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CLASS-r SQUARE-MEAN STEPANOV-LIKE PSEUDO ALMOST AUTOMORPHIC SOLUTIONS

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Abstract. Using measure theory, we propose a new way of studying square-mean Stepanov-like pseudo almost automorphic processes. We begin by defining, through (ν_1, ν_2)-ergodic processes, the spaces of square-mean Stepanov-like (ν_1, ν_2)-pseudo almost automorphic processes of class r, and we prove completeness and composition results for them. Next, the Banach contraction principle allows us to show that a functional differential equation driven by stochastic noise admits one and only one square-mean Stepanov-like (ν_1, ν_2)-pseudo almost automorphic solution belonging to class r. An application closes the paper.

Keywords. *Measure-theoretic approach, (ν_1, ν_2)-pseudo almost automorphy, partial delay differential equations, stochastic processes, fractional Brownian motion.*
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1. Introduction

Recurrent but non-periodic behaviour is described by almost periodic functions (Bohr) and by their extension to almost automorphic functions due to Bochner [10, 11]; see N'Guérékata [32]. Adding an ergodic perturbation leads to the pseudo counterparts of both notions; weighted and measure-theoretic versions were given in [17, 7, 8, 9], and Stepanov-like versions, where the conditions are only required in a local mean sense, in [16, 33, 13]. Blot, Cieutat and Ezzinbi [7, 8] replaced the weight by a pair of positive measures (ν_1, ν_2). This setting was used to obtain class-r solutions of functional partial differential equations whose delay is finite in [23, 38], combined with the Stepanov-like approach in the deterministic case in [34], and studied using the α-norm in [37, 27, 26, 31, 28].

Stochastic versions of these notions are studied in [4, 6, 13, 25, 19, 24, 15, 18, 35, 14, 12, 30, 3, 36, 29]. In particular, Diop, Ezzinbi and Mbaye [19] considered an equation of type (1.1) without delay in the measure-theoretic square-mean setting, and a fractional Brownian forcing was added in [18]; Zabsonre and Kiema in [36] treated the square-mean case where the class is infinite. To the best of

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our knowledge, no previous work combines square-mean structure, the Stepanov-like regularity, general measures (ν_1, ν_2), class r and a noise made of a Brownian motion and a fractional Brownian motion. Our contributions are: (i) the space of square-mean Stepanov-like (ν_1, ν_2)-ergodic processes whose class is r; (ii) the space of square-mean Stepanov-like (ν_1, ν_2)-pseudo almost automorphic processes whose class is r, with completeness, invariance under translations and composition results; (iii) a unique square-mean Stepanov-like $cl(\nu_1, \nu_2)$-pseudo almost automorphic solution in the mild sense, of class r, to (1.1), obtained from the spectral decomposition of Adimy and co-authors [1, 2] and the Banach contraction principle; (iv) an application to a stochastic partial functional differential equation with finite delay.

We consider in a real separable Hilbert space $\mathbb{H}$ the equation

$$\mathrm{d}\vartheta(t) = [A\vartheta(t) + L(\vartheta_t) + F(t)]\mathrm{d}t + G(t)\mathrm{d}W(t) + Q(t)\mathrm{d}B^H(t), \text{ for } t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ with $\|T(t)\| \leq Me^{-\omega t}$ for $t \geq 0$ and some $M, \omega > 0$; $F, G, Q : \mathbb{R} \to \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ are specified later; $W(t)$ is a two-sided standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where $\mathcal{F}_t$ is the σ-field generated by the increments of W up to time t; $B^H(t)$ is an independent two-sided fractional Brownian motion of Hurst parameter $H \in (\frac{1}{2}, 1)$ on the same probability space; $L : \mathcal{C} \to \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ is linear and bounded, where $\mathcal{C} = C([-r, 0], \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ is endowed with the uniform norm, and $\vartheta_t(\tau) = \vartheta(t + \tau)$ for $-r \leq \tau \leq 0$. Here $\mathbb{L}^2(\mathbf{P}, \mathbb{H})$ is the space of $\mathbb{H}$-valued random variables x with $\mathbb{E}\|x\|^2 = \displaystyle\int_\Omega \|x\|^2 \mathrm{d}\mathbf{P} < \infty$.

Section 2 collects the spectral decomposition and (ν_1, ν_2)-ergodic processes, Section 3 gives their properties, Section 4 shows that a solution exists and is unique, and Section 5 presents the application.

2. Spectral decomposition and ergodic processes

To equation (1.1), we associate the initial value problem

$$\begin{cases} \mathrm{d}\vartheta(t) = [A\vartheta(t) + L(\vartheta_t) + F(t)]\mathrm{d}t + G(t)\mathrm{d}W(t) + Q(t)\mathrm{d}B^H(t) \text{ for } t \geq 0, \\ \vartheta_0 = \varphi \in \mathcal{C}. \end{cases} \tag{2.1}$$

Definition 2.1. Let $\varphi \in \mathcal{C}$. An $\mathcal{F}_t$-adapted process $\vartheta : [0, +\infty[\longrightarrow \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ is called a mild solution of (2.1) if $\vartheta_0 = \varphi$ and

$$\begin{aligned} \vartheta(t) = T(t)\varphi(0) &+ \int_0^t T(t-s)\big[L(\vartheta_s) + F(s)\big]\mathrm{d}s \\ &+ \int_0^t T(t-s)G(s)\mathrm{d}W(s) + \int_0^t T(t-s)Q(s)\mathrm{d}B^H(s). \end{aligned}$$

For $t \geq 0$, let $\mathcal{Y}(t)\varphi = v_t(\cdot, \varphi)$, where $v(\cdot, \varphi)$ solves $\mathrm{d}\vartheta(t) = [A\vartheta(t) + L(\vartheta_t)]\mathrm{d}t$ for $t \geq 0$, $\vartheta_0 = \varphi \in \mathcal{C}$.

Proposition 2.2. *The family $(\mathcal{Y}(t))_{t\geq 0}$ forms a C_0-semigroup of linear operators on $\mathcal{C}$, and for $t \geq 0$, $\tau \in [-r, 0]$ it obeys the translation rule*

$$(\mathcal{Y}(t)\varphi)(\tau) = \begin{cases} (\mathcal{Y}(t + \tau)\varphi)(0) \text{ for } t + \tau \geq 0, \\ \varphi(t + \tau) \text{ for } t + \tau \leq 0. \end{cases}$$

Proposition 2.3. [1] *The operator $\mathcal{A}_{\mathcal{Y}}$ defined on $\mathcal{C}$ by $D(\mathcal{A}_{\mathcal{Y}}) = \{\varphi \in \mathcal{C}, \varphi(0) \in D(A), \varphi'(0) \in \overline{D(A)}$ and $\varphi'(0) = A\varphi(0) + L(\varphi)\}$, $\mathcal{A}_{\mathcal{Y}}\varphi = \varphi'$, generates $(\mathcal{Y}(t))_{t\geq 0}$ on $\mathcal{C}$.*

Let $\langle J_0 \rangle = \{J_0 c : c \in \mathbb{L}^2(\mathbf{P}, \mathbb{H})\}$, where $(J_0 c)(\tau) = 0$ for $\tau \in [-r, 0[$ and $(J_0 c)(0) = c$. The space $\mathcal{C} \oplus \langle J_0 \rangle$ with the norm $|\phi + J_0 c|_{\mathcal{C}} = |\phi|_{\mathcal{C}} + |c|$ is a Banach space. The extension $\widetilde{\mathcal{A}}_{\mathcal{Y}}$ is defined on $\mathcal{C} \oplus \langle J_0 \rangle$ by $D(\widetilde{\mathcal{A}}_{\mathcal{Y}}) = \{\varphi \in \mathcal{C},\ \varphi(0) \in D(A)$ and $\varphi'(0) \in \overline{D(A)}\}$ and $\widetilde{\mathcal{A}}_{\mathcal{Y}}\varphi = J_0(A\varphi(0) + L(\varphi) - \varphi'(0))$.
$(\mathbf{H}_0)$ The operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on Hilbert space $\mathbb{H}$.

Lemma 2.4. [2] *Under $(\boldsymbol{H}_0)$, one can find $\widetilde{M} \geq 1$ and $\widetilde{\omega} \in \mathbb{R}$ such that $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}}_{\mathcal{Y}})$ and $\|(\lambda I - \widetilde{\mathcal{A}}_{\mathcal{Y}})^{-n}\| \leq \widetilde{M}(\lambda - \widetilde{\omega})^{-n}$ for $n \in \mathbb{N}$ and $\lambda > \widetilde{\omega}$. Moreover, the part of $\widetilde{\mathcal{A}}_{\mathcal{Y}}$ on $\overline{D(\widetilde{\mathcal{A}}_{\mathcal{Y}})} = \mathcal{C}$ is $\mathcal{A}_{\mathcal{Y}}$.*

Definition 2.5. We say that $(\mathcal{Y}(t))_{t\geq 0}$ is hyperbolic when $\sigma(\mathcal{A}_{\mathcal{Y}}) \cap i\mathbb{R} = \varnothing$.

$(\mathbf{H}_1)$ $T(t)$ is compact on $\mathbb{H}$ for $t > 0$.

Proposition 2.6. [37] *Under $(\boldsymbol{H}_0)$ and $(\boldsymbol{H}_1)$, the operator $\mathcal{Y}(t)$ with $t > r$ is compact.*

Theorem 2.7. [2] *Let $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$ hold and let $(\mathcal{Y}(t))_{t\geq 0}$ be hyperbolic. Then $\mathcal{C} = E_s \oplus E_u$, where the stable space E_s and the unstable space E_u are closed and $\mathcal{Y}(t)$-invariant, the restriction of $(\mathcal{Y}(t))_{t\geq 0}$ to E_u is a group, and there exist $\overline{M}, \omega > 0$ such that*
$\|\mathcal{Y}(t)\varphi\| \leq \overline{M}e^{-\omega t}\|\varphi\|$ *for* $t \geq 0,\ \varphi \in E_s$, $\quad \|\mathcal{Y}(t)\varphi\| \leq \overline{M}e^{\omega t}\|\varphi\|$ *for* $t \leq 0,\ \varphi \in E_u$.
$\mathcal{P}^s$ and $\mathcal{P}^u$ denote the projections onto E_s and E_u.

Denote by $\mathcal{N}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the family of positive measures ν_1 on $\mathcal{N}$ with $\nu_1(\mathbb{R}) = +\infty$ and $\nu_1([a, b]) < \infty$ for $a \leq b$. A process $\vartheta : \mathbb{R} \to \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ is stochastically bounded (resp. continuous) in the square-mean sense if $\sup_{t\in\mathbb{R}} \mathbb{E}\|\vartheta(t)\|^2 < \infty$ (resp. $\lim_{t\to s} \mathbb{E}\|\vartheta(t) - \vartheta(s)\|^2 = 0$ for all s); see [18]. The space $SBC(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ of such bounded continuous processes is a Banach space for $\|\vartheta\|_\infty = \sup_{t\in\mathbb{R}}(\mathbb{E}\|\vartheta(t)\|^2)^{\frac{1}{2}}$.

A continuous process $q : \mathbb{R} \to \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ is square-mean almost automorphic, $q \in SAA(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$, if every real sequence (t'_n) has a subsequence (t_n) such that, for some process $\widetilde{q}$, $\lim_n \mathbb{E}\|q(t + t_n) - \widetilde{q}(t)\|^2 = 0$ and $\lim_n \mathbb{E}\|\widetilde{q}(t - t_n) - q(t)\|^2 = 0$ for all $t \in \mathbb{R}$. When both convergences hold uniformly on every compact set of $\mathbb{R}$, we call q compact almost automorphic, $q \in SAA_c(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$; this space is a Banach space for $\|\cdot\|_\infty$. The same definition, for each fixed $x \in \mathbb{L}^2(\mathbf{P}, \mathbb{H})$, gives $SAA(\mathbb{R} \times \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$.

The Bochner transform of ϑ is $\vartheta^b(t, s) = \vartheta(t + s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, see [5] for instance. The space $BS^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ of processes with $\vartheta^b \in L^\infty(\mathbb{R}, \mathbb{L}^2((0, 1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})))$ is a Banach space for

$$\|\vartheta\|_{S^2} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \mathbb{E}\|\vartheta(s)\|^2 \mathrm{d}s \Big)^{\frac{1}{2}}.$$

A process $\vartheta \in BS^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ is square-mean Stepanov-like almost automorphic, $\vartheta \in S^2AA(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$, if $\vartheta^b \in SAA(\mathbb{R}, \mathbb{L}^2((0, 1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})))$, see [13]; $S^2AA(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ is a Banach space for $\|\cdot\|_{S^2}$ and contains $SAA(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$.

Definition 2.8. Let $\nu_1, \nu_2 \in \mathcal{M}$. A process $q \in SBC(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ is (ν_1, ν_2)-ergodic in the square-mean sense, $q \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2)$, if $\lim_{\ell \to +\infty} \frac{1}{\nu_2([-\ell, \ell])} \int_{-\ell}^{\ell} \mathbb{E}\|q(t)\|^2 \mathrm{d}\nu_1(t) = 0$, and (ν_1, ν_2)-ergodic with class r, $q \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$, if

$$\lim_{\ell \to +\infty} \frac{1}{\nu_2([-\ell, \ell])} \int_{-\ell}^{\ell} \sup_{\tau \in [t-r, t]} \mathbb{E}\|q(\tau)\|^2 \mathrm{d}\nu_1(t) = 0.$$

Proposition 2.9. [19] *Let $\nu_1, \nu_2 \in \mathcal{M}$. Then $\mathcal{E}(\mathbb{R}, \mathbb{L}^2(\boldsymbol{P}, \mathbb{H}), \nu_1, \nu_2, r)$ is a Banach space with the supremum norm $\|\cdot\|_\infty$.*

Definition 2.10. Let $\nu_1, \nu_2 \in \mathcal{M}$. A process $\vartheta \in BS^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ is square-mean Stepanov-like (ν_1, ν_2)-ergodic, $\vartheta \in \mathcal{E}S^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2)$, if $\vartheta^b \in \mathcal{E}(\mathbb{R}; \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2)$, and of class r, $\vartheta \in \mathcal{E}S^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$, if $\vartheta^b \in \mathcal{E}(\mathbb{R}; \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$, that is, $\vartheta^b \in SBC(\mathbb{R}; \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})))$ and

$$\lim_{\ell \to +\infty} \frac{1}{\nu_2([-\ell, \ell])} \int_{-\ell}^{\ell} \sup_{\tau \in [t-r, t]} \Big(\int_{\tau}^{\tau+1} \mathbb{E}\|\vartheta(s)\|^2 \mathrm{d}s \Big)^{\frac{1}{2}} \mathrm{d}\nu_1(t) = 0.$$

Definition 2.11. Let $\nu_1, \nu_2 \in \mathcal{M}$. A process $q \in BS^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ is square-mean Stepanov-like (ν_1, ν_2)-pseudo almost automorphic of class r, $q \in S^2PAA(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$, if $q = q_1 + q_2$ with $q_1^b \in SAA(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})))$ and $q_2^b \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$. If moreover $q_1^b \in SAA_c(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})))$, q is called compact and we write $q \in S^2PAA_c(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$. Replacing $\mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$ by $\mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2)$ gives the space $S^2PAA(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2)$ of [19].

Definition 2.12. Let $\nu_1, \nu_2 \in \mathcal{M}$. A function $q : \mathbb{R} \times \mathbb{L}^2(\mathbf{P}, \mathbb{H}) \to \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ with $q(\cdot, x) \in L^2_{loc}(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ for each $x \in \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ is square-mean Stepanov-like (ν_1, ν_2)-pseudo almost automorphic of class r with respect to t, $q \in S^2PAA(\mathbb{R} \times \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$, if $q = q_1 + q_2$ with $q_1^b \in SAA(\mathbb{R} \times \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})))$ and $q_2^b \in \mathcal{E}(\mathbb{R} \times \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$, i.e.

$$\lim_{\ell \to +\infty} \frac{1}{\nu_2([-\ell, \ell])} \int_{-\ell}^{\ell} \sup_{\tau \in [t-r, t]} \Big(\int_{\tau}^{\tau+1} \mathbb{E}\|q_2(s, x)\|^2 \mathrm{d}s \Big)^{\frac{1}{2}} \mathrm{d}\nu_1(t) = 0.$$

The compact version $S^2PAA_c(\mathbb{R} \times \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$ is defined as above.

3. Properties of square-mean Stepanov-like (ν_1, ν_2)-pseudo almost automorphic processes of class r

The results of this section use some of the following conditions on $\nu_1, \nu_2 \in \mathcal{M}$.
$(\mathbf{H}_2)$ Let $\nu_1, \nu_2 \in \mathcal{M}$ be such that

$$\limsup_{\ell \to +\infty} \frac{\nu_1([-\ell, \ell])}{\nu_2([-\ell, \ell])} = \delta < \infty.$$

Proposition 3.1. *Let $\nu_1, \nu_2 \in \mathcal{M}$. Then $S^2PAA(\mathbb{R}; \mathbb{L}^2((0,1), \mathbb{L}^2(\boldsymbol{P}, \mathbb{H})), \nu_1, \nu_2, r)$ is a Banach space with the norm $\|\cdot\|_{S^2}$.*

Proof. Let $(u_k)_k$ be a sequence in $S^2PAA(\mathbb{R}; \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$ such that $\lim_{k\to+\infty} u_k = u$ in $BS^2(\mathbb{R}; \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$. For each k, let $u_k = v_k + w_k$ with $v_k^b \in SAA(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})))$ and $w_k^b \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$. $(v_k)_k$ converges to some $v \in BS^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ and $(w_k)_k$ also converges to some $w \in BS^2(\mathbb{R}; \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$. It remains to show that $w \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$. By Minkowski's inequality, bounding the supremum over $[t-r, t]$ by the supremum over $\mathbb{R}$, we have

$$\frac{1}{\nu_2([-\ell,\ell])}\int_{-\ell}^{\ell}\sup_{\tau\in[t-r,t]}\Big(\int_{\tau}^{\tau+1}\mathbb{E}\|w(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)$$
$$\leq 2\|w_k - w\|_{S^2}\frac{\nu_1([-\ell,\ell])}{\nu_2([-\ell,\ell])} + \frac{2}{\nu_2([-\ell,\ell])}\int_{-\ell}^{\ell}\sup_{\tau\in[t-r,t]}\Big(\int_{\tau}^{\tau+1}\mathbb{E}\|w_k(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t).$$

Letting $k \to +\infty$ and using $(\mathbf{H}_2)$, we get $w \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$, and therefore
$u \in S^2PAA(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$.

□

The next result characterizes the square-mean Stepanov-like (ν_1, ν_2)-ergodic processes of class r.

Theorem 3.2. *Let $\nu_1, \nu_2 \in \mathcal{M}$ satisfy $(\boldsymbol{H}_2)$, let I be a bounded (possibly empty) interval and let $q \in BS^2(\mathbb{R}, \mathbb{L}^2(\boldsymbol{P}, \mathbb{H}))$. Then the three properties below are equivalent:*
i) $q \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\boldsymbol{P}, \mathbb{H})), \nu_1, \nu_2, r)$,
ii) $\lim_{\ell\to+\infty}\frac{1}{\nu_2([-\ell,\ell]\setminus I)}\int_{[-\ell,\ell]\setminus I}\sup_{\tau\in[t-r,t]}\Big(\int_{\tau}^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t) = 0$,
iii) for every $\epsilon > 0$, $\lim_{\ell\to+\infty}\dfrac{\nu_1\Big(\{t\in[-\ell,\ell]\setminus I : \sup_{\tau\in[t-r,t]}\Big(\int_{\tau}^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}} > \epsilon\}\Big)}{\nu_2([-\ell,\ell]\setminus I)} = 0$.

Proof. The proof follows that of Theorem 2.13 in [8].
i) $\Leftrightarrow$ *ii)*. Set $A = \nu_1(I)$ and $B = \int_I \sup_{\tau\in[t-r,t]}\Big(\int_{\tau}^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)$.
Both A and B are finite, because I is bounded while q is continuous and bounded in the square-mean sense.
Take $\ell > 0$ large enough that $I \subset [-\ell, \ell]$ and $\nu_2([-\ell,\ell]\setminus I) > 0$. Then

$$\frac{1}{\nu_2([-\ell,\ell]\setminus I)}\int_{[-\ell,\ell]\setminus I}\sup_{\tau\in[t-r,t]}\Big(\int_{\tau}^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)$$
$$= \frac{1}{\nu_2([-\ell,\ell]) - A}\Big[\int_{[-\ell,\ell]}\sup_{\tau\in[t-r,t]}\Big(\int_{\tau}^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t) - B\Big]$$
$$= \frac{\nu_2([-\ell,\ell])}{\nu_2([-\ell,\ell]) - A}\Big[\frac{1}{\nu_2([-\ell,\ell])}\int_{[-\ell,\ell]}\sup_{\tau\in[t-r,t]}\Big(\int_{\tau}^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t) \quad -$$

$\frac{B}{\nu_2([-\ell,\ell])}\Big]$.

Since $\nu_2(\mathbb{R}) = +\infty$, these equalities show that $ii)$ holds if and only if

$$\lim_{\ell\to+\infty}\frac{1}{\nu_2([-\ell,\ell])}\int_{[-\ell,\ell]}\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)=0,$$

that is $i)$.

$iii)\Rightarrow ii)$. For $\epsilon>0$ and $\ell>0$, consider the two sets

$$A_\ell^\epsilon=\Big\{t\in[-\ell,\ell]\setminus I:\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}>\epsilon\Big\},$$

and

$$B_\ell^\epsilon=\Big\{t\in[-\ell,\ell]\setminus I:\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\le\epsilon\Big\}.$$

Suppose now that, for every $\epsilon>0$,

$$\lim_{\ell\to+\infty}\frac{\nu_1(A_\ell^\epsilon)}{\nu_2([-\ell,\ell]\setminus I)}=0. \tag{3.1}$$

From the equality

$$\begin{aligned}&\int_{[-\ell,\ell]\setminus I}\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)\\&=\int_{A_\ell^\epsilon}\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)+\int_{B_\ell^\epsilon}\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t),\end{aligned}$$

we deduce that, for ℓ sufficiently large,

$$\begin{aligned}&\frac{1}{\nu_2([-\ell,\ell]\setminus I)}\int_{[-\ell,\ell]\setminus I}\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)\\&\qquad\le\|q\|_{S^2}\times\frac{\nu_1(A_\ell^\epsilon)}{\nu_2([-\ell,\ell]\setminus I)}+\epsilon\frac{\nu_1(B_\ell^\epsilon)}{\nu_2([-\ell,\ell]\setminus I)}.\end{aligned}$$

Since $\nu_1(\mathbb{R})=\nu_2(\mathbb{R})=\infty$, by (3.1) and $(\mathbf{H}_2)$ we have, for all $\epsilon>0$,

$$\limsup_{\ell\to+\infty}\frac{1}{\nu_2([-\ell,\ell]\setminus I)}\int_{[-\ell,\ell]\setminus I}\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)\le\delta\epsilon.$$

Consequently $ii)$ holds.

$ii)\Rightarrow iii)$. Since the integrand exceeds ϵ on A_ℓ^ϵ,

$$\frac{1}{\nu_2([-\ell,\ell]\setminus I)}\int_{[-\ell,\ell]\setminus I}\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)\ge\epsilon\frac{\nu_1(A_\ell^\epsilon)}{\nu_2([-\ell,\ell]\setminus I)},$$

and letting $\ell\to+\infty$ gives (3.1), i.e. $iii)$. □

Definition 3.3. [8] Two measures $\nu,\nu'\in\mathcal{M}$ are called equivalent, written $\nu\sim\nu'$, when one can find $\alpha,\beta>0$ and a bounded (possibly empty) interval I with $\alpha\nu(A)\le\nu'(A)\le\beta\nu(A)$ for every $A\in\mathcal{N}$ disjoint from I. This defines an equivalence relation on $\mathcal{M}$.

Theorem 3.4. *Let $\nu_1,\nu_2,\nu_1',\nu_2'\in\mathcal{M}$. If $\nu_1\sim\nu_1'$ and $\nu_2\sim\nu_2'$, then $S^2PAA(\mathbb{R},\mathbb{L}^2(\boldsymbol{P},\mathbb{H}),\nu_1,\nu_2,r)=S^2PAA(\mathbb{R},\mathbb{L}^2(\boldsymbol{P},\mathbb{H}),\nu_1',\nu_2',r)$.*

Proof. By definition of $\sim$, we can choose $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded (possibly empty) interval I such that, for every $A \in \mathcal{N}$ with $A \cap I = \varnothing$, $\alpha_1\nu_1(A) \leq \nu_1'(A) \leq \beta_1\nu_1(A)$ and $\alpha_2\nu_2(A) \leq \nu_2'(A) \leq \beta_2\nu_2(A)$; in particular

$$\frac{1}{\beta_2\nu_2(A)} \leq \frac{1}{\nu_2'(A)} \leq \frac{1}{\alpha_2\nu_2(A)}.$$

Applying these bounds to $D_\epsilon = \big\{t \in [-\ell, \ell]\backslash I : \sup_{\tau\in[t-r,t]} \big(\int_\tau^{\tau+1} \mathbb{E}\|q(s)\|^2 \mathrm{d}s\big)^{\frac{1}{2}} > \epsilon\big\}$, we obtain, for large ℓ,

$$\frac{\alpha_1\nu_1(D_\epsilon)}{\beta_2\nu_2([-\ell,\ell]\setminus I)} \leq \frac{\nu_1'(D_\epsilon)}{\nu_2'([-\ell,\ell]\setminus I)} \leq \frac{\beta_1\nu_1(D_\epsilon)}{\alpha_2\nu_2([-\ell,\ell]\setminus I)}.$$

Theorem 3.2 then shows that the two ergodic spaces coincide,

$$\mathcal{E}S^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r) = \mathcal{E}S^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1', \nu_2', r),$$

and the equality $S^2PAA(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r) = S^2PAA(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1', \nu_2', r)$ follows directly from their definition. □

For $\nu_1 \in \mathcal{M}$ and $a \in \mathbb{R}$, the translated measure $(\nu_1)_a$ on $(\mathbb{R}, \mathcal{N})$ is

$$(\nu_1)_a(A) = \nu_1(\{a + b : b \in A\}) \text{ for } A \in \mathcal{N}. \tag{3.2}$$

($\mathbf{H}_3$) Whenever $a, b, c \in \mathbb{R}$ satisfy $0 \leq a < b < c$, one can find δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \geq \delta_0 \Rightarrow \nu_1(a + \delta, b + \delta) \geq \alpha_0\nu_1(\delta, c + \delta).$$

($\mathbf{H}_4$) To each $\ell \in \mathbb{R}$ correspond a constant $\beta > 0$ and a bounded interval I with

$$\nu_1(\{a + \ell : a \in A\}) \leq \beta\nu_1(A) \text{ whenever } A \in \mathcal{N} \text{ satisfies } A \cap I = \varnothing.$$

For $\nu_1, \nu_2 \in \mathcal{M}$, set

$$cl(\nu_1, \nu_2) = \Big\{\overline{\omega}_1, \overline{\omega}_2 \in \mathcal{M} : \nu_1 \sim \overline{\omega}_1, \nu_2 \sim \overline{\omega}_2\Big\}.$$

Lemma 3.5. [8] *Let $\nu_1 \in \mathcal{M}$ satisfy ($\boldsymbol{H}_4$). Then ($\boldsymbol{H}_3$) holds, the measures ν_1 and $(\nu_1)_\ell$ are equivalent for all $\ell \in \mathbb{R}$, and $\limsup_{\ell\to+\infty} \dfrac{\nu_1([-\ell-\sigma, \ell+\sigma])}{\nu_1([-\ell,\ell])} < \infty$ for all $\sigma > 0$.*

Given $q \in BS^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ and $\ell \in \mathbb{R}$, the translate of q is $q_\ell(t) = q(t + \ell)$, $t \in \mathbb{R}$.
We call a subset $\mathfrak{F} \subset BS^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ translation invariant when $q_\ell \in \mathfrak{F}$ for every $q \in \mathfrak{F}$ and every $\ell \in \mathbb{R}$.

Theorem 3.6. *If $\nu_1, \nu_2 \in \mathcal{M}$ satisfy ($\boldsymbol{H}_4$), then $S^2PAA(\mathbb{R}, \mathbb{L}^2(\boldsymbol{P}, \mathbb{H}), \nu_1, \nu_2, r)$ is translation invariant.*

Proof. It suffices to verify that $\mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$ is translation invariant. For this, let $q \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$ and

$$\Gamma(t) = \sup_{\tau\in[t-r,t]} \Big(\int_\tau^{\tau+1} \mathbb{E}\|q(s)\|^2 \mathrm{d}s\Big)^{\frac{1}{2}}.$$

Then $\Gamma \in \mathcal{E}(\mathbb{R},\mathbb{R},\nu_1,\nu_2)$. Since $\mathcal{E}(\mathbb{R},\mathbb{R},\nu_1,\nu_2)$ is translation invariant by [34], it follows that, for all $\alpha \in \mathbb{R}$,

$$\begin{aligned}&\lim_{\ell\to+\infty}\frac{1}{\nu_2([-\ell,\ell])}\int_{-\ell}^{\ell}\Gamma(t+\alpha)\mathrm{d}\nu_1(t)\\&=\lim_{\ell\to+\infty}\frac{1}{\nu_2([-\ell,\ell])}\int_{-\ell}^{\ell}\sup_{\tau\in[t-r,t]}\Big(\int_{\tau}^{\tau+1}\mathbb{E}\|q(s+\alpha)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)=0,\end{aligned}$$

so $q_\alpha \in \mathcal{E}S^2(\mathbb{R},\mathbb{L}^2(\mathbf{P},\mathbb{H}),\nu_1,\nu_2,r)$, which is therefore translation invariant. □

Theorem 3.7. *Let $\nu_1,\nu_2 \in \mathcal{M}$ and $y \in S^2PAA(\mathbb{R};\mathbb{L}^2(\boldsymbol{P},\mathbb{H}),\nu_1,\nu_2,r)$. Consider $\psi = \psi_1+\psi_2 \in S^2PAA(\mathbb{R}\times\mathbb{L}^2(\boldsymbol{P},\mathbb{H}),\mathbb{L}^2((0,1),\mathbb{L}^2(\boldsymbol{P},\mathbb{H})),\nu_1,\nu_2,r)$ with $\psi_1^b \in SAA(\mathbb{R}\times\mathbb{L}^2(\boldsymbol{P},\mathbb{H}),\mathbb{L}^2((0,1),\mathbb{L}^2(\boldsymbol{P},\mathbb{H})))$ and $\psi_2^b \in \mathcal{E}(\mathbb{R}\times\mathbb{L}^2(\boldsymbol{P},\mathbb{H}),\mathbb{L}^2((0,1),\mathbb{L}^2(\boldsymbol{P},\mathbb{H})),\nu_1,\nu_2,r)$, and suppose that*
(i) $\psi_1(t,x)$ is, uniformly in $t\in\mathbb{R}$, uniformly continuous in x on every bounded set;
(ii) for some nonnegative $L_\psi \in BS^p(\mathbb{R})$, $1<p\le\infty$,

$$\Big(\int_t^{t+1}\mathbb{E}\|\psi(s,x_1)-\psi(s,x_2)\|^2\mathrm{d}s\Big)^{\frac{1}{2}} \le L_\psi(t)\mathbb{E}\|x_1-x_2\|^2,$$

for $x_1,x_2\in\mathbb{L}^2((0,1),\mathbb{L}^2(\boldsymbol{P},\mathbb{H}))$. If

$$\alpha=\frac{1}{\nu_2([-\ell,\ell])}\int_{-\ell}^{\ell}\sup_{\tau\in[t-r,t]}\big(L_\psi(\tau)\big)\mathrm{d}\nu_1(t)<\infty,$$

then $t\mapsto\psi(t,y(t))$ lies in $S^2PAA(\mathbb{R},\mathbb{L}^2((0,1),\mathbb{L}^2(\boldsymbol{P},\mathbb{H})),\nu_1,\nu_2,r)$.

The proof of Theorem 3.7 relies on the following lemma.

Lemma 3.8. *Suppose that $(\boldsymbol{H}_2)$ is satisfied and let $q\in BS^2(\mathbb{R},\mathbb{L}^2(\boldsymbol{P},\mathbb{H}))$. Then $q^b\in\mathcal{E}(\mathbb{R};\mathbb{L}^2((0,1),\mathbb{L}^2(\boldsymbol{P},\mathbb{H})),\nu_1,\nu_2,r)$ exactly when, for every $\epsilon>0$,*

$$\lim_{\ell\longrightarrow+\infty}\frac{\nu_1(M_{\ell,\epsilon}(q))}{\nu_2([-\ell,\ell])}=0,$$

where

$$M_{\ell,\epsilon}(q)=\Big\{t\in[-\ell,\ell]:\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\ge\epsilon\Big\}.$$

Proof. Necessity. If $q\in\mathcal{E}S^2(\mathbb{R},\mathbb{L}^2(\mathbf{P},\mathbb{H}),\nu_1,\nu_2,r)$, restricting the integral to $M_{\ell,\epsilon}(q)$ gives

$$\frac{1}{\nu_2([-\ell,\ell])}\int_{-\ell}^{\ell}\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)\ge\frac{\epsilon\nu_1(M_{\ell,\epsilon}(q))}{\nu_2([-\ell,\ell])},$$

and the left-hand side tends to 0, which gives the limit above. Sufficiency. Suppose that this limit vanishes for every $\epsilon>0$ and that $\mathbb{E}\|q(t)\|^2\le N$ for all $t\in\mathbb{R}$. Splitting $[-\ell,\ell]$ into $M_{\ell,\epsilon}(q)$ and its complement,

$$\frac{1}{\nu_2([-\ell,\ell])}\int_{-\ell}^{\ell}\sup_{\tau\in[t-r,t]}\Big(\int_\tau^{\tau+1}\mathbb{E}\|q(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)\le\frac{N\nu_1(M_{\ell,\epsilon}(q))}{\nu_2([-\ell,\ell])}+\frac{\epsilon\nu_1([-\ell,\ell])}{\nu_2([-\ell,\ell])},$$

so, by $(\mathbf{H}_2)$, the upper limit of the left-hand side is at most $\epsilon\delta$ for any $\epsilon>0$. That is, $q^b\in\mathcal{E}(\mathbb{R};\mathbb{L}^2((0,1),\mathbb{L}^2(\mathbf{P},\mathbb{H})),\nu_1,\nu_2,r)$. □

We now prove Theorem 3.7.

Proof. Write $\psi = \psi_1 + \psi_2$ and $y = y_1 + y_2$ with $\psi_1^b \in SAA(\mathbb{R} \times \mathbb{L}^2(\mathbf{P},\mathbb{H}); \mathbb{L}^2((0,1),\mathbb{L}^2(\mathbf{P},\mathbb{H})))$, $\psi_2^b \in \mathcal{E}(\mathbb{R} \times \mathbb{L}^2(\mathbf{P},\mathbb{H}); \mathbb{L}^2((0,1),\mathbb{L}^2(\mathbf{P},\mathbb{H})), \nu_1, \nu_2, r)$, $y_1^b \in SAA(\mathbb{R}; \mathbb{L}^2((0,1),\mathbb{L}^2(\mathbf{P},\mathbb{H})))$ and $y_2^b \in \mathcal{E}(\mathbb{R}; \mathbb{L}^2((0,1),\mathbb{L}^2(\mathbf{P},\mathbb{H})), \nu_1, \nu_2, r)$, and split

$$\psi^b(t, y(t)) = \psi_1^b(t, y_1(t)) + [\psi^b(t, y(t)) - \psi^b(t, y_1(t))] + \psi_2^b(t, y_1(t)).$$

By the composition result for almost automorphic processes (Theorem 2.6 in [24]), $\psi_1^b(\cdot, y_1(t)) \in SAA(\mathbb{R}; \mathbb{L}^2((0,1),\mathbb{L}^2(\mathbf{P},\mathbb{H})))$. We still have to show that the two remaining terms $\psi^b(\cdot, y(t)) - \psi^b(\cdot, y_1(t))$ and $\psi_2^b(\cdot, y_1(t))$ lie in $\mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1),\mathbb{L}^2(\mathbf{P},\mathbb{H})), \nu_1, \nu_2, r)$. The difference $\psi(t, y(t)) - \psi(t, y_1(t))$ is bounded and continuous, so some $N > 0$ satisfies $\mathbb{E}\|\psi(t, y(t)) - \psi(t, y_1(t))\|^2 \leq N$ for all $t \in \mathbb{R}$. As y and y_1 are bounded, there is a bounded set $B \subset \mathbb{L}^2(\mathbf{P},\mathbb{H})$ containing $y(\mathbb{R})$ and $y_1(\mathbb{R})$. By (ii), for $\epsilon > 0$, $\mathbb{E}\|x_1 - x_2\|^2 \leq \epsilon$ gives

$$\Big(\int_t^{t+1} \mathbb{E}\|\psi(s, x_1) - \psi(s, x_2)\|^2 \mathrm{d}s\Big)^{\frac{1}{2}} \leq L_\psi(t)\epsilon, \text{ for all } t \in \mathbb{R}.$$

For $\chi \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1),\mathbb{L}^2(\mathbf{P},\mathbb{H})), \nu_1, \nu_2, r)$, Lemma 3.8 gives

$$\lim_{\ell \longrightarrow +\infty} \frac{1}{\nu_2([-\ell,\ell])} \nu_1(M_{\ell,\epsilon}(\chi)) = 0.$$

Hence, splitting $[-\ell, \ell]$ into $M_{\ell,\epsilon}(\chi)$ and its complement,

$$\begin{aligned} &\frac{1}{\nu_2([-\ell,\ell])} \int_{-\ell}^{\ell} \sup_{\tau \in [t-r,t]} \Big(\int_\tau^{\tau+1} \mathbb{E}\|\psi(s, y(s)) - \psi(s, y_1(s))\|^2 \mathrm{d}s\Big)^{\frac{1}{2}} \mathrm{d}\nu_1(t) \\ &\leq \frac{N\nu_1(M_{\ell,\epsilon}(\chi))}{\nu_2([-\ell,\ell])} + \frac{\epsilon}{\nu_2([-\ell,\ell])} \int_{[-\ell,\ell]} \sup_{\tau \in [t-r,t]} \big(L_\psi(\tau)\big) \mathrm{d}\nu_1(t), \end{aligned}$$

so that

$$\limsup_{\ell \longrightarrow +\infty} \frac{1}{\nu_2([-\ell,\ell])} \int_{-\ell}^{\ell} \sup_{\tau \in [t-r,t]} \Big(\int_\tau^{\tau+1} \mathbb{E}\|\psi(s, y(s)) - \psi(s, y_1(s))\|^2 \mathrm{d}s\Big)^{\frac{1}{2}} \mathrm{d}\nu_1(t) \leq \epsilon\alpha.$$

Since $\epsilon > 0$ is arbitrary, $\psi^b(\cdot, y(t)) - \psi^b(\cdot, y_1(t))$ belongs to $\mathcal{E}(\mathbb{R}; \mathbb{L}^2((0,1),\mathbb{L}^2(\mathbf{P},\mathbb{H})), \nu_1, \nu_2, r)$.

It remains to treat $\psi_2^b(t, y_1(t))$.

The map ψ_2^b is uniformly continuous in its second variable on the compact set $\mathcal{K} = \overline{\{y_1^b(t) : t \in \mathbb{R}\}}$, so for every $\epsilon > 0$ there is $\delta > 0$ such that, for all $t \in \mathbb{R}$ and $\xi_1, \xi_2 \in \mathcal{K}$,

$$\|\xi_1 - \xi_2\| \leq \delta \Longrightarrow \|\psi_2^b(t, \xi_1(t)) - \psi_2^b(t, \xi_2(t))\| \leq \epsilon.$$

By compactness, there are $n(\epsilon) \in \mathbb{N}$ and $\{a_i\}_{i=1}^{n(\epsilon)} \subset \mathcal{K}$ with

$$\mathcal{K} \subset \bigcup_{i=1}^{n(\epsilon)} B(a_i, \delta),$$

and therefore

$$\mathbb{E}\|\psi_2(t, y_1(t))\|^2 \leq \epsilon + \sum_{i=1}^{n(\epsilon)} \mathbb{E}\|\psi_2(t, a_i)\|^2.$$

Since, for each $i \in \{1, \dots, n(\epsilon)\}$,

$$\lim_{\ell \longrightarrow +\infty} \frac{1}{\nu_2([-\ell,\ell])} \int_{-\ell}^{\ell} \sup_{\tau \in [t-r,t]} \Big(\int_{\tau}^{\tau+1} \mathbb{E}\|\psi_2(s,a_i)\|^2 \mathrm{d}s \Big)^{\frac{1}{2}} \mathrm{d}\nu_1(t) = 0,$$

we get

$$\limsup_{\ell \to +\infty} \frac{1}{\nu_2([-\ell,\ell])} \int_{-\ell}^{\ell} \sup_{\tau \in [t-r,t]} \Big(\int_{\tau}^{\tau+1} \mathbb{E}\|\psi_2(s,y_1(s))\|^2 \mathrm{d}s \Big)^{\frac{1}{2}} \mathrm{d}\nu_1(t) \leq \epsilon.$$

Hence $\psi_2^b(\cdot, y_1(t)) \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$, which completes the proof. □

Theorem 3.9. *Let $\nu_1, \nu_2 \in \mathcal{M}$ satisfy $(\boldsymbol{H}_4)$. For every $\phi \in S^2PAA_c(\mathbb{R}; \mathbb{L}^2(\boldsymbol{P}, \mathbb{H}), \nu_1, \nu_2, r)$, the history map $t \mapsto \phi_t$ lies in $S^2PAA_c(C([-r,0], \mathbb{L}^2(\boldsymbol{P}, \mathbb{H})), \nu_1, \nu_2, r)$.*

Proof. Assume $\phi = v + h$, where

$$v^b \in SAA(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H}))) \quad \text{and} \quad h^b \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r).$$

Clearly, $\phi_t = v_t + h_t$. First, starting from any real sequence $(c_m)_{m\in\mathbb{N}}$, we extract $(c_n)_{n\in\mathbb{N}}$ and find $w \in BS^2(\mathbb{R}; \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ with $v(s + c_n) \to w(s)$ on every compact set of $\mathbb{R}$, uniformly. Take a compact $K \subset \mathbb{R}$ and $L > 0$ with $K \subset [-L, L]$. Given $\epsilon > 0$, choose $N_{\epsilon,L} \in \mathbb{N}$ so that

$$\mathbb{E}\|v(s + c_n) - w(s)\|_{S^2}^2 \leq \epsilon \quad \text{if } n \geq N_{\epsilon,L} \text{ and } s \in [-L, L].$$

For $t \in K$ and $n \geq N_{\epsilon,L}$, we have $\mathbb{E}\|v_{t+c_n} - w_t\|_{S^2}^2 \leq \sup_{\tau\in[-L,L]} \mathbb{E}\|v(\tau + c_n) - w(\tau)\|_{S^2}^2 \leq \epsilon$. Hence, v_{t+c_n} converges to w_t uniformly on K, and the same argument shows that $w_{t-c_n} \to v_t$ uniformly on K. Thus, the function $s \mapsto v_s$ belongs to $SAA_c(\mathbb{R}, \mathbb{L}^2((0,1), \mathcal{C}))$.
Finally, we show that $h_t \in \mathcal{E}(\mathbb{R}; \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$.
Let us denote

$$M_a(\ell) = \frac{1}{(\nu_2)_a([-\ell,\ell])} \int_{-\ell}^{\ell} \sup_{\tau \in [t-r,t]} \Big(\int_{\tau}^{\tau+1} \mathbb{E}\|h(s)\|^2 \mathrm{d}s \Big)^{\frac{1}{2}} \mathrm{d}(\nu_1)_a(t),$$

with $(\nu_1)_a, (\nu_2)_a$ as in (3.2). By Lemma 3.5, $\nu_1 \sim (\nu_1)_a$ and $\nu_2 \sim (\nu_2)_a$, so Theorem 3.4 gives $\mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r) = \mathcal{E}(\mathbb{R}; \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), (\nu_1)_a, (\nu_2)_a, r)$ and $h \in \mathcal{E}(\mathbb{R}; \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), (\nu_1)_a, (\nu_2)_a, r)$. Consequently, $\lim_{\ell\to\infty} M_a(\ell) = 0$ for all $a \in \mathbb{R}$. Splitting $[t - 2r, t] = [t - 2r, t - r] \cup [t - r, t]$ and changing t into $t + r$ in the first part, we get, for every $\ell > 0$,

$$\begin{aligned}
&\frac{1}{\nu_2([-\ell,\ell])} \int_{-\ell}^{\ell} \sup_{\tau \in [t-r,t]} \Big(\int_{\tau}^{\tau+1} \sup_{\xi\in[-r,0]} \mathbb{E}\|h(\xi + s)\|^2 \mathrm{d}s \Big)^{\frac{1}{2}} \mathrm{d}\nu_1(t) \\
&\leq \frac{1}{\nu_2([-\ell,\ell])} \int_{-\ell}^{\ell} \sup_{\tau \in [t-2r,t]} \Big(\int_{\tau}^{\tau+1} \mathbb{E}\|h(s)\|^2 \mathrm{d}s \Big)^{\frac{1}{2}} \mathrm{d}\nu_1(t) \\
&\leq \frac{\nu_2([-\ell - r, \ell + r])}{\nu_2([-\ell,\ell])} M_a(\ell + r) + \frac{1}{\nu_2([-\ell,\ell])} \int_{-\ell}^{\ell} \sup_{\tau \in [t-r,t]} \Big(\int_{\tau}^{\tau+1} \mathbb{E}\|h(s)\|^2 \mathrm{d}s \Big)^{\frac{1}{2}} \mathrm{d}\nu_1(t).
\end{aligned}$$

By Lemma 3.5, the right-hand side tends to 0 as $\ell \to +\infty$, so $t \mapsto \phi_t$ lies in $S^2PAA_c(C([-r,0], \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$. □

4. Existence and uniqueness of solutions

($\mathbf{H}_5$) G is a stochastically bounded process.

($\mathbf{H}_6$) Q is a stochastically bounded process and $H \in (\frac{1}{2}, 1)$ is the Hurst parameter of B^H.

The integral with respect to B^H is the Wiener integral of Duncan, Hu and Pasik-Duncan [20]; unlike the Itô integral, it does not require an adapted integrand, only $\Psi \in L^2_{\mathbb{H}}([a,b]) := \{\Psi : [a,b] \to \mathbb{H} \text{ measurable} : \int_a^b \mathbb{E}\|\Psi(s)\|^2 \mathrm{d}s < \infty\}$.

Lemma 4.1. [20] *Assume ($\boldsymbol{H}_6$) holds. Let $a < b$ and $\Psi \in L^2_{\mathbb{H}}([a,b])$. Then the Wiener integral $\int_a^b \Psi(s)\mathrm{d}B^H(s)$ is well defined in $\mathbb{L}^2(\boldsymbol{P}, \mathbb{H})$, and*

$$\mathbb{E}\Big\| \int_a^b \Psi(s)\mathrm{d}B^H(s)\Big\|^2 \leq 2H(b-a)^{2H-1} \int_a^b \mathbb{E}\|\Psi(s)\|^2 \mathrm{d}s.$$

Each stochastic convolution below has an integrand $\Psi_t(s) = \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 Q(s))$ (or its unstable counterpart), and by Theorem 2.7 and Lemma 2.4, $\int_a^b \mathbb{E}\|\Psi_t(s)\|^2 \mathrm{d}s \leq \overline{M}^2\widetilde{M}^2|\mathcal{P}^s|^2 \int_a^b e^{-2\omega(t-s)}\mathbb{E}\|Q(s)\|^2\mathrm{d}s < \infty$, so Lemma 4.1 applies.

Theorem 4.2. *Suppose ($\boldsymbol{H}_0$), ($\boldsymbol{H}_1$), ($\boldsymbol{H}_5$), ($\boldsymbol{H}_6$) and the hyperbolicity of $(\mathcal{Y}(t))_{t\geq 0}$. For every bounded continuous F on $\mathbb{R}$, (1.1) admits exactly one solution u that is bounded on $\mathbb{R}$, given by*

$$\begin{aligned} u_t = \lim_{\lambda\to+\infty} \Big[&\int_{-\infty}^t \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 F(s))\mathrm{d}s + \int_{+\infty}^t \mathcal{Y}^u(t-s)\mathcal{P}^u(\widetilde{R}_\lambda J_0 F(s))\mathrm{d}s \\ &+ \int_{-\infty}^t \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 G(s))\mathrm{d}W(s) + \int_{+\infty}^t \mathcal{Y}^u(t-s)\mathcal{P}^u(\widetilde{R}_\lambda J_0 G(s))\mathrm{d}W(s) \\ &+ \int_{-\infty}^t \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 Q(s))\mathrm{d}B^H(s) + \int_{+\infty}^t \mathcal{Y}^u(t-s)\mathcal{P}^u(\widetilde{R}_\lambda J_0 Q(s))\mathrm{d}B^H(s)\Big], \end{aligned} \tag{4.1}$$

where $\widetilde{R}_\lambda = \lambda(\lambda I - \mathcal{A}_{\mathcal{Y}})^{-1}$ for $\lambda > \widetilde{\omega}$ and $\mathcal{P}^s$, $\mathcal{P}^u$ are the projections of Theorem 2.7.

Proof. The existence of the two terms driven by $\mathrm{d}s$ has been proved in [1]. For each $t \in \mathbb{R}$, by Itô's isometry,

$$\begin{aligned} &\mathbb{E}\| \int_{-\infty}^t \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 G(s))\mathrm{d}W(s)\|^2 \\ &\leq (\overline{M}\widetilde{M}|\mathcal{P}^s|)^2 \sum_{n=1}^{+\infty} \Big(e^{-2\omega(n-1)} \int_{t-n}^{t-n+1} \mathbb{E}\|G(s)\|^2\mathrm{d}s\Big) \leq \rho, \end{aligned}$$

where $\rho = (\overline{M}\widetilde{M}|\mathcal{P}^s|)^2\|G\|^2_{S^2}\dfrac{1}{1-e^{-2\omega}}$. Let $\Theta(n,s,t) = \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_n J_0 G(s))$ for $s \leq t$. For n sufficiently large and $\sigma \leq t$, in the same way,

$$\mathbb{E}\| \int_{-\infty}^{\sigma} \Theta(n,s,t)\mathrm{d}W(s)\|^2$$

$$\leq (\overline{M}\widetilde{M}|\mathcal{P}^s|)^2 e^{-2\omega(t-\sigma)} \sum_{k=1}^{+\infty} \Big(e^{-2\omega(k-1)} \int_{\sigma-k}^{\sigma-k+1} \mathbb{E}\|G(s)\|^2 \mathrm{d}s \Big) \leq \rho e^{-2\omega(t-\sigma)}.$$

Consequently, for $\sigma \leq t$ and all large n, m,

$$\mathbb{E}\Big\| \int_{-\infty}^{t} \Theta(n,s,t)\mathrm{d}W(s) - \int_{-\infty}^{t} \Theta(m,s,t)\mathrm{d}W(s) \Big\|^2$$
$$\leq 6\rho e^{-2\omega(t-\sigma)} + 3\mathbb{E}\Big\| \int_{\sigma}^{t} \Theta(n,s,t)\mathrm{d}W(s) - \int_{\sigma}^{t} \Theta(m,s,t)\mathrm{d}W(s) \Big\|^2.$$

Since $\lim_{n\to+\infty} \mathbb{E}\Big\| \int_{\sigma}^{t} \Theta(n,s,t)\mathrm{d}W(s) \Big\|^2$ exists, letting $n, m \to +\infty$ and then $\sigma \to -\infty$, we get

$$\limsup_{n,m\to+\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} \Theta(n,s,t)\mathrm{d}W(s) - \int_{-\infty}^{t} \Theta(m,s,t)\mathrm{d}W(s) \Big\|^2 = 0.$$

Therefore $\lim_{n\to+\infty} \int_{-\infty}^{t} \Theta(n,s,t)\mathrm{d}W(s)$ exists and its second moment is bounded on $\mathbb{R}$; the same holds for the unstable term driven by $\mathrm{d}W$.

For the fractional term, set $Z(n,s,t) = \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_n J_0 Q(s))$ for $s \leq t$. As B^H has no independent increments, we combine Minkowski's inequality with Lemma 4.1 on $[\sigma-k, \sigma-k+1]$, $k \geq 1$, where $(b-a)^{2H-1} = 1$ and $e^{-2\omega(t-s)} \leq e^{-2\omega(t-\sigma+k-1)}$:

$$\mathbb{E}\Big\| \int_{\sigma-k}^{\sigma-k+1} Z(n,s,t)\mathrm{d}B^H(s) \Big\|^2 \leq 2H\overline{M}^2\widetilde{M}^2|\mathcal{P}^s|^2 e^{-2\omega(t-\sigma+k-1)} \int_{\sigma-k}^{\sigma-k+1} \mathbb{E}\|Q(s)\|^2 \mathrm{d}s.$$

Taking square roots, summing over $k \geq 1$ and using Minkowski's inequality,

$$\Big(\mathbb{E}\Big\| \int_{-\infty}^{\sigma} Z(n,s,t)\mathrm{d}B^H(s) \Big\|^2 \Big)^{\frac{1}{2}} \leq \gamma_H^{\frac{1}{2}} e^{-\omega(t-\sigma)}, \qquad \gamma_H = \frac{2H\,\overline{M}^2\,\widetilde{M}^2|\mathcal{P}^s|^2}{(1-e^{-\omega})^2} \|Q\|_{S^2}^2.$$

For n, m sufficiently large and $\sigma \leq t$, Minkowski's inequality gives

$$\Big(\mathbb{E}\Big\| \int_{-\infty}^{t} Z(n,s,t)\mathrm{d}B^H(s) - \int_{-\infty}^{t} Z(m,s,t)\mathrm{d}B^H(s) \Big\|^2 \Big)^{\frac{1}{2}}$$
$$\leq 2\gamma_H^{\frac{1}{2}} e^{-\omega(t-\sigma)} + \Big(\mathbb{E}\Big\| \int_{\sigma}^{t} \big(Z(n,s,t) - Z(m,s,t)\big)\mathrm{d}B^H(s) \Big\|^2 \Big)^{\frac{1}{2}}.$$

The last term tends to 0 as $n, m \to +\infty$; letting $\sigma \to -\infty$, $\lim_{n\to+\infty} \int_{-\infty}^{t} Z(n,s,t)\mathrm{d}B^H(s)$ exists and its second moment is at most γ_H. The unstable term is treated in the same way, so u_t is well defined and bounded on $\mathbb{R}$. □

Theorem 4.3. *Assume that* $(\boldsymbol{H}_5)$ *and* $(\boldsymbol{H}_6)$ *hold. Let* $F, G, Q \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\boldsymbol{P}, \mathbb{H})), \nu_1, \nu_2, r)$ *and let* Φ *be defined by*

$$\Phi(t) = \lim_{\lambda\to+\infty} \Big[\int_{-\infty}^{t} \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 F(s))\mathrm{d}s + \int_{+\infty}^{t} \mathcal{Y}^u(t-s)\mathcal{P}^u(\widetilde{R}_\lambda J_0 F(s))\mathrm{d}s$$
$$+ \int_{-\infty}^{t} \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 G(s))\mathrm{d}W(s) + \int_{+\infty}^{t} \mathcal{Y}^u(t-s)\mathcal{P}^u(\widetilde{R}_\lambda J_0 G(s))\mathrm{d}W(s)$$
$$+ \int_{-\infty}^{t} \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 Q(s))\mathrm{d}B^H(s) + \int_{+\infty}^{t} \mathcal{Y}^u(t-s)\mathcal{P}^u(\widetilde{R}_\lambda J_0 Q(s))\mathrm{d}B^H(s) \Big].$$

Then $\Phi \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2(\boldsymbol{P}, \mathbb{H}), \nu_1, \nu_2, r)$.

Proof. For each $n = 1, 2, 3, \ldots$, let $X_n(t)$ be given by the formula of $\Phi(t)$, the stable integrals being taken over $[t-n, t-n+1]$ and the unstable ones over $[t+n-1, t+n]$. By the inequality $\|\sum_{i=1}^{6} a_i\|^2 \leq 6\sum_{i=1}^{6}\|a_i\|^2$, Itô's isometry and Lemma 4.1, we get, with $K = \max((\widetilde{M}\widetilde{M}|\mathcal{P}^s|)^2, (\widetilde{M}\widetilde{M}|\mathcal{P}^u|)^2)$,

$$\begin{aligned}\mathbb{E}\|X_n(t)\|^2 \leq{}& \frac{6K}{2\omega}\left(e^{2\omega}-1\right)e^{-2\omega n}\left[\int_{t-n}^{t-n+1}\mathbb{E}\|F(s)\|^2\,\mathrm{d}s + \int_{t+n-1}^{t+n}\mathbb{E}\|F(s)\|^2\,\mathrm{d}s\right]\\ &+ 6Ke^{-2\omega(n-1)}\left[\int_{t-n}^{t-n+1}\mathbb{E}\|G(s)\|^2\,\mathrm{d}s + \int_{t+n-1}^{t+n}\mathbb{E}\|G(s)\|^2\,\mathrm{d}s\right]\\ &+ 12HKe^{-2\omega(n-1)}\left[\int_{t-n}^{t-n+1}\mathbb{E}\|Q(s)\|^2\,\mathrm{d}s + \int_{t+n-1}^{t+n}\mathbb{E}\|Q(s)\|^2\,\mathrm{d}s\right],\end{aligned}$$

where we used $\int_{t-n}^{t-n+1} e^{-2\omega(t-s)}\mathrm{d}s = \int_{t+n-1}^{t+n} e^{2\omega(t-s)}\mathrm{d}s = \frac{1}{2\omega}(e^{2\omega}-1)e^{-2\omega n}$. Since the series $\frac{6K}{2\omega}\left(e^{2\omega}+1\right)\sum_{n=1}^{\infty} e^{-2\omega n}$, $6K\sum_{n=1}^{\infty} e^{-2\omega n}$ and $12HK\sum_{n=1}^{\infty} e^{-2\omega n}$ are convergent, the Weierstrass test shows that $\sum_{n=1}^{\infty} X_n(t)$ converges in $\|\cdot\|_\infty$ uniformly on $\mathbb{R}$. Moreover,

$$\mathbb{E}\|X_n(t)\|^2 \leq S(K,H,\omega)\sum_{\varphi\in\{F,G,Q\}}\Big[\int_{t-n}^{t-n+1}\mathbb{E}\|\varphi(s)\|^2\mathrm{d}s + \int_{t+n-1}^{t+n}\mathbb{E}\|\varphi(s)\|^2\mathrm{d}s\Big], \quad (4.2)$$

where $S(K,H,\omega) = \max\left(C^1(K,\omega), C^2(K,\omega), C^3(K,H,\omega)\right)$ with $C^1(K,\omega) = \frac{6K}{2\omega}\left(e^{2\omega}+1\right)\frac{e^{-2\omega}}{1-e^{-2\omega}}$, $C^2(K,\omega) = 6K\frac{e^{-2\omega}}{1-e^{-2\omega}}$ and $C^3(K,H,\omega) = 12HK\frac{e^{-2\omega}}{1-e^{-2\omega}}$. Since $F, G, Q \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$, we conclude that $X_n \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$. Thus $\sum_{n=1}^{N} X_n$ and its uniform limit $\Phi = \sum_{n=1}^{\infty} X_n$ belong to $\mathcal{E}(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$ by Proposition 2.9. □

Theorem 4.4. *Assume that* $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_5)$ *and* $(\boldsymbol{H}_6)$ *hold. Let* $F, G, Q \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\boldsymbol{P}, \mathbb{H})), \nu_1, \nu_2, r)$ *and let* Φ *be defined as in Theorem 4.3. Then* $\Phi \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\boldsymbol{P}, \mathbb{H})), \nu_1, \nu_2, r)$.

Proof. For each $n = 1, 2, 3, \ldots$, let X_n be defined as in the proof of Theorem 4.3 and set $J_n = \{-n, -n+1, n-1, n\}$. By (4.2) and Minkowski's inequality, for $\tau \in \mathbb{R}$,

$$\Big(\int_{\tau}^{\tau+1}\mathbb{E}\|X_n(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}} \leq S^{\frac{1}{2}}(K,H,\omega)\sum_{\varphi\in\{F,G,Q\}}\sum_{j\in J_n}\Big(\int_{\tau+j}^{\tau+j+1}\mathbb{E}\|\varphi(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}.$$

It follows that

$$\begin{aligned}&\frac{1}{\nu_2([-\ell,\ell])}\int_{-\ell}^{\ell}\sup_{\tau\in[t-r,t]}\Big(\int_{\tau}^{\tau+1}\mathbb{E}\|X_n(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t)\\ &\leq \frac{S^{\frac{1}{2}}(K,H,\omega)}{\nu_2([-\ell,\ell])}\sum_{\varphi\in\{F,G,Q\}}\sum_{j\in J_n}\int_{-\ell}^{\ell}\sup_{\tau\in[t-r,t]}\Big(\int_{\tau+j}^{\tau+j+1}\mathbb{E}\|\varphi(s)\|^2\mathrm{d}s\Big)^{\frac{1}{2}}\mathrm{d}\nu_1(t).\end{aligned}$$

Hence $X_n \in \mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$, and by Proposition 2.9 so is the uniform limit $\Phi = \sum_{n=1}^{+\infty} X_n$.

□

Theorem 4.5. *Let $F, G, Q \in SAA_c(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\boldsymbol{P}, \mathbb{H})))$ and let Φ be defined by*

$$\begin{aligned}\Phi(t) = \lim_{\lambda\to+\infty} \Big[&\int_{-\infty}^{t} \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 F(s))\mathrm{d}s + \int_{+\infty}^{t} \mathcal{Y}^u(t-s)\mathcal{P}^u(\widetilde{R}_\lambda J_0 F(s))\mathrm{d}s \\ &+ \int_{-\infty}^{t} \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 G(s))\mathrm{d}W(s) + \int_{+\infty}^{t} \mathcal{Y}^u(t-s)\mathcal{P}^u(\widetilde{R}_\lambda J_0 G(s))\mathrm{d}W(s) \\ &+ \int_{-\infty}^{t} \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 Q(s))\mathrm{d}B^H(s) + \int_{+\infty}^{t} \mathcal{Y}^u(t-s)\mathcal{P}^u(\widetilde{R}_\lambda J_0 Q(s))\mathrm{d}B^H(s)\Big].\end{aligned}$$

Then $\Phi \in SAA_c(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\boldsymbol{P}, \mathbb{H})))$.

Proof. Given a real sequence $(c_m)_{m\in\mathbb{N}}$, we extract $(c_n)_{n\in\mathbb{N}}$ and find $\widetilde{F}, \widetilde{G}, \widetilde{Q} \in SBC(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}))$ for which $F(t+c_n)$ converges to $\widetilde{F}(t)$, $G(t+c_n)$ converges to $\widetilde{G}(t)$ and $Q(t+c_n)$ converges to $\widetilde{Q}(t)$ uniformly on every compact subset Λ of $\mathbb{R}$. Using Theorem 2.7, for $\varphi \in \{F, G, Q\}$,

$$\lim_{\lambda\to+\infty} \left\| \mathcal{Y}^s(t-s)\mathcal{P}^s\big(\widetilde{R}_\lambda J_0 \varphi(s)\big)\right\|^2 \le \overline{M}^2\widetilde{M}^2 \left\|\mathcal{P}^s\right\|^2 e^{-2\omega(t-s)} \left\|\varphi(s)\right\|^2, \tag{4.3}$$

$$\lim_{\lambda\to+\infty} \left\| \mathcal{Y}^u(t-s)\mathcal{P}^u\big(\widetilde{R}_\lambda J_0 \varphi(s)\big)\right\|^2 \le \overline{M}^2\widetilde{M}^2 \left\|\mathcal{P}^u\right\|^2 e^{2\omega(t-s)} \left\|\varphi(s)\right\|^2. \tag{4.4}$$

For $n \in \mathbb{N}$, define the shifted noise processes

$$W_n(t) := W(t+c_n) - W(c_n) \quad \text{and} \quad B_n^H(t) := B^H(t+c_n) - B^H(c_n), \qquad t \in \mathbb{R}.$$

Since W and B^H have stationary increments, W_n and B_n^H have the same laws as W and B^H, so Itô's isometry and Lemma 4.1 hold for them with the same constants. The change of variable $s = \sigma + c_n$ shows that $\zeta(t+c_n) := \Phi(t+c_n)$ is given by the formula of Φ with $F(\cdot+c_n)$, $G(\cdot+c_n)$, $Q(\cdot+c_n)$, W_n, B_n^H in place of F, G, Q, W, B^H. By (4.3)–(4.4) and dominated convergence, $\zeta(t+c_n)$ converges to the process $w(t)$ given by the same formula with $\widetilde{F}$, $\widetilde{G}$, $\widetilde{Q}$.

We now check that this convergence is uniform on compact sets. Let $t \in \Lambda$ and take X_k as in the proof of Theorem 4.3, built from $F(\cdot+c_n) - \widetilde{F}$, $G(\cdot+c_n) - \widetilde{G}$, $Q(\cdot+c_n) - \widetilde{Q}$, so that $\zeta(t+c_n) - w(t) = \sum_{k=1}^{+\infty} X_k(t)$. By (4.2) and Minkowski's inequality, as in the proof of Theorem 4.4,

$$\begin{aligned}&\left(\int_t^{t+1} \mathbb{E}\|\zeta(s+c_n) - w(s)\|^2 \mathrm{d}s\right)^{\frac{1}{2}} \\ &\le S^{\frac{1}{2}}(K, H, \omega) \sum_{\varphi\in\{F,G,Q\}} \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \mathbb{E}\|\varphi(\tau+c_n) - \widetilde{\varphi}(\tau)\|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}.\end{aligned}$$

Given $\epsilon > 0$, fix $L > 0$ with $\Lambda \subset [-\frac{L}{2}, \frac{L}{2}]$ and $N_\epsilon \in \mathbb{N}$ such that $\mathbb{E}\|\varphi(\tau+c_n) - \widetilde{\varphi}(\tau)\|^2 \le \frac{\epsilon}{3}$ for $\varphi \in \{F, G, Q\}$, $n \ge N_\epsilon$ and $\tau \in [-L, L]$. Then the left-hand side above is at most $S^{\frac{1}{2}}(K, H, \omega)\epsilon$ for every $t \in \Lambda$. Since ϵ is arbitrary and the bound does not depend on $t \in \Lambda$, $\zeta(\cdot+c_n) \to w$ uniformly on Λ in the above sense. In the same manner, $w(\cdot - c_n) \to \zeta$ uniformly on Λ, which implies that $\Phi \in SAA_c(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})))$. □

For the existence of a square-mean Stepanov-like (ν_1, ν_2)-pseudo almost automorphic solution, we assume:

(**A**), (**B**), (**C**): F, G, $Q : \mathbb{R} \to \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ are respectively square-mean Stepanov-like $cl(\nu_1, \nu_2)$-pseudo almost automorphic processes.

Theorem 4.6. *Assume that* $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_4)$, $(\boldsymbol{H}_5)$, $(\boldsymbol{H}_6)$, $(\boldsymbol{A})$, $(\boldsymbol{B})$ *and* $(\boldsymbol{C})$ *hold. Then equation* (1.1) *has a unique square-mean Stepanov-like* $cl(\nu_1, \nu_2)$*-pseudo almost automorphic solution of class* r.

Proof. By (**A**)–(**C**), each $\varphi \in \{F, G, Q\}$ splits as $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1^b \in SAA\left(\mathbb{R}, \mathbb{L}^2\left((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})\right)\right)$ and $\varphi_2^b \in \mathcal{E}\left(\mathbb{R}, \mathbb{L}^2\left((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})\right), \nu_1, \nu_2, r\right)$. The conclusion then follows from Theorem 4.2, Theorem 4.4 and Theorem 4.5. □

We now establish the existence of a square-mean Stepanov-like (ν_1, ν_2)-pseudo almost automorphic solution of class r for the nonlinear problem

$$\mathrm{d}\vartheta(t) = [A\vartheta(t) + L(\vartheta_t) + F(t, \vartheta_t)]\,\mathrm{d}t + G(t, \vartheta_t)\mathrm{d}W(t) + Q(t)\mathrm{d}B^H(t), \quad \text{for } t \in \mathbb{R}, \tag{4.5}$$

where $F, G : \mathbb{R} \times \mathcal{C} \to \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ and $Q : \mathbb{R} \to \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ are three stochastic processes. We assume:

($\mathbf{H}_7$) (resp. ($\mathbf{H}_8$)) Let $\nu_1, \nu_2 \in \mathcal{M}$; F (resp. G): $\mathbb{R} \times C\left([-r, 0], \mathbb{L}^2(\mathbf{P}, \mathbb{H})\right) \to \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ is square-mean Stepanov-like $cl(\nu_1, \nu_2)$-pseudo almost automorphic of class r and there exists $L_F > 0$ (resp. $L_G > 0$) such that $\mathbb{E}\left\|F(t, \phi_1) - F(t, \phi_2)\right\|^2 \leq L_F \mathbb{E}\left\|\phi_1 - \phi_2\right\|^2$ (resp. the same with G, L_G) for all $t \in \mathbb{R}$ and $\phi_1, \phi_2 \in C\left([-r, 0], \mathbb{L}^2(\mathbf{P}, \mathbb{H})\right)$.

($\mathbf{H}_9$) The unstable space $E_u \equiv \{0\}$.

Recall that $K = \max\left((\overline{M}\widetilde{M}|\mathcal{P}^s|)^2, (\overline{M}\widetilde{M}|\mathcal{P}^u|)^2\right)$, with $\overline{M}, \omega$ as in Theorem 2.7 and $\widetilde{M}, \widetilde{\omega}$ as in Lemma 2.4.

Theorem 4.7. *Assume that* $(\boldsymbol{H}_0)$–$(\boldsymbol{H}_9)$ *and* $(\boldsymbol{C})$ *hold. If* $C(\Delta, \omega)\,(L_F + L_G) < 1$, *where* $\Delta = \overline{M}^2\widetilde{M}^2|\mathcal{P}^s|^2$ *and* $C(\Delta, \omega) = \max\left(C^1(\Delta, \omega), C^2(\Delta, \omega)\right)$ *with* $C^1(\Delta, \omega) = \dfrac{\Delta}{2\omega}\left(e^{2\omega} + 1\right)\dfrac{e^{-2\omega}}{1 - e^{-2\omega}}$ *and* $C^2(\Delta, \omega) = 2K\dfrac{e^{-2\omega}}{1 - e^{-2\omega}}$, *then equation* (4.5) *has a unique square-mean Stepanov-like* $cl(\nu_1, \nu_2)$*-pseudo almost automorphic solution of class* r.

Proof. Fix $\vartheta \in S^2PAA_c(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$. Since F, G act on the history segment ϑ_t, Theorem 3.9 gives that $t \mapsto \vartheta_t$ belongs to $S^2PAA_c\left(C([-r, 0], \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r\right)$, and Theorem 3.7 implies that $F(\cdot, \vartheta_\cdot)$ and $G(\cdot, \vartheta_\cdot)$ are in $S^2PAA_c(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$. Since $E_u \equiv \{0\}$, we get $\mathcal{P}^u \equiv 0$. Consider the mapping $\mathcal{H}$ on $S^2PAA_c(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$ defined by

$$\begin{aligned}(\mathcal{H}\vartheta)(t) = \lim_{\lambda \to +\infty} \Big[&\int_{-\infty}^{t} \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 F(s, \vartheta_s))\mathrm{d}s \\ &+ \int_{-\infty}^{t} \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 G(s, \vartheta_s))\mathrm{d}W(s) \\ &+ \int_{-\infty}^{t} \mathcal{Y}^s(t-s)\mathcal{P}^s(\widetilde{R}_\lambda J_0 Q(s))\mathrm{d}B^H(s)\Big](0).\end{aligned}$$

In view of Theorems 4.2, 4.4 and 4.5, the claim reduces to showing that $\mathcal{H}$ has exactly one fixed point in $S^2PAA_c(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$. Let $\vartheta_1, \vartheta_2 \in S^2PAA_c(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$. Then $(\mathcal{H}\vartheta_1)(t) - (\mathcal{H}\vartheta_2)(t)$ is given by the first two terms above, computed with $F(s, \vartheta_{1s}) - F(s, \vartheta_{2s})$ and $G(s, \vartheta_{1s}) - G(s, \vartheta_{2s})$.

For each $n = 1, 2, 3, \ldots$, let $X_n(t)$ denote the same expression with integrals restricted to $[t-n, t-n+1]$. Using Itô's isometry,

$$\mathbb{E}\|X_n(t)\|^2 \le \Delta\mathbb{E}\left[\int_{t-n}^{t-n+1} e^{-\omega(t-s)}\|F(s,\vartheta_{1s}) - F(s,\vartheta_{2s})\|\,\mathrm{d}s\right]^2 + \Delta\mathbb{E}\left[\int_{t-n}^{t-n+1} e^{-2\omega(t-s)}\|G(s,\vartheta_{1s}) - G(s,\vartheta_{2s})\|^2\,\mathrm{d}s\right],$$

and by the Cauchy–Schwarz inequality

$$\mathbb{E}\|X_n(t)\|^2 \le C(\Delta,\omega)\big(L_F + L_G\big)\int_{t-n}^{t-n+1}\mathbb{E}\|\vartheta_{1s} - \vartheta_{2s}\|^2\,\mathrm{d}s.$$

Since $\sum_{n=1}^{\infty} X_n = \mathcal{H}\vartheta_1 - \mathcal{H}\vartheta_2$, we have

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}\mathbb{E}\|(\mathcal{H}\vartheta_1)(s) - (\mathcal{H}\vartheta_2)(s)\|^2\,\mathrm{d}s \le C(\Delta,\omega)(L_F+L_G)\|\vartheta_1 - \vartheta_2\|_{S^2}^2.$$

So $\mathcal{H}$ is a strict contraction, and the contraction mapping principle yields exactly one fixed point $u \in S^2PAA_c(\mathbb{R}, \mathbb{L}^2(\mathbf{P},\mathbb{H}), \nu_1, \nu_2, r)$, which is the unique square-mean Stepanov-like $cl(\nu_1,\nu_2)$-pseudo almost automorphic solution of class r of (4.5). □

5. Application

To illustrate our results, we consider the model

$$\begin{cases} \mathrm{d}w(t,x) = \dfrac{\partial^2}{\partial x^2}w(t,x)\,\mathrm{d}t + \Big[\displaystyle\int_{-r}^{0}\kappa(\tau)w(t+\tau,x)\,\mathrm{d}\tau + \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right) \\ \qquad\qquad + \arctan(t) + \displaystyle\int_{-r}^{0} e^{\omega\tau}h\big(\tau, w(t+\tau,x)\big)\,\mathrm{d}\tau\Big]\mathrm{d}t \\ \qquad\quad + \Big[\cos\left(\dfrac{1}{2+\sin(t)+\sin(\sqrt{2}t)}\right) + \arctan(t) \\ \qquad\qquad + \displaystyle\int_{-r}^{0} e^{\omega\tau}h\big(\tau, w(t+\tau,x)\big)\,\mathrm{d}\tau\Big]\mathrm{d}W(t) \\ \qquad\quad + \Big[\cos(t) + \cos(\sqrt{2}t) + \arctan(t)\Big]\,\mathrm{d}B^H(t), \\ w(t,0) = w(t,\pi) = 0, \qquad t\in\mathbb{R}. \end{cases} \tag{5.1}$$

Here $r \ge 0$, the kernel $\kappa : [-r, 0] \to \mathbb{R}$ is continuous, and $h : [-r, 0] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz in its second variable and stochastically bounded in the square-mean sense.

$$\kappa(\tau) = \frac{\tau^2 - 1}{(\tau^2+1)^2} \text{ for } \tau \in [-r, 0] \quad \text{and} \quad h(\tau, x) = \tau^2 + \sin\left(\frac{x}{4}\right) \text{ for } (\tau, x) \in [-r, 0]\times\mathbb{R}.$$

These functions κ and h are those used in [28]; κ is continuous and $\mathbb{E}\|h(\tau,x_1) - h(\tau,x_2)\|^2 \le \frac{1}{16}\mathbb{E}\|x_1 - x_2\|^2$, so h is Lipschitz in x, uniformly in τ. W and B^H are as in Section 1. We work in $\mathbb{H} = L^2((0,\pi))$ with $Ay = y''$, $D(A) = H^2(0,\pi)\cap H_0^1(0,\pi)$. The operator A generates the C_0-semigroup $(T(t)x)(y) = \sum_{n=1}^{+\infty} e^{-n^2t}\langle x, e_n\rangle_{L^2} e_n(y)$, where $e_n(y) = \sqrt{2/\pi}\,\sin(ny)$, and $\|T(t)\| \le e^{-t}$ for $t \ge 0$. Hence, one may take $M = \omega = 1$, and $(\mathbf{H}_0)$, $(\mathbf{H}_1)$ hold.

We define $F, G : \mathbb{R} \times \mathcal{C} \to L^2((0,\pi))$, $Q : \mathbb{R} \to L^2((0,\pi))$ and $L : \mathcal{C} \to \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ by

$$F(t,\varphi)(x) = \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right) + \arctan(t) + \int_{-r}^{0} e^{\omega\tau} h(\tau, \varphi(\tau)(x))\,\mathrm{d}\tau,$$

$$G(t,\varphi)(x) = \cos\left(\frac{1}{2+\sin(t)+\sin(\sqrt{2}t)}\right) + \arctan(t) + \int_{-r}^{0} e^{\omega\tau} h(\tau, \varphi(\tau)(x))\,\mathrm{d}\tau,$$

$$Q(t)(x) = \cos(t) + \cos(\sqrt{2}t) + \arctan(t), \qquad L(\phi)(x) = \int_{-r}^{0} \kappa(\tau)\phi(\tau)(x)\,\mathrm{d}\tau,$$

for $x \in (0,\pi)$ and $t \in \mathbb{R}$.

Setting $\vartheta(t) = w(t,x)$, equation (5.1) takes the abstract form (4.5). Let $\mathrm{d}\nu_1(t) = \rho_1(t)\mathrm{d}t$ and $\mathrm{d}\nu_2(t) = \rho_2(t)\mathrm{d}t$, with $\rho_1(t) = 1$ for $t > 0$, $\rho_1(t) = e^t$ for $t \le 0$, and $\rho_2(t) = |t|$. From [8], $\nu_1, \nu_2 \in \mathcal{M}$ satisfy $(\mathbf{H}_3)$, and $(\mathbf{H}_2)$ holds since $\dfrac{\nu_1([-\ell,\ell])}{\nu_2([-\ell,\ell])} = \dfrac{1-e^{-\ell}+\ell}{\ell^2} \to 0$. Since $2+\cos(t)+\cos(\sqrt{2}t)$ never vanishes, $t \mapsto \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right)$ is bounded, continuous and compact almost automorphic, hence square-mean Stepanov-like almost automorphic (see [22]).

For $s \in [0,1]$ and $\tau \in [t-r, t]$, we have $\arctan(t-r) \le \arctan(\tau+s) \le \frac{\pi}{2}$, hence

$$\sup_{\tau\in[t-r,t]} \Big(\int_0^1 \mathbb{E}|\arctan(\tau+s) - \frac{\pi}{2}|^2 \mathrm{d}s\Big)^{\frac{1}{2}} \le \frac{\pi}{2} - \arctan(t-r) \le \pi.$$

Since $\mathrm{d}\nu_1(t) = \mathrm{d}t$ for $t > 0$, $\mathrm{d}\nu_1(t) = e^t\mathrm{d}t$ for $t \le 0$ and $\nu_2([-\ell,\ell]) = \ell^2$, we get

$$\frac{1}{\nu_2([-\ell,\ell])}\int_{-\ell}^{\ell} \sup_{\tau\in[t-r,t]} \Big(\int_0^1 \mathbb{E}|\arctan(\tau+s) - \frac{\pi}{2}|^2 \mathrm{d}s\Big)^{\frac{1}{2}} \mathrm{d}\nu_1(t) \le \frac{\pi}{\ell} + \frac{\pi(1-e^{-\ell})}{\ell^2} \longrightarrow 0$$

as $\ell \to +\infty$. Hence $t \mapsto \arctan(t) - \frac{\pi}{2}$ lies in $\mathcal{E}S^2(\mathbb{R}, \mathbb{L}^2(\mathbf{P}, \mathbb{H}), \nu_1, \nu_2, r)$, since it belongs to $\mathcal{E}(\mathbb{R}, \mathbb{L}^2((0,1), \mathbb{L}^2(\mathbf{P}, \mathbb{H})), \nu_1, \nu_2, r)$.

For $\varphi_1, \varphi_2 \in \mathcal{C}$, the deterministic parts of F cancel in the difference, and the Cauchy–Schwarz inequality on $[-r, 0]$ gives

$$\begin{aligned}
&\|F(t,\varphi_1)(x) - F(t,\varphi_2)(x)\|^2 \\
&\le \Big(\int_{-r}^{0} e^{2\omega\tau}\mathrm{d}\tau\Big)\Big(\int_{-r}^{0} \|h(\tau,\varphi_1(\tau)(x)) - h(\tau,\varphi_2(\tau)(x))\|^2 \mathrm{d}\tau\Big) \\
&\le C(\omega) \sup_{\tau\in[-r,0]} \|\varphi_1(\tau)(x) - \varphi_2(\tau)(x)\|^2,
\end{aligned}$$

where $C(\omega) := \frac{rL_h(1-e^{-2\omega r})}{2\omega}$. Taking expectations, $\mathbb{E}\|F(t,\varphi_1)(x) - F(t,\varphi_2)(x)\|^2 \le C(\omega)\|\varphi_1(x) - \varphi_2(x)\|_{\mathcal{C}}^2$. The history term of G is the same as that of F, so the same bound holds for G. In the same way, $t \mapsto \cos\left(\frac{1}{2+\sin(t)+\sin(\sqrt{2}t)}\right)$ is square-mean Stepanov-like almost automorphic. Thus F and G are uniformly square-mean Stepanov-like (ν_1,ν_2)-pseudo almost automorphic of class r and satisfy $(\mathbf{H}_7)$ and $(\mathbf{H}_8)$ with $L_F = L_G = C(\omega)$. Finally, $L : \mathcal{C} \to \mathbb{L}^2(\mathbf{P}, \mathbb{H})$ is linear and bounded, since κ is continuous on $[-r, 0]$.

Moreover, h is stochastically bounded, say $\mathbb{E}\|h(t,\phi(t))\|^2 \le \beta_h$, and the Cauchy–Schwarz inequality gives, as above, $\mathbb{E}\big\|\int_{-r}^{0} e^{\omega\tau} h(\tau,\varphi(\tau)(x))\,\mathrm{d}\tau\big\|^2 \le \frac{r\beta_h(1-e^{-2\omega r})}{2\omega} =: \gamma_G$. Since $|\cos| \le 1$ and $|\arctan(t)| \le \frac{\pi}{2}$, the inequality

$(a+b)^2 \le 2a^2+2b^2$ yields $\mathbb{E}\|G(t,\varphi)(x)\|^2 \le 2\big(1+\frac{\pi}{2}\big)^2 + 2\gamma_G =: \beta_1$, so G satisfies $(\mathbf{H}_5)$.

We write $Q(t)(x) = \big[\cos(t)+\cos(\sqrt{2}t)+\frac{\pi}{2}\big]+\big[\arctan(t)-\frac{\pi}{2}\big]$. The first bracket is almost periodic, hence almost automorphic (see [32]) and square-mean Stepanov-like almost automorphic, and the second one lies in $\mathcal{ES}^2(\mathbb{R},\mathbb{L}^2(\mathbf{P},\mathbb{H}),\nu_1,\nu_2,r)$ by the above. Hence Q is uniformly square-mean Stepanov-like (ν_1,ν_2)-pseudo almost automorphic of class r. Moreover, $\mathbb{E}\|Q(t)\|^2 = \pi\big(\cos(t)+\cos(\sqrt{2}t)+\arctan(t)\big)^2 \le \pi\big(2+\frac{\pi}{2}\big)^2 =: \beta_2$, so Q satisfies $(\mathbf{H}_6)$.

Lemma 5.1. [21] *If* $\int_{-r}^{0}|\kappa(\tau)|\mathrm{d}\tau < 1$, *then the semigroup* $(\mathcal{Y}(t))_{t\ge0}$ *is hyperbolic and the unstable space* $E_u \equiv \{0\}$, *i.e.* $(\boldsymbol{H}_9)$ *holds.*

Since $-\kappa(\tau)$ is the derivative of $\frac{\tau}{\tau^2+1}$, we get $\displaystyle\int_{-r}^{0}|\kappa(\tau)|\mathrm{d}\tau = \frac{r}{r^2+1}$ for $0\le r<1$ and, splitting at $\tau=-1$, $\displaystyle\int_{-r}^{0}|\kappa(\tau)|\mathrm{d}\tau = 1-\frac{r}{r^2+1}$ for $r\ge1$. In either case $\displaystyle\int_{-r}^{0}|\kappa(\tau)|\mathrm{d}\tau < 1$.

Theorem 5.2. *Suppose that the conditions above are satisfied. If the Lipschitz constant* L_h *of* h *is sufficiently small, then* (5.1) *admits exactly one square-mean Stepanov-like* $cl(\nu_1,\nu_2)$*-pseudo almost automorphic solution of class* r.

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