $u_{1j}^1, u_{1k}^1 \in V(H_1^1)$.

Since $r(u_{1j}^1 \mid B_1^1) \neq r(u_{1k}^1 \mid B_1^1)$ and $B_1^1 \subseteq W$, then $r(u_{1j}^1 \mid W) \neq r(u_{1k}^1 \mid W)$.

Subcase 3.2: $v_x^0, v_y^0 \in V(S_n^0)$ where $x \neq y$.

Since $v_x^0, v_y^0 \in V(S_n^0)$, $d(v_x^0, v_y^0) = s, \quad s \in \{1, 2\}$. For any chosen $u_{xj}^x \in W$. We know that $d(v_x^0, u_{xj}^x) = 1$, for $j = 1, 2, \ldots, |V(H^i)|$. Therefore, $d(v_y^0, u_{xj}^x) = d(v_x^0, v_y^0) + d(v_x^0, u_{xj}^x) = s + 1$. Because $d(v_y^0, u_{xj}^x) > d(v_x^0, u_{xj}^x)$, it holds that $r(v_x^0 \mid W) \neq r(v_y^0 \mid W)$.

Subcase 3.3: $v_x^0 \in V(S_n^0)$ and $u_{1j}^1 \in V(H_1^1)$.

If $x = 1$, For any chosen $u_{zk}^z \in V(H_z^z) \subseteq W$, for $z \neq 1$. Then, $d(v_1^0, u_{zk}^z) = d(v_1^0, v_z^0) + d(v_z^0, u_{zk}^z) = 1 + 1 = 2$, Whereas $d(u_{1j}^1, u_{zk}^z) = d(u_{1j}^1, v_1^0) + d(v_1^0, v_z^0) + d(v_z^0, u_{zk}^z) = 1 + 1 + 1 = 3$. Hence, $d(v_1^0, u_{zk}^z) \neq d(u_{1j}^1, u_{zk}^z)$.

If $x \neq 1$, For any chosen $u_{xk}^x \in V(H_x^x) \subseteq W$. Then, $d(v_x^0, u_{xk}^x) = 1$, whereas $d(u_{1j}^1, u_{xk}^x) = d(u_{1j}^1, v_1^0) + d(v_1^0, v_x^0) + d(v_x^0, u_{xk}^x) = 1 + 1 + 1 = 3$. Therefore, $d(v_x^0, u_{xk}^x) \neq d(u_{1j}^1, u_{xk}^x)$. Thus, $r(v_x^0 \mid W) \neq r(u_{1j}^1 \mid W)$.

Next, to prove that W is a diametral resolving set with minimum cardinality, let $T \subseteq V(S_n \odot \mathcal{H})$ be a set containing the diametral set such that $|T| < |W|$. Let $|T| = |W| - 1$. Then there is an index $i = 1$ such that at most $|B_1^1| - 1$ vertices in H_1^1 are elements of T. Consequently, there are two vertices $u_{1j}^1, u_{1k}^1 \in V(H_1^1), \quad u_{1j}^1, u_{1k}^1 \notin T$, such that $r(u_{1j}^1 \mid B_1^1) = r(u_{1k}^1 \mid B_1^1)$. Moreover, for every vertex $z \in V(S_n \odot \mathcal{H}) \setminus V(H_1^1)$, every path from z to a vertex of H_1^1 must pass through v_1^0. Since both u_{1j}^1 and u_{1k}^1 are adjacent to v_1^0, we have $d(z, u_{1j}^1) = d(z, v_1^0) + 1 = d(z, u_{1k}^1)$. Therefore, no vertex outside H_1^1 can distinguish u_{1j}^1 and u_{1k}^1. Consequently, T is not a diametral resolving set. By Lemma 3.3, W is the diametral resolving set with minimum cardinality. Thus, $\dim_{\text{diam}}(S_n \odot \mathcal{H}) = |D(S_n \odot \mathcal{H})| + \dim(H^1)$. □

Next, the subcase of the main graph from specific graphs are the cycle graph and the complete graph.

Theorem 3.9. *The diametral metric dimension of* $S_n \odot \mathcal{H}$ *is*

$$\dim_{\text{diam}}(S_n \odot \mathcal{H}) = |D(S_n \odot \mathcal{H})| + \dim(K_1 + H^1). \tag{3.5}$$

for $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ *is a sequence of path graphs or cycle graphs for* $|V(H^i)| > 6$.

Proof. Let $V(S_n) = \{v_i \mid i = 1, 2, 3, \ldots, n\}$ and $\mathcal{H}$ be a sequence of n path graphs or cycle graphs, namely $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ with $V(H^i) = \{u_{ij} \mid j = 1, 2, 3, \ldots, |V(H^i)|\}, \quad i = 1, 2, 3, \ldots, n$. The vertex set of the graph $S_n \odot \mathcal{H}$ is given by $V(S_n \odot \mathcal{H}) = V(S_n^0) \cup \bigcup_{i=1}^{n} V(H_i^i)$, where $V(S_n^0) = \{v_i^0 \in V(S_n \odot \mathcal{H}) \mid v_i \in V(S_n)\}$ and $V(H_i^i) = \{u_{ij}^i \mid u_{ij} \in V(H^i)\}, \quad i = 1, 2, 3, \ldots, n$. Since H^1 is either a path graph or a cycle graph, by Lemma 2.3, a metric basis of $K_1 + H^1$ can be chosen with cardinality $\dim(K_1 + H^1)$, so

$$B^1 = \begin{cases} \{u_{1m_1}, u_{1m_2}, \ldots, u_{1m_r}\}, & H^1 \cong P_m, \\ \{u_{1q_1}, u_{1q_2}, \ldots, u_{1q_s}\}, & H^1 \cong C_q, \end{cases}$$

such that $|B^1| = \left\lfloor \frac{2m_r+2}{5} \right\rfloor$ or $|B^1| = \left\lfloor \frac{2q_s+2}{5} \right\rfloor$. Let $B_1^1 = \{u_{1j}^1 \in V(S_n \odot \mathcal{H}) \mid u_{1j} \in B^1\}$. Define $W = \{u_{ij}^i \in V(H_i^i) \mid v_i \in D(S_n)\} \cup \{u_{1j}^1 \in V(S_n \odot \mathcal{H}) \mid u_{1j} \in B^1\}$, so that $|W| = |D(S_n \odot \mathcal{H})| + \dim(K_1 + H^1)$. The next step, we can prove that any two vertices in $S_n \odot \mathcal{H}$ have the distinct representation to W. We have three cases for two possibilities of vertices using the same logic as in the proof of Theorem 3.8. Next, by Lemma 3.3, we can prove $\dim_{\text{diam}}(S_n \odot \mathcal{H}) = |D(S_n \odot \mathcal{H})| + \dim(K_1 + H^1)$. □

The final specific graph for the main graph is the case where G is a cycle graph or complete graph.

3.3. G is a Cycle or Complete Graph.

Theorem 3.10. *The diametral metric dimension of* $G \odot \mathcal{H}$ *is*

$$\dim_{\text{diam}}(G \odot \mathcal{H}) = |D(G \odot \mathcal{H})|. \tag{3.6}$$

for G *is a cycle graph or a complete graph.*

Proof. Let $V(G) = \{v_i \mid i = 1, 2, 3, \ldots, n\}$ and H be a sequence of graphs, namely $\mathcal{H} = (H^1, H^2, \ldots, H^n)$, with $V(H^i) = \{u_{ij} \mid j = 1, 2, 3, \ldots, |V(H^i)|\}, \quad i = 1, 2, 3, \ldots, n$. The vertex set of the graph $G \odot \mathcal{H}$ is given by $V(G \odot \mathcal{H}) = V(G^0) \cup \bigcup_{i=1}^{n} V(H_i^i)$, where $V(G^0) = \{v_i^0 \in V(G \odot \mathcal{H}) \mid v_i \in V(G)\}$ and $V(H_i^i) = \{u_{ij}^i \mid u_{ij} \in V(H^i)\}, \quad i = 1, 2, 3, \ldots, n$. Define $W = \{u_{ij}^i \in V(H_i^i) \mid v_i \in D(G)\}$ so that $|W| = |D(G \odot \mathcal{H})|$. For any distinct vertices $x, y \in V(G \odot \mathcal{H})$ where $x \neq y$, there are three possible cases: (i) $x, y \in W$; (ii) $x \in W$ and $y \in V(G \odot \mathcal{H}) \setminus W$; (iii) $x, y \in V(G \odot \mathcal{H}) \setminus W$. For cases (1) and (2), by Lemma 3.5, it is proven that $r(x \mid W) \neq r(y \mid W)$. For case (3), let $u_{xk}^x \in W$. Then $d(u_{xk}^x, v_x^0) = 1$. Since $d(v_x^0, v_y^0) = s, \quad 1 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $d(v_x^0, u_{xk}^x) = 1 \quad k = 1, 2, \ldots, |V(H^i)|$, then $d(v_y^0, u_{xk}^x) = d(v_x^0, v_y^0) + d(v_x^0, u_{xk}^x) = s + 1$. Because $d(v_y^0, u_{xk}^x) > d(v_x^0, u_{xk}^x)$, it follows that $r(v_x^0 \mid W) \neq r(v_y^0 \mid W)$. Because W is a diametral set and also a resolving set, W is a diametral basis. Thus, $\dim_{\text{diam}}(G \odot \mathcal{H}) = |D(G \odot \mathcal{H})|$. □

We give an example in the following from Theorem 3.10 below.

Example 3.11. Figure 1 presents the graph C_4, while Figure 2 presents the sequence of graphs $\mathcal{H} = (P_3, C_3, K_4, S_4)$. The vertex set of C_4 is denoted by $V(C_4) = \{v_i \mid i = 1, 2, 3, 4\}$, and the vertex set of each graph H^i in $\mathcal{H}$ is denoted by $V(H^i) = \{u_{ij} \mid j = 1, 2, \ldots, |V(H^i)|\}, \qquad i = 1, 2, 3, 4.$

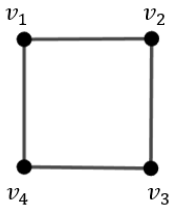


Fig. 1. The graph C_4.

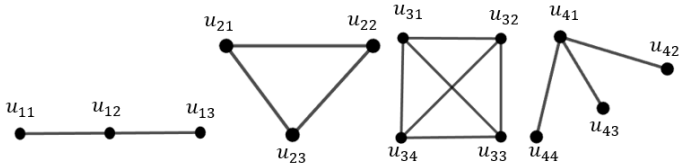


Fig. 2. The sequence of graphs $\mathcal{H} = (P_3, C_3, K_4, S_4)$.

The generalized corona $C_4 \odot \mathcal{H}$ is shown in Figure 3. The vertex set of $C_4 \odot \mathcal{H}$ is given by $V(C_4 \odot \mathcal{H}) = V(C_4^0) \cup \bigcup_{i=1}^{4} V(H_i^i)$, where $V(C_4^0) = \{v_i^0 \mid v_i \in V(C_4)\}$, and $V(H_i^i) = \{u_{ij}^i \mid u_{ij} \in V(H^i),\ i = 1, 2, 3, 4\}$.

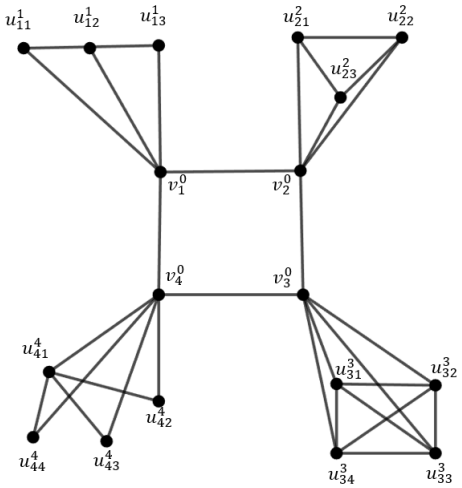


Fig. 3. The generalized corona graph $C_4 \odot \mathcal{H}$.

The diametral set of $C_4 \odot \mathcal{H}$ is given by $D(C_4 \odot \mathcal{H}) = \{u_{ij}^i \in V(H_i^i) \mid i = 1, 2, 3, 4\}$. Since $\mathrm{diam}(C_4) = \mathrm{rad}(C_4) = 2$, it follows from Theorem 3.10 that $\dim_{\mathrm{diam}}(C_4 \odot \mathcal{H}) = |D(C_4 \odot \mathcal{H})| = 14$.

Based on the results above, we obtain the theorem for the diametral metric dimension of the generalized corona graph $G \odot \mathcal{H}$ and $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ is a sequence of graphs from cycle graph, path graph, star graph, and also complete graph. In the following theorem, we obtain the diametral metric dimension of the generalized corona graph $G \odot \mathcal{H}$ for graph G of order at least 3. The result shows that the diametral metric dimension depends on whether the radius and diameter of G are equal. If they are not equal, the result also depends on whether a basis of $K_1 + H^i$ contains the vertex K_1.

Theorem 3.12. *The diametral metric dimension of $G \odot \mathcal{H}$ is*

$$\dim_{\text{diam}}(G \odot \mathcal{H}) = \begin{cases} |D(G \odot \mathcal{H})|, & \text{rad}(G) = \text{diam}(G), \\ |D(G \odot \mathcal{H})| + \sum\limits_{i, v_i \notin D(G)} \left(\dim(K_1 + H^i) - \alpha^i\right), & \textit{otherwise}, \end{cases} \tag{3.7}$$

where $\alpha^i = \begin{cases} 1, & \textit{if there exists a basis of } K_1 + H^i \textit{ containing the vertex of } K_1, \\ 0, & \textit{otherwise}. \end{cases}$

Proof. Let $V(G) = \{v_i \mid i = 1, 2, \ldots, n\}$, and $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ be a sequence of n graphs, where $V(H^i) = \{u_{ij} \mid j = 1, 2, \ldots, |V(H^i)|\}$, $i = 1, 2, \ldots, n$. The vertex set of $G \odot \mathcal{H}$ is $V(G \odot \mathcal{H}) = V(G^0) \cup \bigcup_{i=1}^{n} V(H_i^i)$, where $V(G^0) = \{v_i^0 \mid v_i \in V(G)\}$ and $V(H_i^i) = \{u_{ij}^i \mid u_{ij} \in V(H^i)\}$, $i = 1, 2, \ldots, n$. We consider two cases.

(1) **Case 1.** $\text{rad}(G) = \text{diam}(G)$**.**
By Lemma 3.4, if $\text{rad}(G) = \text{diam}(G)$, then $D(G) = V(G)$. Consequently, $D(G \odot \mathcal{H}) = \bigcup_{i=1}^{n} V(H_i^i)$. Let $W = D(G \odot \mathcal{H}) = \bigcup_{i=1}^{n} V(H_i^i)$. Thus, $|W| = |D(G \odot \mathcal{H})|$. For any distinct vertices $x, y \in V(G \odot \mathcal{H})$ where $x \neq y$, there are three possible cases: (i) $x, y \in W$; (ii) $x \in W$ and $y \in V(G \odot \mathcal{H}) \setminus W$; (iii) $x, y \in V(G \odot \mathcal{H}) \setminus W$. For cases (1) and (2), by Lemma 3.5, it is proven that $r(x \mid W) \neq r(y \mid W)$. For case (3), let $u = v_x^0$ and $v = v_y^0$, where $x \neq y$. For any chosen a vertex $u_{xk}^x \in W$. Since u_{xk}^x is adjacent to v_x^0, we have $d(v_x^0, u_{xk}^x) = 1$. Let $d(v_x^0, v_y^0) = s$, where $1 \leq s \leq \lfloor n/2 \rfloor$ and $d(v_x^0, u_{xk}^x) = 1$ for $k = 1, 2, \ldots, |V(H^i)|$, we have $d(v_y^0, u_{xk}^x) = d(v_x^0, v_y^0) + d(v_x^0, u_{xk}^x) = s + 1$. Since $d(v_y^0, u_{xk}^x) > d(v_x^0, u_{xk}^x)$, it follows that $r(v_x^0 \mid W) \neq r(v_y^0 \mid W)$. Because W is a diametral set and also a resolving set, W is a diametral basis. Thus,

$$\dim_{\text{diam}}(G \odot \mathcal{H}) = |D(G \odot \mathcal{H})|.$$

(2) **Case 2.** $\text{rad}(G) < \text{diam}(G)$**.**
In this case, not every vertex of G is a diametral vertex; hence, $D(G) \subsetneq V(G)$.
   (a) If $v_i \in D(G)$, then all vertices of H_i^i belong to $D(G \odot \mathcal{H})$.
   (b) If $v_i \notin D(G)$, then the vertices of H_i^i do not belong to $D(G \odot \mathcal{H})$. In this case, $D(G \odot \mathcal{H})$ is not sufficient to be a candidate for a diametral basis because there are vertices in H_i^i, where $v_i \notin D(G)$, that have the same representation with respect to $D(G \odot \mathcal{H})$. The vertices

required in the i-th copy are determined by a resolving set of the subgraph consisting of v_i^0 and H_i^i. Since $\langle v_i^0\rangle + H_i^i \cong K_1 + H^i$, then $\dim(v_i^0 + H_i^i) = \dim(K_1 + H^i)$, for all i.

For this case, there are two possibilities for a basis of $K_1 + H^i$, namely, either there exists a basis of $K_1 + H^i$ containing the vertex of K_1, or there is no basis of $K_1 + H^i$ containing the vertex of K_1.

(a) **If there exists a basis of $K_1 + H^i$ containing the vertex of K_1** Let B^i be a basis of the graph $K_1 + H^i$, and let $B_i^i = \{u_{is}^i \in V(G \odot \mathcal{H}) \mid u_{is} \in B^i,\ v_i \notin D(G)\} \setminus \{v_i^0\}, \quad i = 1, 2, 3, \ldots, n$. Define

$$W = \{u_{ij}^i \in V(H^i) \mid v_i \in D(G)\} \cup \bigcup_{i=1}^{n} \Big(\{u_{is}^i \in V(G \odot \mathcal{H}) \mid u_{is}^i \in B^i, \\ v_i \notin D(G)\} \setminus \{v_i^0\} \Big),$$

so that $|W| = |D(G\odot\mathcal{H})| + \sum_{v_i \notin D(G)} \big(\dim(K_1+H^i) - 1\big)$. Take any two distinct vertices $x, y \in V(G \odot \mathcal{H})$. There are three possibilities: (i) $x, y \in W$; (ii) $x \in W$ and $y \in V(G\odot\mathcal{H})\setminus W$; (iii) $x, y \in V(G\odot\mathcal{H})\setminus W$. For Cases (1) and (2), by Lemma 3.5, it has been proved that $r(x \mid W) \neq r(y \mid W)$. For Case (3), there are four subcases as follows.

Subcase 3.1: $u_{ij}^i, u_{ik}^i \in V(H_i^i)$.

Since $r(u_{ij}^i \mid B_i^i) \neq r(u_{ik}^i \mid B_i^i)$ and $B_i^i \subseteq W$, we have $r(u_{ij}^i \mid W) \neq r(u_{ik}^i \mid W)$.

Subcase 3.2: $v_x^0, v_y^0 \in V(G \odot \mathcal{H}) \setminus W$, where $x \neq y$.

Let $d(v_x^0, v_y^0) = s$, where $1 \leq s \leq n-1$. Let $u_{xj}^x \in W$. We obtain $d(v_x^0, u_{xj}^x) = 1$ for $j = 1, 2, \ldots, |V(H^i)|$. Hence, $d(v_y^0, u_{xj}^x) = d(v_x^0, v_y^0) + d(v_x^0, u_{xj}^x) = s + 1$. Since $d(v_y^0, u_{xj}^x) > d(v_x^0, u_{xj}^x)$, it follows that $r(v_x^0 \mid W) \neq r(v_y^0 \mid W)$.

Subcase 3.3: $v_x^0 \in V(G^0)$ and $u_{ij}^i \in V(H_i^i)$.

There are two possibilities, namely, $x = i$ or $x \neq i$.

If $x = i$, let $u_{pj}^p \in W$. Then $d(v_p^0, u_{pj}^p) = 1 = d(v_i^0, u_{ij}^i)$. Since $d(v_p^0, v_i^0) = s$, where $1 \leq s \leq n-1$, we have $d(v_i^0, u_{pj}^p) = d(v_p^0, u_{pj}^p) + d(v_p^0, v_i^0) = 1 + s$. Furthermore, $d(u_{ij}^i, u_{pj}^p) = d(v_p^0, u_{pj}^p) + d(v_i^0, u_{ij}^i) = s + 1 + 1 = s + 2$. Thus, $d(v_i^0, u_{pj}^p) < d(u_{ij}^i, u_{pj}^p)$.

If $x \neq i$, let $u_{ik}^i \in W$. Then$d(u_{ij}^i, v_i^0) = d(u_{ik}^i, v_i^0) = 1$ and $d(u_{ij}^i, u_{ik}^i) \leq 2$. Since $d(v_x^0, v_i^0) = s$, where $1 \leq s \leq n-1$, we have $d(v_x^0, u_{ik}^i) = d(v_x^0, v_i^0) + d(u_{ik}^i, v_i^0) = s + 1$. Thus, $d(u_{ij}^i, u_{ik}^i) < d(v_x^0, u_{ik}^i)$.

Therefore, for $v_x^0, u_{ij}^i \in V(G\odot\mathcal{H})\setminus W$, we have $r(v_x^0 \mid W) \neq r(u_{ij}^i \mid W)$.

Subcase 3.4: $u_{xj}^x \in V(H_x^x)$ and $u_{yj}^y \in V(H_y^y)$, where $x \neq y$.

Let $d(u_{xj}^x, v_x^0) = 1$ and $d(v_x^0, v_y^0) = s$, where $1 \leq s \leq n-1$. Let $u_{yk}^y \in W$. We obtain $d(u_{yj}^y, u_{yk}^y) \leq 2$ and $d(v_y^0, u_{yj}^y) = d(v_y^0, u_{yk}^y) = 1$. It is known that $d(u_{xj}^x, v_y^0) = d(u_{xj}^x, v_x^0) + d(v_x^0, v_y^0) = 1 + s$ and $d(u_{xj}^x, u_{yk}^y) = d(u_{xj}^x, v_y^0) + d(v_y^0, u_{yk}^y) = (1 + s) + 1 = 2 + s$. Therefore, $d(u_{yj}^y, u_{yk}^y < d(u_{xj}^x, u_{yk}^y)$. Hence, for $u_{xj}^x, u_{yj}^y \in V(G \odot \mathcal{H}) \setminus W$, we have $r(u_{xj}^x \mid W) \neq r(u_{yj}^y \mid W)$.

Next, we prove that W is a diametral resolving set with minimum cardinality. let $T \subseteq V(G \odot \mathcal{H})$ be a set containing the diametral set such that $|T| < |W|$. Let $|T| = |W| - 1$. This condition implies that there is an index $i = 1$ such that at most $|(B_i^i \setminus \{v_i^0\})| - 1 = |B_i^i| - 2$ vertices of $K_1 + H^i$ are elements of T. Since $D(G \odot \mathcal{H}) \subseteq T$, there consequently exist two vertices $u_{ij}^i, u_{ik}^i \in V(H_i^i)$ such that $u_{ij}^i, u_{ik}^i \notin T$, and $r(u_{ij}^i \mid B_i^i \setminus \{v_i^0\}) = r(u_{ik}^i \mid B_i^i \setminus \{v_i^0\})$. Moreover, for every vertex $z \in V(G \odot \mathcal{H}) \setminus V(H_i^i)$, every path from z to a vertex in H_i^i must pass through v_i^0. Since both u_{ij}^i and u_{ik}^i are adjacent to v_i^0, we have $d(z, u_{ij}^i) = d(z, v_i^0) + 1 = d(z, u_{ik}^i)$. Consequently, $r(u_{ij}^i \mid T) = r(u_{ik}^i \mid T)$. Consequently, T is not a diametral resolving set. By Lemma 3.3, W is the diametral resolving set with minimum cardinality. Thus, $\dim_{\text{diam}}(G \odot \mathcal{H}) = |D(G \odot \mathcal{H})| \ + \sum_{i,\, v_i \notin D(G)} \big(\dim(K_1 + H^i) - 1\big)$.

(b) **If there is no basis of $K_1 + H^i$ containing the vertex of K_1**

Let B^i be a basis of the graph $K_1 + H^i$, and let $B_i^i = \{u_{is}^i \in V(G \odot \mathcal{H}) \mid u_{is} \in B^i,\ v_i \notin D(G)\}, \quad i = 1, 2, 3, \ldots, n$. Define $W = \{u_{ij}^i \in V(H_i^i) \mid v_i \in D(G)\} \cup \bigcup_{i=1}^n \{u_{is}^i \in V(G \odot \mathcal{H}) \mid u_{is} \in B^i,\ v_i \notin D(G)\}$, so that $|W| = |D(G \odot \mathcal{H})| + \sum_{i:\, v_i \notin D(G)} \dim(K_1 + H^i)$.

Take any two distinct vertices $x, y \in V(G \odot \mathcal{H})$. There are three possibilities: (i) $x, y \in W$; (ii) $x \in W$ and $y \in V(G \odot \mathcal{H}) \setminus W$; (iii) $x, y \in V(G \odot \mathcal{H}) \setminus W$. For Cases (1) and (2), by Lemma 3.5, it has been proved that $r(x \mid W) \neq r(y \mid W)$. For Case (3), there are four subcases as follows.

Subcase 3.1: $u_{ij}^i, u_{ik}^i \in V(H_i^i)$.

Since $r(u_{ij}^i \mid B_i^i) \neq r(u_{ik}^i \mid B_i^i)$ and $B_i^i \subseteq W$, we have $r(u_{ij}^i \mid W) \neq r(u_{ik}^i \mid W)$.

Subcase 3.2: $v_x^0, v_y^0 \in V(G \odot \mathcal{H}) \setminus W$, where $x \neq y$.

Let $d(v_x^0, v_y^0) = s$, where $1 \leq s \leq n - 1$. Let $u_{xj}^x \in W$. We obtain $d(v_x^0, u_{xj}^x) = 1$ for $j = 1, 2, \ldots, |V(H^i)|$. Hence, $d(v_y^0, u_{xj}^x) = d(v_x^0, v_y^0) + d(v_x^0, u_{xj}^x) = s + 1$. Since $d(v_y^0, u_{xj}^x) > d(v_x^0, u_{xj}^x)$, it follows that $r(v_x^0 \mid W) \neq r(v_y^0 \mid W)$.

Subcase 3.3: $v_x^0 \in V(G^0)$ and $u_{ij}^i \in V(H_i^i)$.

There are two possibilities, namely, $x = i$ or $x \neq i$.

If $x = i$, let $u_{pj}^p \in W$. Then $d(v_p^0, u_{pj}^p) = 1 = d(v_i^0, u_{ij}^i)$. Since $d(v_p^0, v_i^0) = s$, where $1 \leq s \leq n - 1$, we have $d(v_i^0, u_{pj}^p) = d(v_p^0, u_{pj}^p) + d(v_p^0, v_i^0) = 1 + s$. Furthermore, $d(u_{ij}^i, u_{pj}^p) = d(v_p^0, u_{pj}^p) + d(v_i^0, u_{ij}^i) = s + 1 + 1 = s + 2$. Thus, $d(v_i^0, u_{pj}^p) < d(u_{ij}^i, u_{pj}^p)$.

If $x \neq i$, let $u_{ik}^i \in W$. Then$d(u_{ij}^i, v_i^0) = d(u_{ik}^i, v_i^0) = 1$ and $d(u_{ij}^i, u_{ik}^i) \leq 2$. Since $d(v_x^0, v_i^0) = s$, where $1 \leq s \leq n - 1$, we have $d(v_x^0, u_{ik}^i) = d(v_x^0, v_i^0) + d(u_{ik}^i, v_i^0) = s + 1$. Thus, $d(u_{ij}^i, u_{ik}^i) < d(v_x^0, u_{ik}^i)$.

Therefore, for $v_x^0, u_{ij}^i \in V(G \odot \mathcal{H}) \setminus W$, we have $r(v_x^0 \mid W) \neq r(u_{ij}^i \mid W)$.

Subcase 3.4: $u_{xj}^x \in V(H_x^x)$ and $u_{yj}^y \in V(H_y^y)$, where $x \neq y$.

Let $d(u^x_{xj}, v^0_x) = 1$ and $d(v^0_x, v^0_y) = s$, where $1 \leq s \leq n-1$. Let $u^y_{yk} \in W$. We obtain $d(u^y_{yj}, u^y_{yk}) \leq 2$ and $d(v^0_y, u^y_{yj}) = d(v^0_y, u^y_{yk}) = 1$. It is known that $d(u^x_{xj}, v^0_y) = d(u^x_{xj}, v^0_x) + d(v^0_x, v^0_y) = 1 + s$ and $d(u^x_{xj}, u^y_{yk}) = d(u^x_{xj}, v^0_y) + d(v^0_y, u^y_{yk}) = (1 + s) + 1 = 2 + s$. Therefore, $d(u^y_{yj}, u^y_{yk} < d(u^x_{xj}, u^y_{yk})$. Hence, for $u^x_{xj}, u^y_{yj} \in V(G \odot \mathcal{H}) \setminus W$, we have $r(u^x_{xj} \mid W) \neq r(u^y_{yj} \mid W)$.

Next, we prove that W is a diametral resolving set with minimum cardinality. let $T \subseteq V(G \odot \mathcal{H})$ be a set containing the diametral set such that $|T| < |W|$. Let $|T| = |W| - 1$. This condition implies that there is an index $i = 1$ such that at most $|B^i_i| - 1$ vertices of $K_1 + H^i$ are elements of T. Since $D(G \odot \mathcal{H}) \subseteq T$, there consequently exist two vertices $u^i_{ij}, u^i_{ik} \in V(H^i_i)$ such that $u^i_{ij}, u^i_{ik} \notin T$, and $r(u^i_{ij} \mid B^i_i) = r(u^i_{ik} \mid B^i_i)$. Moreover, for every vertex $z \in V(G \odot \mathcal{H}) \setminus V(H^i_i)$, every path from z to a vertex in H^i_i must pass through v^0_i. Since both u^i_{ij} and u^i_{ik} are adjacent to v^0_i, we have $d(z, u^i_{ij}) = d(z, v^0_i) + 1 = d(z, u^i_{ik})$. Consequently, $r(u^i_{ij} \mid T) = r(u^i_{ik} \mid T)$. Consequently, T is not a diametral resolving set. By Lemma 3.3, W is the diametral resolving set with minimum cardinality. Thus, $\dim_{\text{diam}}(G \odot \mathcal{H}) = |D(G \odot \mathcal{H})| + \sum_{i:\, v_i \notin D(G)} \big(\dim(K_1 + H^i)\big)$.

□

4. Conclusion

The concept of diametral metric dimension is introduced to extend the notion of metric dimension by requiring a resolving set to contain all diametral vertices of the graph. This study investigated the diametral metric dimension of the generalized corona graph $G \odot \mathcal{H}$, where G is a connected graph and $\mathcal{H} = (H^1, H^2, \ldots, H^n)$ is a sequence of connected graphs. The results show that the diametral metric dimension of a generalized corona graph depends on the relationship between the radius and diameter of the main graph G. If $\text{rad}(G) = \text{diam}(G)$, the diametral metric dimension is equal to the number of diametral vertices of the generalized corona graph. Otherwise, when $\text{rad}(G) < \text{diam}(G)$, additional contributions from the metric dimensions of the graphs $K_1 + H^i$ are required for vertices $v_i \notin D(G)$. These contributions depend on whether a basis of $K_1 + H^i$ contains the vertex corresponding to K_1. The diametral set identifies the vertices that are most distant in the network, while the attached graphs contribute to the resolving requirements of the resulting generalized corona graph. In the context of network planning, these findings provide theoretical insights into how different local structures attached to the vertices of a main network can affect the identification and resolution of remote locations. Further studies could explore other families of graphs in the sequence $\mathcal{H}$ and investigate the diametral metric dimension under other graph operations.

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