Gulf Journal of Mathematics
Vol 24, Issue 2 (2026) 1-23

ISSN: 2309-4966
https://doi.org/10.56947/7e7ygm18

STOCHASTIC DENGUE ENDEMICITY UNDER LÉVY JUMP FLUCTUATIONS

SAWADOGO RAGNIMWENDE[1*] , OUEDRAOGO HAROUNA[2] and KONANE FOURTOUA VICTORIEN[2]

Abstract. Dengue resurgence poses major public-health challenges. Because transmission is affected by environmental and random fluctuations, realistic models are needed. We propose a stochastic dengue model, with a susceptible infectious recovered structure for humans and a susceptible infectious structure for mosquitoes, incorporating white noise and Lévy jumps. We establish the existence and uniqueness of a global positive solution, prove moment and almost sure exponential stability of the disease-free equilibrium, derive sufficient conditions for persistence in mean, and illustrate the results numerically. The analysis clarifies how random fluctuations affect dengue transmission and its long-term variability.

Keywords. White noise; Lévy jumps; Almost surely exponentially stable; Extinction; Persistence in mean; Dengue
2020 Mathematics Subject Classification. 60G05; 60G07; 60G51

1. Introduction

Dengue fever is a mosquito-borne viral disease, endemic in many tropical and subtropical regions [9], and transmitted mainly by female *Aedes aegypti* mosquitoes. The virus belongs to the genus *Flavivirus* and has four serotypes, DENV-1 to DENV-4 [3]. It affects all age groups, ranks among the most severe vector-borne diseases, after malaria [10], and imposes a heavy economic burden [12]. Since no specific antiviral treatment exists [11], understanding transmission dynamics is crucial. This has motivated numerous deterministic models [13, 8, 26], which, however, cannot capture random environmental effects. Stochastic dengue models have therefore been developed [18, 21, 23]. White noise is adequate for small fluctuations but cannot represent sudden, strong disturbances such as floods or heatwaves [19, 28]. Lévy processes, which allow jumps, are well suited to such events [2, 24, 27].
This work differs from the existing literature in three respects. First, instead of adding independent noise to each compartment, we introduce white and Lévy

Date: Received: Sep 27, 2026; Revised: Oct 25, 2026; Accepted: Oct 31, 2026.
* Corresponding author
© The Author(s) 2026. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit https://creativecommons.org/licenses/by-nc-nd/4.0/.

noise directly into the incidence function, so that randomness acts on the rate of passage from susceptible to infected. Second, this noise enters the equations of S and J with opposite signs and cancels in their sum; hence the total human and vector populations remain deterministic, and the randomness creates no artificial births or migrations. Third, the Lévy jumps model sudden one-time events, such as a flood altering mosquito populations, an exceptional vector-control campaign, or a massive influx of people from an endemic area.
The paper is organized as follows. Section 2 gives the necessary preliminaries. Section 3 recalls the basic model [20] and introduces the new one. Section 4 proves the existence of a unique global positive solution lying in a positively invariant set. Section 5 establishes moment and almost sure exponential stability of the disease-free equilibrium, and Section 6 gives sufficient conditions for persistence in mean. Numerical simulations are presented in Section 7.

2. Preliminaries

Throughout, let $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\big)$ be a complete filtered probability space satisfying the usual conditions, and set

$$\Gamma_0 := \big\{(y_1, \dots, y_5) \in \mathbb{R}^5_+ : y_1 + y_2 + y_3 < 1,\ y_4 + y_5 < 1\big\}.$$

We consider the n-dimensional Lévy-driven stochastic differential equation

$$dY(t) = \alpha(t, Y(t))\, dt + \sigma(t, Y(t))\, dZ(t) + \int_{\mathbb{R}^n\setminus\{0\}} \gamma(t, Y(t-), z)\, \widetilde{N}(dt, dz), \quad (2.1)$$

with initial condition $Y(t_0) = Y_0 \in \mathbb{R}^n$, where Z is an m-dimensional standard Brownian motion, N is a Poisson random measure with intensity ν independent of Z, and $\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)\, dt$ is the associated compensated measure. For $V \in C^{2,1}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$, the infinitesimal generator $\mathcal{L}$ of the solution of (2.1) is given by the Itô formula for jump-diffusion processes; we refer to Theorem 6.7.4 in [1] for its explicit expression.

3. Model Stochastic Formulation

For convenience, we recall the baseline model of [20]. The compartments $\bar{S}_{Hu}, \bar{J}_{Hu}, \bar{R}_{Hu}$ denote susceptible, infectious and recovered humans, and $\bar{S}_{Ve}, \bar{J}_{Ve}$ susceptible and infectious vectors; N_{Hu} and N_{Ve} are the total human and vector population sizes. The parameters are $\beta_H, \gamma_H, \theta_H, \mu_H, \sigma_1, \gamma_1$ for humans and $\beta_V, \mu_V, \sigma_2, \gamma_2$ for vectors, with their usual meanings (contact, recovery, disease-induced death, natural mortality, noise amplitude and jump intensity).
We work with the proportions

$$S_{Hu} = \frac{\bar{S}_{Hu}}{N_{Hu}}, \quad J_{Hu} = \frac{\bar{J}_{Hu}}{N_{Hu}}, \quad R_{Hu} = \frac{\bar{R}_{Hu}}{N_{Hu}}, \quad S_{Ve} = \frac{\bar{S}_{Ve}}{N_{Ve}}, \quad J_{Ve} = \frac{\bar{J}_{Ve}}{N_{Ve}}.$$

Adding Brownian and Lévy noises to the contact rates β_H and β_V, the stochastic model reads

$$\begin{cases} dS_{Hu} = \big[\mu_H - \beta_H S_{Hu} J_{Ve} - \mu_H S_{Hu}\big]dt - \sigma_1 S_{Hu} J_{Hu} dW_1 - \displaystyle\int_{\mathbb{U}} \gamma_1(u) S_{Hu} J_{Hu}\, \tilde{N}(dt, du), \\ dJ_{Hu} = \big[-(\mu_H + \gamma_H + \theta_H) J_{Hu} + \beta_H S_{Hu} J_{Ve}\big]dt + \sigma_1 S_{Hu} J_{Hu} dW_1 + \displaystyle\int_{\mathbb{U}} \gamma_1(u) S_{Hu} J_{Hu}\, \tilde{N}(dt, du), \\ dR_{Hu} = -\big[\mu_H R_{Hu} - \gamma_H J_{Hu}\big]dt, \\ dS_{Ve} = \big[\mu_V - \beta_V S_{Ve} J_{Hu} - \mu_V S_{Ve}\big]dt - \sigma_2 S_{Ve} J_{Ve} dW_2 - \displaystyle\int_{\mathbb{U}} \gamma_2(u) S_{Ve} J_{Ve}\, \tilde{N}(dt, du), \\ dJ_{Ve} = \big[\beta_V S_{Ve} J_{Hu} - \mu_V J_{Ve}\big]dt + \sigma_2 S_{Ve} J_{Ve} dW_2 + \displaystyle\int_{\mathbb{U}} \gamma_2(u) S_{Ve} J_{Ve}\, \tilde{N}(dt, du), \end{cases} \tag{3.1}$$

where W_1, W_2 are independent Brownian motions, N is a Poisson counting measure on $\mathbb{U} \subset (0, \infty)$ with intensity ν, and $\tilde{N}(dt, du) = N(dt, du) - \nu(du)dt$ is the compensated measure; all integrands are evaluated at t^-. We assume W_1, W_2, N independent and

(H_1) $\gamma_i : \mathbb{U} \to \mathbb{R}_+$ is bounded and continuous, and $\bar{\Gamma}_i := \sup_{u \in \mathbb{U}} \gamma_i(u) < 1, \quad i = 1, 2;$

(H_2) $\nu(\mathbb{U}) < \infty$.

Perturbing the transmission rates would instead involve the cross terms $S_{Hu} J_{Ve}$ and $S_{Ve} J_{Hu}$, for which the standard positivity proof does not apply directly; lagged correlations between J_{Hu} and J_{Ve} could then be exploited.

4. Existence and Uniqueness of the Positive solution

In this section, we shall prove the existence and uniqueness of the global positive solution of system (3.1) for any positive initial value.

Theorem 4.1. *For all initial values* $Y(0) = (S_{Hu}(0), J_{Hu}(0), R_{Hu}(0), S_{Ve}(0), J_{Ve}(0)) \in \Gamma_0$, *there is a unique solution* $Y(t) = (S_{Hu}(t), J_{Hu}(t), R_{Hu}(t), S_{Ve}(t), J_{Ve}(t))$ *for the system* (3.1) *such that* $\mathbb{P}((t) \in \Gamma_0) = 1$ *for all* $t \geq 0$.

Proof. This proof follows the approach of Eric N'zi and Modeste N'zi [25]. Let

$$N_{Hu}(t) = S_{Hu}(t) + J_{Hu}(t) + R_{Hu}(t), \qquad N_{Ve}(t) = S_{Ve}(t) + J_{Ve}(t).$$

For any $x(s) = \big(S_{Hu}(s), J_{Hu}(s), R_{Hu}(s), S_{Ve}(s), J_{Ve}(s)\big) \in \mathbb{R}^5_+$, we have

$$\begin{aligned} N_{Hu}(s) &\leq 1 + \big(N_{Hu}(0) - 1\big)e^{-\mu_H s} \quad a.s., \\ N_{Ve}(s) &= 1 + \big(N_{Ve}(0) - 1\big)e^{-\mu_V s} \quad a.s. \end{aligned}$$

Since the initial condition lies in Γ_0, it follows that $N_{Hu}(s) < 1$ and $N_{Ve}(s) < 1$ a.s., hence $x(s) \in (0, 1)^5$ for all $s \in [0, t]$. As the coefficients of system (3.1) are locally Lipschitz, there exists a unique local solution on $[0, t]$. Denoting this solution by

$$Y(t) = \big(S_{Hu}(t), J_{Hu}(t), R_{Hu}(t), S_{Ve}(t), J_{Ve}(t)\big),$$

it is defined on $[0, \tau_e)$, where τ_e is the explosion time.
Let us introduce, for the purpose of the proof, the following relations $j \in \{1, 2\}$ et $p \geq 1$:

$$\tilde{M}_j(p) := \int_U \bar{\Gamma}_j^p \, \nu(du) < +\infty \quad \text{and} \quad \varepsilon_j = 1 - \bar{\Gamma}_k.$$

So :

$$\begin{aligned} 1 - \gamma_1(u) J_{Hu}(s-) \geq \varepsilon_1 \quad \text{and} \quad 1 - \gamma_2(u) J_{Ve}(s-) \geq \varepsilon_2, \\ 1 + \gamma_1(u) S_{Hu}(s-) \geq \varepsilon_1 \quad \text{and} \quad 1 + \gamma_2(u) S_{Ve}(s-) \geq \varepsilon_2. \end{aligned}$$

Let the stopping time τ^* be defined by :

$$\tau^* = \inf \left\{ t \in [0, \tau_e) : \min \left(S_{Hu}(t), J_{Hu}(t), R_{Hu}(t), S_{Ve}(t), J_{Ve}(t) \right) \leq 0 \right\} \wedge \tau_e$$
$$\text{where } \inf\{\varnothing\} = \infty.$$

Suppose for the sake of contradiction that $\mathbb{P}(\tau^* < \tau_e) > 0$. Let us define the $\mathcal{C}^2$-function by

$$\tilde{\mathcal{Q}}(Y(t)) = \ln \left(S_{Hu}(t) J_{Hu}(t) R_{Hu}(t) S_{Ve}(t) J_{Ve}(t) \right).$$

Applying Itô's formula, we obtain for all $t \in [0, \tau^*)$, we get :

$$\begin{aligned} \tilde{\mathcal{Q}}(Y(t)) - \tilde{\mathcal{Q}}(Y(0)) = \int_0^t \Bigg[& \frac{\mu_H}{S_{Hu}(s)} + \beta_H \frac{S_{Hu}(s) J_{Ve}(s)}{J_{Hu}(s)} + \gamma_H \frac{J_{Hu}(s)}{R_{Hu}(s)} + \frac{\mu_V}{S_{Ve}(s)} \\ & + \beta_V \frac{S_{Ve}(s) J_{Hu}(s)}{J_{Ve}(s)} - (3\mu_H + \gamma_H + \theta_H + 2\mu_V) - \beta_H J_{Ve}(s) \\ & - \beta_V J_{Hu}(s) - \frac{1}{2}\sigma_1^2 \left(S_{Hu}^2(s) + J_{Hu}^2(s) \right) - \frac{1}{2}\sigma_2^2 \left(S_{Ve}^2(s) + J_{Ve}^2(s) \right) \Bigg] ds \\ & + \mathcal{M}(t) + \int_0^t \tilde{\mathcal{D}}\left(Y(s) \right) ds, \end{aligned}$$

where

$$\begin{aligned} \mathcal{M}(t) = \int_0^t \int_{\mathbb{U}} \Bigg[& \ln \Big(\big(1 - \gamma_1(u) J_{Hu}(s-)\big) \big(1 + \gamma_1(u) S_{Hu}(s-)\big) \Big) \\ & + \ln \Big(\big(1 - \gamma_2(u) J_{Ve}(s-)\big) \big(1 + \gamma_2(u) S_{Ve}(s-)\big) \Big) \Bigg] \tilde{N}(ds, du) \\ & + \int_0^t \sigma_1 \big(S_{Hu}(s) - J_{Hu}(s) \big) \, dW_1(s) + \int_0^t \sigma_2 \big(S_{Ve}(s) - J_{Ve}(s) \big) \, dW_2(s), \end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{D}}\big(Y(s)\big) = \int_{\mathbb{U}} \Big[&\ln\Big(1-\gamma_1(u)J_{Hu}(s-)\Big) + \gamma_1(u)J_{Hu}(s-) \\ &+ \ln\Big(1+\gamma_1(u)S_{Hu}(s-)\Big) + \gamma_1(u)S_{Hu}(s-) \\ &+ \ln\Big(1-\gamma_2(u)J_{Ve}(s-)\Big) + \gamma_2(u)J_{Ve}(s-) \\ &+ \ln\Big(1+\gamma_2(u)S_{Ve}(s-)\Big) + \gamma_2(u)S_{Ve}(s-)\Big]\,\nu(du).\end{aligned}$$

Set :

$$\begin{aligned}\tilde{\mathcal{G}}(Y(s-)) = &\ln\Big(\big(1-\gamma_1(u)J_{Hu}(s-)\big)\big(1+\gamma_1(u)S_{Hu}(s-)\big)\Big) \\ &+ \ln\Big(\big(1-\gamma_2(u)J_{Ve}(s-)\big)\big(1+\gamma_2(u)S_{Ve}(s-)\big)\Big).\end{aligned}$$

Using the fact that for all $t \in [0, \tau^*)$,
$\min\Big(S_{Hu}(t), J_{Hu}(t), R_{Hu}(t), S_{Ve}(t), J_{Ve}(t)\Big) > 0$, and in view of $S_{Hu}(t) < 1$, $J_{Hu}(t) < 1$, $R_{Hu}(t) < 1$, $S_{Ve}(t) < 1$ and $J_{Ve}(t) < 1$, we get :

$$\begin{aligned}\tilde{\mathcal{Q}}(Y(t)) - \tilde{\mathcal{Q}}(Y(0)) &\geq \int_0^{t\wedge\tau^*} \tilde{\mathcal{K}}(Y(s))\,ds + \int_0^{t\wedge\tau^*} \tilde{\mathcal{D}}(Y(s))\,ds \\ &\quad + \mathcal{M}(Y(t\wedge\tau^*)) \\ &\equiv \tilde{\mathcal{H}}(t), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned}\tilde{\mathcal{K}}(Y(s)) = &-\frac{1}{2}\sigma_1^2\left(S_{Hu}^2(s) + J_{Hu}^2(s)\right) - \frac{1}{2}\sigma_2^2\left(S_{Ve}^2(s) + J_{Ve}^2(s)\right) \\ &- (3\mu_H + \gamma_H + \theta_H + 2\mu_V) - \beta_H J_{Ve}(s) - \beta_V J_{Hu}(s).\end{aligned}$$

On the set $\{\tau^* < \tau_e\}$, we have :

$$S_{Hu}(\tau^*)J_{Hu}(\tau^*)R_{Hu}(\tau^*)S_{Ve}(\tau^*)J_{Ve}(\tau^*) = 0.$$

Hence, by path continuity of the process, $(S_{Hu}(t), J_{Hu}(t), R_{Hu}(t), S_{Ve}(t), J_{Ve}(t))$, we have :

$$\lim_{t\to\tau^*} \tilde{\mathcal{Q}}(Y(t)) = -\infty. \tag{4.2}$$

From (4.1) and (4.2), we obtain on $\{\tau^* < \tau_e, \}$, $-\infty = \tilde{\mathcal{H}}(\tau^*)$.
Therefore,

$$\Big\{-\infty = \tilde{\mathcal{H}}(\tau^*)\Big\} \supset \{\tau^* < \tau_e, \}. \tag{4.3}$$

Let $\varepsilon > 0$ and $y \in (-1, \infty)$ satisfy $1 + y \geq \varepsilon$. By the mean value theorem, we deduce that

$$|\ln(1+y)| \leq \frac{1}{\varepsilon}|y|. \tag{4.4}$$

By applying relation (4.4) with $y \in \{-\gamma_1 J_{Hu}, \gamma_1 S_{Hu}, -\gamma_2 J_{Ve}, \gamma_2 S_{Ve}\}$ to the various logarithmic terms of the integral over $\mathbb{U}$, combined with the inequality

$(\alpha+\beta)^2 \leq 2(\alpha^2+\beta^2)$ and taking into account the lower bounds by ε_1 and ε_2, we obtain:

$$\begin{aligned}
\Big|\int_{\mathbb{U}} \tilde{\mathcal{G}}(Y(s-))\nu(du)\Big|^2 &\leq 2\left[\frac{1}{\varepsilon_1}\int_{\mathbb{U}} \gamma_1(u)\big(J_{Hu}(s-)+S_{Hu}(s-)\big)\,\nu(du)\right]^2 \\
&\quad + 2\left[\frac{1}{\varepsilon_2}\int_{\mathbb{U}} \gamma_2(u)\big(J_{Ve}(s-)+S_{Ve}(s-)\big)\,\nu(du)\right]^2 \\
&\leq \frac{8}{\varepsilon_1^2}\left(\int_{\mathbb{U}} \gamma_1(u)\,\nu(du)\right)^2 + \frac{8}{\varepsilon_2^2}\left(\int_{\mathbb{U}} \gamma_2(u)\,\nu(du)\right)^2 \\
&\leq \frac{8}{\varepsilon_1^2}\tilde{M}_1(1)^2 + \frac{8}{\varepsilon_2^2}\tilde{M}_2(1)^2 \\
&\equiv \bar{D}_0.
\end{aligned}$$

By combining the Cauchy-Schwarz inequality with Itô's isometry, it follows that:

$$\begin{aligned}
&\mathbb{E}\left[\tilde{\mathcal{H}}(\tau^*) - \int_0^{\tau^*} \tilde{\mathcal{K}}(Y(s))\,ds - \int_0^{\tau^*} \tilde{\mathcal{D}}(Y(s-))\,ds\right]^2 \\
&\leq 3\sigma_1^2\mathbb{E}\left(\int_0^{\tau^*}\big(S_{Hu}(s)-J_{Hu}(s)\big)\,dW_1(s)\right)^2 + 3\sigma_2^2\mathbb{E}\left(\int_0^{\tau^*}\big(S_{Ve}(s)-J_{Ve}(s)\big)\,dW_2(s)\right)^2 \\
&\quad + 3\mathbb{E}\left(\int_0^{\tau^*}\int_{\mathbb{U}} \tilde{\mathcal{G}}(Y(s-))\,\tilde{N}(ds,du)\right)^2 \\
&\leq 3\sigma_1^2\mathbb{E}\int_0^{\tau^*}\big(S_{Hu}(s)-J_{Hu}(s)\big)^2\,ds + 3\sigma_2^2\mathbb{E}\int_0^{\tau^*}\big(S_{Ve}(s)-J_{Ve}(s)\big)^2\,ds \\
&\quad + 3\mathbb{E}\int_0^{\tau^*}\int_{\mathbb{U}}\big|\tilde{\mathcal{G}}(Y(s-))\big|^2\,\nu(du)\,ds \\
&\leq 6\sigma_1^2\mathbb{E}\int_0^{\tau^*}\big(S_{Hu}^2(s)+J_{Hu}^2(s)\big)\,ds + 6\sigma_2^2\mathbb{E}\int_0^{\tau^*}\big(S_{Ve}^2(s)+J_{Ve}^2(s)\big)\,ds \\
&\quad + 3\mathbb{E}\int_0^{\tau^*}\int_{\mathbb{U}}\big|\tilde{\mathcal{G}}(Y(s-))\big|^2\,\nu(du)\,ds \\
&\leq 12\big(\sigma_1^2+\sigma_2^2\big)\mathbb{E}[\tau^*] + 3\bar{D}_0\mathbb{E}[\tau^*] \\
&\leq \big[12\big(\sigma_1^2+\sigma_2^2\big)+3\bar{D}_0\big]\,\mathbb{E}[\tau^*].
\end{aligned}$$

For all $s\in[0;\tau^*]$, we have :

$$\int_0^{\tau^*}\Big|\tilde{\mathcal{K}}(Y(s))\Big|\,ds \leq \left[(3\mu_H+\gamma_H+\theta_H+2\mu_V)+\beta_H+\beta_V+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2\right]\tau^*,$$

$$\mathbb{E}\left[\int_0^{\tau^+}\Big|\tilde{\mathcal{K}}(Y(s))\Big|\,ds\right] \leq \left[(3\mu_H+\gamma_H+\theta_H+2\mu_V)+\beta_H+\beta_V+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2\right]\mathbb{E}\left(\tau^*\right).$$

Furthermore, combining a second-order Taylor–Lagrange expansion with the lower bounds ε_i and the soundness of the variables by 1, we get:

$$\begin{aligned}\int_0^{\tau^*} \left|\tilde{\mathcal{D}}(Y(s))\right| ds &\leq \frac{1}{2\varepsilon_1^2}\int_0^{\tau^*}\int_{\mathbb{U}} \gamma_1^2(u)\Big[J_{Hu}^2(s-)+S_{Hu}^2(s-)\Big]\,\nu(du)\,ds \\ &\quad+\frac{1}{2\varepsilon_2^2}\int_0^{\tau^*}\int_{\mathbb{U}} \gamma_2^2(u)\Big[J_{Ve}^2(s-)+S_{Ve}^2(s-)\Big]\,\nu(du)\,ds \\ &\leq \left[\frac{\tilde{M}_1(2)}{\varepsilon_1^2}+\frac{\tilde{M}_2(2)}{\varepsilon_2^2}\right]\tau^*.\end{aligned}$$

It follows that:

$$\mathbb{E}\left[\int_0^{\tau^*} \left|\tilde{\mathcal{D}}(Y(s))\right| ds\right] \leq \left[\frac{\tilde{M}_1(2)}{\varepsilon_1^2}+\frac{\tilde{M}_2(2)}{\varepsilon_2^2}\right]\mathbb{E}[\tau^*].$$

On $\{\tau^* < \tau_e\}$, we have $\mathbb{E}[\tau^*] < \infty$.
This implies that :

$$\mathbb{E}\left[\tilde{\mathcal{H}}\left(\tau^+\right) - \int_0^{\tau^*}\left(\tilde{\mathcal{K}}(Y(s))+\tilde{\mathcal{D}}(Y(s))\right)ds\right]^2 < \infty,$$

$$\mathbb{E}\left[\int_0^{\tau^*}\left|\tilde{\mathcal{K}}(Y(s))\right| ds\right] < +\infty \text{ and } \mathbb{E}\left[\int_0^{\tau^*}\left|\tilde{\mathcal{D}}(Y(s))\right| ds\right] < \infty.$$

Consequently,

$$\begin{aligned}\mathbb{E}\left[\left|\tilde{\mathcal{H}}\left(\tau^*\right)\right|\right] &\leq \mathbb{E}\left[\left|\tilde{\mathcal{H}}\left(\tau^*\right) - \int_0^{\tau^*}\left(\tilde{\mathcal{K}}(Y(s))+\tilde{\mathcal{D}}(Y(s))\right)ds\right|\right] \\ &\quad+\mathbb{E}\left[\int_0^{\tau^+}\left|\left(\tilde{\mathcal{K}}(Y(s))+\tilde{\mathcal{D}}(Y(s))\right)\right| ds\right] \\ &< \infty.\end{aligned}$$

Thus, $\tilde{\mathcal{H}}\left(\tau^*\right)$ is therefore almost surely finite on the event $\{\tau^+ < \tau_e\}$. Relation (4.3) implies that

$$\begin{aligned}\mathbb{P}\left\{\tau^* < \tau_e\right\} &\leq \mathbb{P}\left\{-\infty = H\left(\tau^*\right)\right\} \\ &\leq 0,\end{aligned}$$

yields a contradiction with the initial hypothesis. Hence, $\mathbb{P}\left(\{\tau^* < \tau_e\}\right) = 0$ and $\tau^* = \tau_e$ a.s.
To prove that the solution is global, let k_0 be sufficiently large such that $S_{Hu}(0), J_{Hu}(0), R_{Hu}(0), S_{Ve}(0), J_{Ve}(0) \in]\frac{1}{k_0}, 1]$. For each $k \geq k_0$, let us define the stopping time

$$\tau_k := \inf\left\{t \in [0, \tau_e[\colon \min\left(S_{Hu}(t), J_{Hu}(t), R_{Hu}(t), S_{Ve}(t), J_{Ve}(t)\right) \leq \frac{1}{k}\right\} \wedge \tau_e,$$

where $\inf \varnothing = \infty$. The sequence $(\tau_k)_{k\geq k_0}$ is increasing almost surely and converges to τ^*. Hence, global existence follows once we show that $\tau^* = \infty$ a.s. Define the

function $\mathcal{Q}$ for all $Y(t) = (S_{Hu}(t), J_{Hu}(t), R_{Hu}(t), S_{Ve}(t), J_{Ve}(t)) \in \mathbb{R}^5_+$ by

$$\mathcal{Q}(Y(t)) = \sum_{i=1}^{5} \frac{1}{Y_i(t)}.$$

By Itô's formula, for all $t \geq 0$ and $s \in [0, t \wedge \tau_k]$, we have:

$$\begin{aligned}
\mathrm{d}\mathcal{Q}(Y(s)) = \Bigg[& -\frac{\mu_H}{S^2_{Hu}(s)} - \frac{\beta_H S_{Hu}(s) J_{Ve}(s)}{J^2_{Hu}(s)} - \frac{\gamma_H J_{Hu}(s)}{R^2_H(s)} - \frac{\mu_V}{S^2_{Ve}(s)} - \frac{\beta_V S_{Ve}(s) J_{Hu}(s)}{J^2_{Ve}(s)} \\
& + \frac{\mu_H}{R_H(s)} + \frac{\beta_H J_{Ve}(s) + \mu_H + \sigma_1^2 J^2_{Hu}(s)}{S_{Hu}(s)} + \frac{\mu_H + \gamma_H + \theta_H + \sigma_1^2 S^2_{Hu}(s)}{J_{Hu}(s)} \\
& + \frac{\beta_V J_{Hu}(s) + \mu_V + \sigma_2^2 J^2_{Ve}(s)}{S_{Ve}(s)} + \frac{\mu_V + \sigma_2^2 S^2_{Ve}(s)}{J_{Ve}(s)} \Bigg] \mathrm{d}s + \mathrm{d}\mathcal{M}(s) \\
& + \int_{\mathbb{U}} \Bigg(\Bigg[\frac{\gamma_1^2(u) J^2_{Ve}(s-)}{S_{Hu}(s-)\big(1 - \gamma_1(u) J_{Hu}(s-)\big)} + \frac{\gamma_1^2(u) S^2_{Hu}(s-)}{J_{Hu}(s-)\big(1 + \gamma_1(u) S_{Hu}(s-)\big)} \\
& + \frac{\gamma_2^2(u) J^2_{Ve}(s-)}{S_{Ve}(s-)\big(1 - \gamma_2(u) J_{Ve}(s-)\big)} + \frac{\gamma_2^2(u) S^2_{Ve}(s-)}{J_{Ve}(s-)\big(1 + \gamma_2(u) S_{Ve}(s-)\big)} \Bigg] \nu(\mathrm{d}u) \Bigg) \mathrm{d}s,
\end{aligned} \tag{4.5}$$

where $\mathcal{M}(s)$ is given by

$$\begin{aligned}
\mathrm{d}\mathcal{M}(s) = & \sigma_1 \left(\frac{J_{Hu}(s)}{S_{Hu}(s)} - \frac{S_{Hu}(s)}{J_{Hu}(s)} \right) \mathrm{d}W_1(s) + \sigma_2 \left(\frac{J_{Ve}(s)}{S_{Ve}(s)} - \frac{S_{Ve}(s)}{J_{Ve}(s)} \right) \mathrm{d}W_2(s) \\
& + \int_{\mathbb{U}} \Bigg[\frac{\gamma_1(u) J_{Hu}(s-)}{S_{Hu}(s-)\big(1 - \gamma_1(u) J_{Hu}(s-)\big)} - \frac{\gamma_1(u) S_{Hu}(s-)}{J_{Hu}(s-)\big(1 + \gamma_1(u) S_{Hu}(s-)\big)} \\
& + \frac{\gamma_2(u) J_{Ve}(s-)}{S_{Ve}(s-)\big(1 - \gamma_2(u) J_{Ve}(s-)\big)} - \frac{\gamma_2(u) S_{Ve}(s-)}{J_{Ve}(s-)\big(1 + \gamma_2(u) S_{Ve}(s-)\big)} \Bigg] \tilde{N}(\mathrm{d}s, \mathrm{d}u).
\end{aligned}$$

By omitting the five negative drift terms in (4.5) and invoking the upper bounds

$$1 - \gamma_1(u) J_{Hu} \geq \varepsilon_1, \quad 1 + \gamma_1(u) S_{Hu} \geq \varepsilon_1, \quad 1 - \gamma_2(u) J_{Ve} \geq \varepsilon_2, \quad 1 + \gamma_2(u) S_{Ve} \geq \varepsilon_2,$$

together with the observation that

$$S_{Hu}(t), J_{Hu}(t), R_{Hu}(t), S_{Ve}(t), J_{Ve}(t) \in (0, 1),$$

it follows that

$$\begin{aligned}
\mathrm{d}\mathcal{Q}(Y(s)) \leq & \Big[\left(\beta_H + \mu_H + \sigma_1^2\right) S^{-1}_{Hu}(s) + \left(\mu_H + \gamma_H + \theta_H + \sigma_1^2\right) J^{-1}_{Hu}(s) + \mu_H R^{-1}_{Hu}(s) \\
& + \left(\beta_V + \mu_V + \sigma_2^2\right) S^{-1}_{Ve}(s) + \left(\mu_V + \sigma_2^2\right) J^{-1}_{Ve}(s) \Big] \mathrm{d}s + \mathrm{d}\mathcal{M}(s) \\
& + \left(\int_{\mathbb{U}} \left[\frac{\gamma_1^2(u)}{\varepsilon_1 S_{Hu}(s)} + \frac{\gamma_1^2(u)}{\varepsilon_1 J_{Hu}(s)} + \frac{\gamma_2^2(u)}{\varepsilon_2 S_{Ve}(s)} + \frac{\gamma_2^2(u)}{\varepsilon_2 J_{Ve}(s)} \right] \nu(\mathrm{d}u) \right) \mathrm{d}s.
\end{aligned}$$

Let us define:

$$\tilde{R} := \left(\beta_H + \mu_H + \sigma_1^2 + \frac{\tilde{M}_1(2)}{\varepsilon_1}\right) \vee \left(\mu_H + \gamma_H + \theta_H + \sigma_1^2 + \frac{\tilde{M}_1(2)}{\varepsilon_1}\right) \vee \mu_H \\ \vee \left(\beta_V + \mu_V + \sigma_2^2 + \frac{\tilde{M}_2(2)}{\varepsilon_2}\right) \vee \left(\mu_V + \sigma_2^2 + \frac{\tilde{M}_2(2)}{\varepsilon_2}\right).$$

It follows that

$$\mathrm{d}\mathcal{Q}(Y(s)) \leq \tilde{R}\mathcal{Q}(Y(s))\,\mathrm{d}s + \mathrm{d}\mathcal{M}(s). \tag{4.6}$$

Taking the expectation on both sides of (4.6) and applying Fubini's theorem yields

$$\mathbb{E}[\mathcal{Q}(Y(s))] \leq \mathcal{Q}(Y(0)) + \tilde{R}\int_0^s \mathbb{E}[\mathcal{Q}(Y(\zeta))]\,\mathrm{d}\zeta.$$

By Gronwall's inequality, we obtain

$$\mathbb{E}[\mathcal{Q}(Y(s))] \leq \mathcal{Q}(Y(0))\exp(\tilde{R}s), \quad \forall s \in [0, t\wedge\tau_k].$$

Consequently, for all $t \geq 0$, we have

$$\begin{aligned}\mathbb{E}[\mathcal{Q}(Y(t\wedge\tau_k))] &\leq \mathcal{Q}(Y(0))\exp\left(\tilde{R}\,(t\wedge\tau_k)\right)\\ &\leq \mathcal{Q}(Y(0))\exp(\tilde{R}t).\end{aligned} \tag{4.7}$$

Since $\mathcal{Q}(Y(t\wedge\tau_k)) > 0$, it follows that

$$\begin{aligned}\mathbb{E}[\mathcal{Q}(Y(t\wedge\tau_k))] &= \mathbb{E}\left[\mathcal{Q}(Y(t\wedge\tau_k))\mathbf{1}_{\{\tau_k\leq t\}}\right] + \mathbb{E}\left[\mathcal{Q}(Y(t\wedge\tau_k))\mathbf{1}_{\{\tau_k> t\}}\right]\\ &\geq \mathbb{E}\left[\mathcal{Q}(Y(t\wedge\tau_k))\mathbf{1}_{\{\tau_k\leq t\}}\right].\end{aligned} \tag{4.8}$$

where $\mathbf{1}_A$ denotes the indicator function of the set A. Since at least one component of $(S_{Hu}(\tau_k), J_{Hu}(\tau_k), R_{Hu}(\tau_k), S_{Ve}(\tau_k), J_{Ve}(\tau_k))$ equals either $\frac{1}{k}$ or k on the event $\{\tau_k \leq t\}$, it follows from (4.8) that

$$\mathbb{E}[\mathcal{Q}(Y(t\wedge\tau_k))] \geq C_k\mathbb{P}\left(\tau_k \leq t\right), \tag{4.9}$$

where C_k is a positive constant depending on k (e.g., $C_k = 5k$). Combining inequalities (4.7) and (4.9), we obtain for all $t \geq 0$

$$\mathbb{P}\left(\tau_k \leq t\right) \leq \frac{\mathcal{Q}(Y(0))\exp(\tilde{R}t)}{5k}.$$

Taking the limit as $k \to +\infty$ yields $\mathbb{P}\left(\tau^* \leq t\right) = 0$ for all $t \geq 0$. Consequently, $\mathbb{P}\left(\tau^* = \infty\right) = 1$. Finally, since $\tau_e \geq \tau^*$ a.s., we conclude that $\tau_e = +\infty$ a.s. □

5. Moment Exponential Stability

In this section, we study the q-th moment exponential stability of the disease-free equilibrium $E_0 = (1, 0, 0, 1, 0)$.
Set

$$\tilde{S}_{Hu}(t) = 1 - S_{Hu}(t) \text{ and } \tilde{S}_{Ve}(t) = 1 - S_{Ve}(t).$$

Then system (3.1) can be written in the form

$$dY(t) = f(Y(t))dt + g(Y(t))dW(t) + \int_{\mathbb{U}} \gamma(Y(t-), u)\tilde{N}(dt, du), \tag{5.1}$$

where

$$f(0) = 0,\ g(0) = 0 \text{ and } \gamma(0, u) = 0 \text{ for all } u \in \mathbb{U}.$$

Let us first introduce the following auxiliary constants:

$$\mathfrak{M} := -\beta_H\beta_V + (\mu_H + \gamma_H + \theta_H)\mu_V > 0,$$
$$u_1 := \frac{\mu_V + \beta_V}{\mathfrak{M}}, \qquad u_2 := \frac{\mu_H + \gamma_H + \beta_H}{\mathfrak{M}}, \qquad \phi_0 := \frac{1}{u_1 \vee u_2}.$$

Let us introduce the following technical assumption:

$$(\mathcal{A}_\phi) \qquad \mathfrak{M} > 0, \quad \Xi := \phi_0 - \frac{1}{2}(q-1)(\sigma_1^2 + \sigma_2^2) - \widetilde{\mathcal{K}}_q^\gamma > 0, \tag{5.2}$$

where, $\forall q > 2$

$$\tilde{\mathcal{K}}_q^\gamma = 2^{q-3}(q-1)\Big[\int_{\mathbb{U}} \bar{\Gamma}_1^2(u)\nu(du) + \int_{\mathbb{U}} \bar{\Gamma}_1^q(u)\nu(du) + \int_{\mathbb{U}} \bar{\Gamma}_2^2(u)\nu(du) + \int_{\mathbb{U}} \bar{\Gamma}_2^q(u)\nu(du)\Big].$$

Lemma 5.1. *Let $V \geq 0$ and assume that for all $u \in \mathbb{U}$,*

$$\Delta V(t-) \leq \big(\gamma_1(u) + \gamma_2(u)\big)V(t-).$$

Then, for every $q > 2$,

$$\int_{\mathbb{U}} \Big[(V + \Delta V)^q - V^q - qV^{q-1}\Delta V\Big]\nu(du) \leq q\widetilde{\mathcal{K}}_q^\gamma V^q,$$

where $\widetilde{\mathcal{K}}_q^\gamma$ is the constant defined in $(\mathcal{A}_\phi)$.

Proof. Taylor's formula with integral remainder gives

$$(V + \Delta V)^q - V^q - qV^{q-1}\Delta V = q(q-1)(\Delta V)^2 \int_0^1 (1-s)(V + s\Delta V)^{q-2}\, ds.$$

Using $\Delta V \leq (\gamma_1 + \gamma_2)V$, $(a+b)^{q-2} \leq 2^{q-3}(a^{q-2} + b^{q-2})$, and Young's inequality, we obtain

$$\int_{\mathbb{U}} \Big[(V + \Delta V)^q - V^q - qV^{q-1}\Delta V\Big]\nu(du) \leq q\widetilde{\mathcal{K}}_q^\gamma V^q.$$

This proves the lemma. □

Lemma 5.2. *Let*

$$V(t) = u_1 J_{Hu}(t) + u_2 J_{Ve}(t), \qquad u_1 = \frac{\mu_V + \beta_V}{\mathfrak{M}}, \quad u_2 = \frac{\mu_H + \gamma_H + \beta_H}{\mathfrak{M}}, \quad \phi_0 = \frac{1}{u_1 \vee u_2}.$$

Then, under $(\mathcal{A}_\phi)$,

$$\mathcal{L}(V^q(t)) \leq -q\Xi V^q(t), \qquad \Xi = \phi_0 - \frac{1}{2}(q-1)(\sigma_1^2 + \sigma_2^2) - \widetilde{\mathcal{K}}_q^\gamma.$$

Proof. By Itô's formula,

$$\mathcal{L}(V^q) = qV^{q-1}b_V + \tfrac{1}{2}q(q-1)V^{q-2}\big(\sigma_H^2 + \sigma_V^2\big) + \mathcal{I},$$

where $\mathcal{I}$ is the jump term. Since $u^T = \mathbf{1}^T(-A)^{-1}$, we have $b_V \le -(J_{Hu} + J_{Ve}) \le -\phi_0 V$, while $1 - \tilde{S}_{Hu}, 1 - \tilde{S}_{Ve} \le 1$, $u_1 J_{Hu} \le V$ and $u_2 J_{Ve} \le V$ give

$$\sigma_H^2 + \sigma_V^2 \le (\sigma_1^2 + \sigma_2^2)V^2,$$

and Lemma 5.1 gives $\mathcal{I} \le q\widetilde{\mathcal{K}}_q^{\gamma} V^q$. Hence

$$\mathcal{L}(V^q) \le -q\phi_0 V^q + \tfrac{1}{2}q(q-1)(\sigma_1^2 + \sigma_2^2)V^q + q\widetilde{\mathcal{K}}_q^{\gamma} V^q = -q\Xi V^q.$$

□

Proposition 5.3. *Let $Y(t) = (S_{Hu}(t), J_{Hu}(t), R_{Hu}(t), S_{Ve}(t), J_{Ve}(t))$ be a solution of system (3.1). If $(\mathcal{A}_\phi)$ is satisfied, then, for all $q > 2$ and all $t \ge 0$,*

$$\mathbb{E}\big(J_{Hu}^q(t)\big) + \mathbb{E}\big(J_{Ve}^q(t)\big) \le \kappa_0 \exp\left(-q\,\Xi\,t\right), \tag{5.3}$$

where

$$\kappa_0 := \left(\frac{1}{u_1^q} + \frac{1}{u_2^q}\right) V^q(0),$$

with

$$V(t) := u_1 J_{Hu}(t) + u_2 J_{Ve}(t).$$

Proof. Taking expectations in Itô's formula (the martingale term has zero mean) and using Lemma 5.2,

$$\frac{d}{dt}\mathbb{E}\big(V^q(t)\big) \le -q\Xi\,\mathbb{E}\big(V^q(t)\big).$$

Gronwall's lemma yields $\mathbb{E}(V^q(t)) \le V^q(0)e^{-q\Xi t}$. Since $V = u_1 J_{Hu} + u_2 J_{Ve}$ implies

$$J_{Hu}^q + J_{Ve}^q \le \Big(\frac{1}{u_1^q} + \frac{1}{u_2^q}\Big)V^q,$$

we obtain

$$\mathbb{E}\big(J_{Hu}^q(t)\big) + \mathbb{E}\big(J_{Ve}^q(t)\big) \le \kappa_0 e^{-q\Xi t}, \qquad \kappa_0 = \Big(\frac{1}{u_1^q} + \frac{1}{u_2^q}\Big)V^q(0).$$

□

Proposition 5.4. *Let $Y(t) = (S_{Hu}(t), J_{Hu}(t), R_{Hu}(t), S_{Ve}(t), J_{Ve}(t))$ be a solution of system (3.1). Then, there exist three positive constants κ_1, κ_2, and κ_3 such that*

$$\begin{aligned}
\mathcal{L}\left(\tilde{S}_{Hu}^q(t)\right) &\le -\frac{q\mu_H}{2}\tilde{S}_{Hu}^q(t) + \kappa_1\left(J_{Hu}^q(t) + J_{Ve}^q(t)\right),\\
\mathcal{L}\left(\tilde{S}_{Ve}^q(t)\right) &\le -\frac{q\mu_V}{2}\tilde{S}_{Ve}^q(t) + \kappa_2\left(J_{Hu}^q(t) + J_{Ve}^q(t)\right),\\
\mathcal{L}\left(R_{Hu}^q(t)\right) &\le -\frac{q\mu_H}{2}R_{Hu}^q(t) + \kappa_1 J_{Hu}^q(t).
\end{aligned}$$

Proof. It suffices to prove the first inequality; the other two are analogous. By Itô's formula, and since $1 - \tilde{S}_{Hu} \leq 1$,

$$\mathcal{L}(\tilde{S}_{Hu}^q) \leq q\tilde{S}_{Hu}^{q-1}\left[-\mu_H \tilde{S}_{Hu} + \beta_H J_{Ve}\right] + \tfrac{1}{2}q(q-1)\sigma_1^2 \tilde{S}_{Hu}^{q-2} J_{Hu}^2 + H,$$

where H is the jump term. Applying Taylor's formula to H and Young's inequality to the three terms with parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, we obtain

$$\mathcal{L}(\tilde{S}_{Hu}^q) \leq -K_1 \tilde{S}_{Hu}^q + K_2 J_{Ve}^q + K_3 J_{Hu}^q,$$

with

$$K_1 = q\mu_H - (q-1)\beta_H \varepsilon_1 - \tfrac{(q-1)(q-2)}{2}\sigma_1^2 \varepsilon_2 - C_S,$$
$$K_2 = \beta_H \varepsilon_1^{-(q-1)}, \qquad K_3 = (q-1)\sigma_1^2 \varepsilon_2^{-\frac{q-2}{2}} + C_J,$$

and

$$C_S = (q-1)(q-2)2^{q-4}\varepsilon_3 \int_{\mathbb{U}} \gamma_1^2 \, d\nu, \quad C_J = (q-1)2^{q-3}\varepsilon_3^{-\frac{q-2}{2}} \int_{\mathbb{U}} \gamma_1^2 \, d\nu + q(q-1)2^{q-4} \int_{\mathbb{U}} \gamma_1^q \, d\nu.$$

Choosing $\varepsilon_1, \varepsilon_2, \varepsilon_3$ small enough that $K_1 \geq \dfrac{q\mu_H}{2}$ yields the first inequality, with $\kappa_1 = \max(K_2, K_3)$. The other two follow analogously. □

Theorem 5.5. *Under the assumption $\mathcal{A}_\phi$, the global solution of the system (3.1) is exponentially stable in the q-th mean in the region Γ_0. More precisely, for every decay rate η satisfying*

$$0 < \eta < \min\left(q\Xi, \frac{q\mu_H}{2}, \frac{q\mu_V}{2}\right), \tag{5.4}$$

there exists a constant $\hat{C} > 0$ such that, for every initial condition in Γ_0,

$$\mathbb{E}\left[\left\|\left(\tilde{S}_{Hu}(t),\, J_{hu}(t),\, R_{Hu}(t),\, \tilde{S}_{Ve}(t),\, J_{Ve}(t)\right)\right\|^q\right] \leq \hat{C}e^{-\eta t}, \quad \forall t \geq 0.$$

Proof. Set

$$\mathfrak{M}_H(t) = \mathbb{E}[\tilde{S}_{Hu}^q(t)], \quad \mathfrak{M}_V(t) = \mathbb{E}[\tilde{S}_{Ve}^q(t)], \quad \mathfrak{M}_R(t) = \mathbb{E}[R_{Hu}^q(t)].$$

Using Proposition 5.3 and Proposition 5.4, we obtain

$$\frac{d}{dt}\mathfrak{M}_H(t) \leq -\frac{q\mu_H}{2}\,\mathfrak{M}_H(t) + \kappa_1\kappa_0 \exp(-q\Xi t),$$
$$\frac{d}{dt}\mathfrak{M}_V(t) \leq -\frac{q\mu_V}{2}\,\mathfrak{M}_V(t) + \kappa_2\kappa_0 \exp(-q\Xi t),$$
$$\frac{d}{dt}\mathfrak{M}_R(t) \leq -\frac{q\mu_H}{2}\,\mathfrak{M}_R(t) + \kappa_3\kappa_0 \exp(-q\Xi t).$$

Let

$$0 < \eta < \min\left(q\Xi, \frac{q\mu_H}{2}, \frac{q\mu_V}{2}\right).$$

By the comparison principle for linear differential inequalities, there exist constants $\kappa_1', \kappa_2', \kappa_3' > 0$ such that

$$\mathfrak{M}_H(t) \leq \kappa_1' e^{-\eta t}, \qquad \mathfrak{M}_V(t) \leq \kappa_2' e^{-\eta t}, \qquad \mathfrak{M}_R(t) \leq \kappa_3' e^{-\eta t}.$$

Combining these estimates with Proposition 5.3 gives

$$\mathbb{E}\left[\left\|(\tilde{S}_{Hu}, J_{Hu}, R_{Hu}, \tilde{S}_{Ve}, J_{Ve})\right\|^q\right] \leq \widehat{C}e^{-\eta t},$$

for all $t \geq 0$, where $\widehat{C} = \kappa_0 + \kappa_1' + \kappa_2' + \kappa_3'$. □

Lemma 5.6. *For all $X, Y \in \Gamma_0$, there exists a constant $C > 0$ such that*

$$|f(X) - f(Y)| \vee \|g(X) - g(Y)\| \leq C|X - Y|. \tag{5.5}$$

Moreover, there exists $\tilde{R} > 0$ such that

$$\int_{\mathbb{U}} |h(X,u) - h(Y,u)|\, \nu(du) \leq \tilde{R}\, |X - Y|. \tag{5.6}$$

Finally, for every $q \geq 2$, there exist constants $\mathcal{R}_2(q) > 0$ and $\mathcal{R}(q) > 0$ such that

$$\int_{\mathbb{U}} |h(Y,u)|^q\, \nu(du) \leq \mathcal{R}_2(q)|Y|^q, \tag{5.7}$$

and, for the solution $Y(t) \in \Gamma_0$,

$$\mathbb{E}|Y(t)|^q \leq \mathcal{R}(q), \qquad \forall t \geq 0. \tag{5.8}$$

Proof. The functions f and g are Lipschitz on the compact set Γ_0, hence (5.5) holds. Since h is Lipschitz in Y and the jump intensities satisfy

$$M_i(q) := \int_{\mathbb{U}} \gamma_i(u)^q\, \nu(du) < \infty, \qquad i = 1, 2,\ q \geq 1,$$

we obtain (5.6) and (5.7) by standard estimates. Finally, Γ_0 is bounded, so $|Y(t)| \leq \mathcal{K}$ for some $\mathcal{K} > 0$, which gives (5.8) with $\mathcal{R}(q) = \mathcal{K}^q$. □

Theorem 5.7. *Under the assumptions of exponential stability of order $q > 2$ for the zero solution of (5.1), it follows that the equilibrium point under consideration is almost surely exponentially stable.*

Proof. Our approach is inspired by the works of Can Chen and Yanmei Kang (see [17]) and Mao (see [5]). Fix any
$Y(0) = (\tilde{S}_H(0), J_{Hu}(0), R_{Hu}(0), \tilde{S}_m(0), J_{Ve}(0)) \neq 0_{\mathbb{R}^5}$. Using the results of Lemma 5.6, we obtain the following inequality:

$$\big|f(Y) - f(Z)\big| \vee \big|\big|g(Y) - g(Z)\big|\big| \vee \int_{\mathbb{U}} \big|h(Y,u) - h(Z,u)\big|\nu(du) \leq K_1\big|Y - Z\big| \tag{5.9}$$

where $K_1 = R_2 \vee \tilde{R}$. Let $q \geq 2$. Then for any $r > t$, by Itô's formula, we have

$$\begin{aligned}
\big|Y(r)\big|^q &= \big|Y(t)\big|^q + \int_t^r q\big|Y(s)\big|^{q-2}Y^T(s)f(Y(s))ds \\
&\quad + \frac{1}{2}q\int_t^r \big|Y(s)\big|^{q-2}\big|g(Y(s))\big|^2 ds \\
&\quad + \frac{1}{2}q(q-2)\int_t^r \big|Y(s)\big|^{q-4}\big|Y^T(s)g(Y(s))\big|^2 ds \\
&\quad + \int_t^r \int_{\mathbb{U}} \big[\big|Y(s-) + h(Y(s-),u)\big|^q - \big|Y(s-)\big|^q \\
&\quad - q\big|Y(s-)\big|^{q-2}Y^T(s-)h(Y(s-),u)\big]\nu(du)ds \\
&\quad + \int_t^r q\big|Y(s)\big|^{q-2}Y^T(s)g(Y(s))dB(s) \\
&\quad + \int_t^r \int_{\mathbb{U}} \big[\big|Y(s-) + h(Y(s-),u)\big|^q - \big|Y(s-)\big|^q\big]\tilde{N}(ds,du).
\end{aligned}$$

Then, it follows from (5.9) that

$$\begin{aligned}
\big|Y(r)\big|^q &\leq \big|Y(t)\big|^q + c_1\int_t^r \big|Y(s)\big|^q ds \\
&\quad + \int_t^r \int_{\mathbb{U}} \big[\big|Y(s-) + h(Y(s-),u)\big|^q - \big|Y(s-)\big|^q \\
&\quad - q\big|Y(s-)\big|^{q-2}Y^T(s-)h(Y(s-),u)\big]\nu(du)ds \\
&\quad + \int_t^r q\big|Y(s)\big|^{q-2}Y^T(s)g(Y(s))dB(s) \\
&\quad + \int_t^r \int_{\mathbb{U}} \big[\big|Y(s-) + h(Y(s-),u)\big|^q - \big|Y(s-)\big|^q\big]\tilde{N}(ds,du),
\end{aligned}$$

where

$$c_1 = qK_1 + \frac{1}{2}q(q-1)K_1^2.$$

Hence, for any $k = 1, 2, \cdots,$

$$\begin{aligned}\mathbb{E}\big(\sup_{k-1\le t\le k}\big|Y(t)\big|^q\big) \;\le\; & \mathbb{E}\big|Y(k-1)\big|^q + c_1\int_{k-1}^{k}\mathbb{E}\big|Y(s)\big|^q ds \\ & +\mathbb{E}\big(\sup_{k-1\le t\le k}\int_{k-1}^{t} q\big|Y(s)\big|^{q-2}Y^T(s)g(Y(s))dB(s)\big) \\ & +\mathbb{E}\Big\{\sup_{k-1\le t\le k}\Big[\int_{k-1}^{t}\int_{\mathbb{U}}\big[\big|Y(s-)+h(Y(s-),u)\big|^q \\ & -\big|Y(s-)\big|^q\big]\tilde{N}(ds,du)+\int_{k-1}^{t}\int_{\mathbb{U}}\big[\big|Y(s-)+h(Y(s-),u)\big|^q \\ & -\big|Y(s-)\big|^q - q\big|Y(s-)\big|^{q-2}Y^T(s-)h(Y(s-),u)\big]\nu(du)ds\big]\Big\}.\end{aligned} \tag{5.10}$$

By the Burkholder-Davis-Gundy inequality (see [7] p. 129-130), we deduce that

$$\begin{aligned}&\mathbb{E}\big(\sup_{k-1\le t\le k}\int_{k-1}^{t} q\big|Y(s)\big|^{q-2}Y^T(s)g(Y(s))dB(s)\big) \\ &\le \frac{1}{2}\mathbb{E}\big(\sup_{k-1\le t\le k}\big|Y(t)\big|^q\big)+16q^2K_1^2\int_{k-1}^{k}\mathbb{E}\big|Y(s)\big|^q ds.\end{aligned} \tag{5.11}$$

Applying Kunita's estimates for $f=0$ and $g=0$ and employing Hölder's inequality, it follows that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{k-1\le t\le k}\Big(\int_{k-1}^{t}\int_{\mathbb{U}}\big[\big|Y(s-)+h(Y(s-),u)\big|^q \\ &-\big|Y(s-)\big|^q\big]\tilde{N}(ds,du)+\int_{k-1}^{t}\int_{\mathbb{U}}\big[\big|Y(s-)+h(Y(s-),u)\big|^q \\ &-\big|Y(s-)\big|^q - q\big|Y(s-)\big|^{q-2}Y^T(s-)h(Y(s-),u)\big]\nu(du)ds\Big)\Big] \\ &\le c_3\mathbb{E}\Big[\Big(\int_{k-1}^{k}\int_{\mathbb{U}}\big|h(Y(s-),u)\big|^2\nu(du)\Big)^{\frac{q}{2}}ds\Big] \\ &+c_4\mathbb{E}\Big[\Big(\int_{k-1}^{k}\int_{\mathbb{U}}\big|h(Y(s-),u)\big|^q\nu(du)ds\Big)\Big] \\ &\le \Big[c_3K_2^{\frac{q}{2}}+c_4K_2\Big]\int_{k-1}^{k}\mathbb{E}\big|Y(s)\big|^q ds.\end{aligned}$$

So, (5.10) becomes

$$\mathbb{E}\big(\sup_{k-1\le t\le k}\big|Y(t)\big|^q\big) \;\le\; 2\mathbb{E}\big|Y(k-1)\big|^q + C(q)\int_{k-1}^{k}\mathbb{E}\big|Y(s)\big|^q ds,$$

where

$$C(q) \quad = \quad \left[2k_1q + q(q-1)K_1 + qK_1^2 + 32q^2K_1^2 + 2c_3K_2^{\frac{q}{2}} + 2c_4K_2\right].$$

Since the trivial solution of (5.1) is q-th moment exponentially stable, there exists a pair of positive constants λ and C such that

$$\mathbb{E}\left(\left|Y(t)\right|^q\right) \leq C\left|Y(0)\right|^q \exp\left(-\lambda t\right) \text{ for all } t \geq 0 \text{ and } Y(0) \neq 0. \tag{5.12}$$

Using (5.12), we obtain

$$\mathbb{E}\Big(\sup_{k-1\leq t\leq k}\left|Y(t)\right|^q\Big) \quad \leq \quad \gamma(q)e^{-\lambda(k-1)}$$

where

$$\gamma(q) = C\left|Y(0)\right|^q(2 + C(q)).$$

Let $\epsilon \in (0,1)$ be arbitrary, then

$$\begin{aligned}\mathbb{P}\left\{\sup_{k-1\leq t\leq k}\left|Y(t)\right|^q > e^{-(\lambda-\epsilon)(k-1)}\right\} \quad &\leq \quad e^{-(\lambda-\epsilon)(k-1)}\mathbb{E}\Big(\sup_{k-1\leq t\leq k}\left|Y(t)\right|^q\Big)\\ &\leq \quad \gamma(q)e^{-\epsilon(k-1)}.\end{aligned}$$

Since the series $\sum_{k=1}^{\infty} e^{-\epsilon(k-1)}$ is convergent, by the Borel-Cantelli lemma, for almost all $\omega \in \Omega$,

$$\sup_{k-1\leq t\leq k}\left|Y(t)\right|^q \leq e^{-(\lambda-\epsilon)(k-1)}. \tag{5.13}$$

Consequently, for almost all $\omega \in \Omega$, $k-1 \leq t \leq k$,

$$\frac{1}{t}\ln\left|Y(t)\right| \leq \frac{1}{qt}\ln\left|Y(t)\right|^q \leq -\frac{1}{qt}(\lambda-\epsilon)t \leq -\frac{1}{q(k-1)}(\lambda-\epsilon)(k-1).$$

So,

$$\limsup_{t\longrightarrow\infty}\frac{1}{t}\ln\left|Y(t)\right| \leq -\frac{1}{q}(\lambda-\epsilon), \ a.s.$$

Since $\epsilon > 0$ is arbitrary, we get

$$\limsup_{t\longrightarrow\infty}\frac{1}{t}\ln\left|Y(t)\right| \leq -\frac{\lambda}{q}, \ a.s.$$

By definition, the trivial solution of equation (5.1) is almost surely exponentially stable. The proof is complete. □

6. Persistence in Mean

In this subsection, we investigate the sufficient conditions for the persistence in the mean of the dengue fever. As a result, we will present a definition of persistence in the mean (see [14, 22]).

Definition 6.1. We say that the system (3.1) is persistent in mean if $\liminf_{t\to\infty}\frac{1}{t}\int_0^t (J_{Hu}(s))\,ds > 0$ or $\liminf_{t\to\infty}\frac{1}{t}\int_0^t (J_{Ve}(s))\,ds > 0$ almost surely.

Before stating the persistence theorem, let us prepare three useful lemmas.

Lemma 6.2. *Let $(S_{Hu}(.), J_{Hu}(.), R_{Hu}(.), S_{Ve}(.), J_{Ve}(.))$ be the solution of the system (3.1) with any initial value $(S_{Hu}(0), J_{Hu}(0), R_{Hu}(0), S_{Ve}(0), J_{Ve}(0)) \in \Gamma_0$. Then*

$$\lim_{t\to\infty} \frac{X(t)}{t} = 0, \ \ a.s,$$

for all $X(t) \in \{S_{Hu}, J_{Hu}, R_{Hu}, S_{Ve}, J_{Ve}\}$.

Proof. The proof of this lemma is the same as that of Lemma 3.2 in [20]. □

Lemma 6.3. *(see [7]) Let $M = \{M\}_{t\geq 0}$ be a continuous real valued local martingale and vanishing at $t = 0$, then*

$$\lim_{t\to\infty} \langle M, M\rangle_t = \infty, \ a.s, \ \text{implies that } \lim_{t\to\infty} \frac{M_t}{\langle M, M\rangle_t} = 0, \ a.s, \ \text{and also}$$

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} < 0, \ \ a.s, \ \text{implies that } \lim_{t\to\infty} \frac{M_t}{t} = 0 \ a.s.$$

Lemma 6.4. *Let $(S_{Hu}(.), J_{Hu}(.), R_{Hu}(.), S_{Ve}(.), J_{Ve}(.))$ a solution of the system (3.1) with initial conditions*
$(S_{Hu}(0), J_{Hu}(0), R_{Hu}(0), S_{Ve}(0), J_{Ve}(0)) \in (0; 1)^5$. Then

$$\begin{aligned}
\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t \sigma_1 J_{Hu}(s)dW_1(s)ds &= 0, \ a.s,\\
\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t \sigma_2 J_{Ve}(s)dW_2(s)ds &= 0, \ a.s,
\end{aligned}$$

and

$$\begin{aligned}
\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t \sigma_1 S_{Hu}(s)dW_1(s)ds &= 0, \ a.s,\\
\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t \sigma_2 S_{Ve}(s)dW_2(s)ds &= 0, \ a.s.
\end{aligned}$$

Proof. The proof of this lemma is the same as that of Lemma 3.3, (see [20]). □

Lemma 6.5. *Let $\left(S_{Hu}, J_{Hu}, R_{Hu}, S_{Ve}, J_{Ve}\right)$ be a solution of (3.1) with initial condition in $(0, 1)^5$. Then*

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathbb{U}} \ln\big(1 - \gamma_1(u)J_{Hu}(s-)\big)\tilde{N}(ds, du) = 0, \quad a.s,\\
\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathbb{U}} \ln\big(1 - \gamma_2(u)J_{Ve}(s-)\big)\tilde{N}(ds, du) = 0, \quad a.s,\\
\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathbb{U}} \ln\big(1 + \gamma_1(u)S_{Hu}(s-)\big)\tilde{N}(ds, du) = 0, \quad a.s,\\
\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathbb{U}} \ln\big(1 + \gamma_2(u)S_{Ve}(s-)\big)\tilde{N}(ds, du) = 0, \quad a.s.
\end{aligned}$$

Proof. Set

$$k(t) = \int_0^t \int_{\mathbb{U}} g(s-,u)\,\tilde{N}(ds,du),$$

where g denotes any of the four integrands above. Since $S_{Hu}, J_{Hu}, S_{Ve}, J_{Ve} \in (0,1)$ and the γ_i, $i = \overline{1,2}$ are bounded, g is bounded. Applying Proposition 2.4 of Kunita (see [6]), we have

$$\langle k,k\rangle_t = \int_0^t \int_{\mathbb{U}} |g(s-,u)|^2 \nu(du)\,ds \le \hbar t \quad a.s.,$$

for some constant $\hbar > 0$. Hence

$$\lim_{t\to\infty} \frac{\langle k,k\rangle_t}{t} \le \hbar \quad a.s.$$

By the martingale strong law (Lemma 6.3),

$$\lim_{t\to\infty} \frac{k(t)}{t} = 0 \quad a.s.$$

Applying this to the four integrands yields the announced limits. □

For future needs, set

$$\alpha_1 := \mu_H + \frac{\sigma_1^2}{2} + \bar{\Phi}_1, \qquad \Phi_1(J_{Hu}) := \int_{\mathbb{U}} \big[-\ln(1-\gamma_1(u)J_{Hu}) - \gamma_1(u)J_{Hu}\big]\nu(du),$$

$$\alpha_2 := \mu_H + \gamma_H + \theta_H + \frac{\sigma_1^2}{2} + \bar{\Phi}_2, \quad \Phi_2(S_{Hu}) := \int_{\mathbb{U}} \big[-\ln(1+\gamma_1(u)S_{Hu}) + \gamma_1(u)S_{Hu}\big]\nu(du),$$

$$\alpha_3 := \mu_V + \frac{\sigma_2^2}{2} + \bar{\Phi}_3, \qquad \Phi_3(J_{Ve}) := \int_{\mathbb{U}} \big[-\ln(1-\gamma_2(u)J_{Ve}) - \gamma_2(u)J_{Ve}\big]\nu(du),$$

$$\alpha_4 := \mu_V + \frac{\sigma_2^2}{2} + \bar{\Phi}_4, \qquad \Phi_4(S_{Ve}) := \int_{\mathbb{U}} \big[-\ln(1+\gamma_2(u)S_{Ve}) + \gamma_2(u)S_{Ve}\big]\nu(du),$$

where $\bar{\Phi}_i := \sup_{[0,1]} \Phi_i$, $i = 1,\dots,4$. Set

$$\mathcal{R}_0^H := \frac{2\sqrt{\mu_H\beta_H\mu_V\beta_V}}{(\mu_H+\beta_H)(\alpha_1\alpha_2\alpha_3\alpha_4)^{1/4}}, \qquad \mathcal{R}_0^{\mathcal{V}} := \frac{2\sqrt{\mu_H\beta_H\mu_V\beta_V}}{(\mu_V+\beta_V)(\alpha_1\alpha_2\alpha_3\alpha_4)^{1/4}}. \tag{6.1}$$

Theorem 6.6. *Let $(S_{Hu}, J_{Hu}, R_{Hu}, S_{Ve}, J_{Ve})$ be a solution of (3.1) with initial condition in Γ_0. Then:*

(i) *if $\mathcal{R}_0^H < 1$, then $\liminf_{t\to\infty} \langle J_{Hu}(t)\rangle > 0$ a.s.;*
(ii) *if $\mathcal{R}_0^{\mathcal{V}} < 1$, then $\liminf_{t\to\infty} \langle J_{Ve}(t)\rangle > 0$ a.s.*

Proof. Let

$$\mathcal{G}(t) = -\sum_{j=1}^{4} \lambda_j \ln \chi_j(t), \qquad \chi_1 = S_{Hu}, \quad \chi_2 = J_{Hu}, \quad \chi_3 = S_{Ve}, \quad \chi_4 = J_{Ve},$$

where $\lambda_1,\dots,\lambda_4>0$ will be chosen later. By Itô's formula, after using $0<\chi_j(t)<1$ and $\Phi_i\le\bar\Phi_i$, there exist constants $\alpha_j>0$ such that

$$\mathrm{d}\mathcal{G}(t)\le\Big[-\frac{\lambda_1\mu_H}{S_{Hu}}-\frac{\lambda_2\beta_H S_{Hu}J_{Ve}}{J_{Hu}}-\frac{\lambda_3\mu_V}{S_{Ve}}-\frac{\lambda_4\beta_V S_{Ve}J_{Hu}}{J_{Ve}}+\lambda_1\beta_H J_{Ve}+\lambda_3\beta_V J_{Hu}+\sum_{j=1}^{4}\alpha_j\lambda_j\Big]\mathrm{d}t+\mathrm{d}M(t),$$

where M is the martingale term (Brownian and compensated Poisson parts). By the arithmetic–geometric mean inequality,

$$\frac{\lambda_1\mu_H}{S_{Hu}}+\frac{\lambda_2\beta_H S_{Hu}J_{Ve}}{J_{Hu}}+\frac{\lambda_3\mu_V}{S_{Ve}}+\frac{\lambda_4\beta_V S_{Ve}J_{Hu}}{J_{Ve}}\ge 4\big(\lambda_1\lambda_2\lambda_3\lambda_4\mu_H\mu_V\beta_H\beta_V\big)^{1/4}.$$

Choosing

$$\lambda_1=\frac{\mu_H}{\alpha_1},\qquad \lambda_2=\frac{\beta_H}{\alpha_2},\qquad \lambda_3=\frac{\mu_V}{\alpha_3},\qquad \lambda_4=\frac{\beta_V}{\alpha_4},$$

and using $J_{Hu},J_{Ve}\in(0,1)$, we obtain

$$\mathrm{d}\mathcal{G}(t)\le\Big[-2\frac{\sqrt{\mu_H\beta_H\mu_V\beta_V}}{(\alpha_1\alpha_2\alpha_3\alpha_4)^{1/4}}+\frac{\mu_V\beta_V}{\alpha_3}J_{Hu}(t)+\mu_H+\beta_H\Big]\mathrm{d}t+\mathrm{d}M(t),$$

and similarly

$$\mathrm{d}\mathcal{G}(t)\le\Big[-2\frac{\sqrt{\mu_H\beta_H\mu_V\beta_V}}{(\alpha_1\alpha_2\alpha_3\alpha_4)^{1/4}}+\frac{\mu_H\beta_H}{\alpha_1}J_{Ve}(t)+\mu_V+\beta_V\Big]\mathrm{d}t+\mathrm{d}M(t).$$

Integrating from 0 to t gives

$$\int_0^t J_{Hu}(s)\,\mathrm{d}s\ge\frac{\alpha_3}{\mu_V\beta_V}\big(\mathcal{G}(t)-\mathcal{G}(0)\big)-\Big[-2\frac{\alpha_3\sqrt{\mu_H\beta_H}}{\sqrt{\mu_V\beta_V}(\alpha_1\alpha_2\alpha_3\alpha_4)^{1/4}}+\frac{\alpha_3(\mu_H+\beta_H)}{\mu_V\beta_V}\Big]t-\frac{\alpha_3}{\mu_V\beta_V}M(t),$$

$$\int_0^t J_{Ve}(s)\,\mathrm{d}s\ge\frac{\alpha_1}{\mu_H\beta_H}\big(\mathcal{G}(t)-\mathcal{G}(0)\big)-\Big[-2\frac{\alpha_1\sqrt{\mu_V\beta_V}}{\sqrt{\mu_H\beta_H}(\alpha_1\alpha_2\alpha_3\alpha_4)^{1/4}}+\frac{\alpha_1(\mu_V+\beta_V)}{\mu_H\beta_H}\Big]t-\frac{\alpha_1}{\mu_H\beta_H}M(t).$$

Since $x-1-\ln x\ge 0$ for $x>0$,

$$\mathcal{G}(t)\ge-\bar\lambda\,\bar Y(t),\qquad \bar Y(t)=S_{Hu}(t)+J_{Hu}(t)+S_{Ve}(t)+J_{Ve}(t),\quad \bar\lambda=\bigwedge_{j=1}^{4}\lambda_j.$$

Using Lemmas 6.2, 6.4 and 6.5, we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t J_{Hu}(s)\,\mathrm{d}s\ge-\Big[-2\frac{\alpha_3\sqrt{\mu_H\beta_H}}{\sqrt{\mu_V\beta_V}(\alpha_1\alpha_2\alpha_3\alpha_4)^{1/4}}+\frac{\alpha_3(\mu_H+\beta_H)}{\mu_V\beta_V}\Big],\quad a.s,$$

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t J_{Ve}(s)\,\mathrm{d}s\ge-\Big[-2\frac{\alpha_1\sqrt{\mu_V\beta_V}}{\sqrt{\mu_H\beta_H}(\alpha_1\alpha_2\alpha_3\alpha_4)^{1/4}}+\frac{\alpha_1(\mu_V+\beta_V)}{\mu_H\beta_H}\Big],\quad a.s.$$

Using the parameters defined in (6.1), it follows that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t J_{Hu}(s)\,\mathrm{d}s\ge\frac{\alpha_3(\mu_H+\beta_H)}{\mu_V\beta_V}\big(1-\mathcal{R}_0^H\big),\quad a.s,$$

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t J_{Ve}(s)\,\mathrm{d}s\ge\frac{\alpha_1(\mu_V+\beta_V)}{\mu_H\beta_H}\big(1-\mathcal{R}_0^{\mathcal{V}}\big),\quad a.s.$$

This finishes the proof of Theorem 6.6. □

7. Numerical Simulations

We illustrate the theoretical results by simulating system (3.1) with the Milstein scheme of [4], which has strong order 1.0 and preserves the multiplicative structure of the noise. The numerical values are as follows. In the extinction scenario we take

$$\beta_H = 0.150, \quad \gamma_H = 0.640, \quad \theta_H = 0.100, \quad \mu_H = 0.880, \quad \beta_V = 0.021, \quad \mu_V = 0.980,$$

while in the persistence scenario

$$\beta_H = 1.000, \quad \gamma_H = 0.040, \quad \theta_H = 0.200, \quad \mu_H = 0.200, \quad \beta_V = 0.200, \quad \mu_V = 0.100.$$

In both cases we set $\sigma_1 = \sigma_2 = 0.269$, $\gamma_1 = 0.850$ and $\gamma_2 = 0.800$. The values of $\beta_H, \gamma_H, \theta_H, \mu_H, \beta_V, \mu_V$ are taken from [16]; the noise and jump parameters are assumed.

Example 7.1. With the extinction values above, the conditions of Theorem 5.7 hold, so J_{Hu} and J_{Ve} converge to zero almost surely, as confirmed by Figure 1.

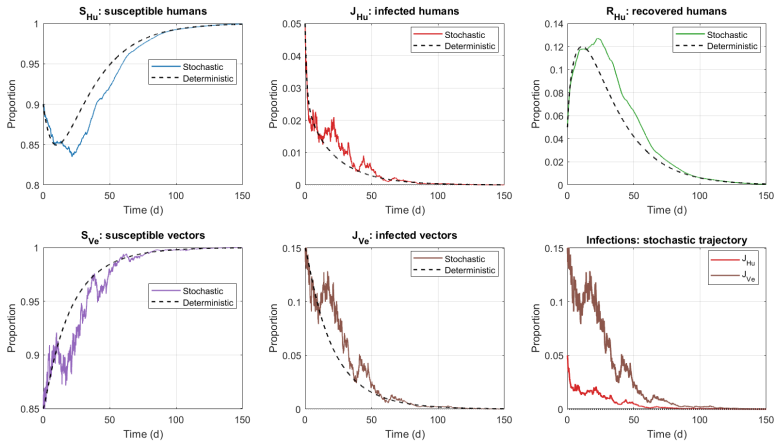


Figure 1. Sample paths in the extinction regime.

Example 7.2. With the persistence values above, the conditions of Theorem 6.6 hold, so dengue persists in both populations, as confirmed by Figure 2.

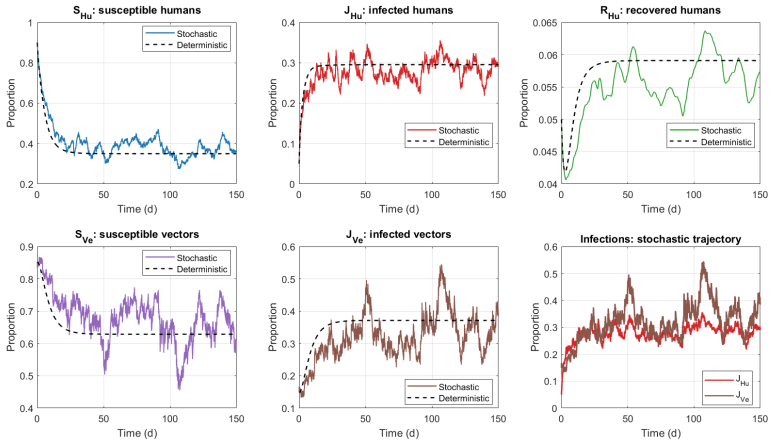


FIGURE 2. Sample paths in the persistence regime.

8. Conclusion

In this work, we formulated and analyzed a compartmental model of dengue incorporating both white noise and Lévy jumps, in order to describe the spread of the disease under small environmental fluctuations and sudden, massive external disturbances. The main findings are as follows. First, by constructing a suitable Lyapunov function, we obtained sufficient conditions for the existence and uniqueness of a positive solution. We then derived sufficient conditions for the q-th moment exponential stability and the almost-sure exponential stability of the disease-free equilibrium, and established conditions for persistence in mean. Several questions remain open. In particular, during an outbreak individuals may change their behavior as they become aware of the infection. In future work, we therefore plan to study the effect of memory on the dynamics of the model, using generalized fractional and fractal-fractional derivatives, together with the numerical scheme for fractional differential equations involving the generalized Hattaf fractional (GHF) derivative introduced in [15].

Acknowledgement. The authors thank the anonymous referees for their valuable comments and suggestions.

References

[1] Applebaum, D. (2009). *Lévy processes and stochastic calculus* (2nd ed.). Cambridge University Press. https://doi.org/10.1017/CBO9780511809781

[2] Cont, R., & Tankov, P. (2004). *Financial modelling with jump processes*. Chapman & Hall/CRC. https://doi.org/10.1201/9780203485217

[3] Gubler, D. J. (2002). Epidemic dengue/dengue hemorrhagic fever as a public health, social and economic problem in the 21st century. *Trends in Microbiology*, *10*(2), 100–103. https://doi.org/10.1016/S0966-842X(01)02288-0

[4] Higham, D. J. (2001). An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Review*, *43*(3), 525–546. https://doi.org/10.1137/S0036144500378302

[5] Kolmanovskii, V., Koroleva, N., Maizenberg, T., Mao, X., & Matasov, A. (2003). Neutral stochastic differential delay equations with Markovian switching. *Stochastic Analysis and Applications*, *21*(4), 819–847. https://doi.org/10.1081/SAP-120022865

[6] Kunita, H. (2010). Itô's stochastic calculus: Its surprising power for applications. *Stochastic Processes and their Applications*, *120*(5), 622–652. https://doi.org/10.1016/j.spa.2010.01.013

[7] Mao, X. (2007). *Stochastic differential equations and applications* (2nd ed.). Elsevier. https://doi.org/10.1533/9780857099402

[8] Newton, E. A. C., & Reiter, P. (1992). A model of the transmission of dengue fever with an evaluation of the impact of ultra-low volume (ULV) insecticide applications on dengue epidemics. *The American Journal of Tropical Medicine and Hygiene*, *47*(6), 709–720. https://doi.org/10.4269/ajtmh.1992.47.709

[9] Racloz, V., Ramsey, R., Tong, S., & Hu, W. (2012). Surveillance of dengue fever virus: A review of epidemiological models and early warning systems. *PLoS Neglected Tropical Diseases*, *6*(5), e1648. https://doi.org/10.1371/journal.pntd.0001648

[10] Bhatt, S., Gething, P. W., Brady, O. J., Messina, J. P., Farlow, A. W., Moyes, C. L., Drake, J. M., Brownstein, J. S., Hoen, A. G., Sankoh, O., et al. (2013). The global distribution and burden of dengue. *Nature*, *496*(7446), 504–507. https://doi.org/10.1038/nature12060

[11] Stanaway, J. D., Shepard, D. S., Undurraga, E. A., Halasa, Y. A., Coffeng, L. E., Brady, O. J., Hay, S. I., Bedi, N., Bensenor, I. M., Castañeda-Orjuela, C. A., et al. (2016). The global burden of dengue: An analysis from the Global Burden of Disease Study 2013. *The Lancet Infectious Diseases*, *16*(6), 712–723. https://doi.org/10.1016/S1473-3099(16)00026-8

[12] Wichmann, O., Yoon, I.-K., Vong, S., Limkittikul, K., Gibbons, R. V., Mammen, M. P., Ly, S., Buchy, P., Sirivichayakul, C., Buathong, R., et al. (2011). Dengue in Thailand and Cambodia: An assessment of the degree of underrecognized disease burden based on reported cases. *PLoS Neglected Tropical Diseases*, *5*(3), e996. https://doi.org/10.1371/journal.pntd.0000996

[13] Agusto, F. B., & Khan, M. A. (2018). Optimal control strategies for dengue transmission in Pakistan. *Mathematical Biosciences*, *305*, 102–121. https://doi.org/10.1016/j.mbs.2018.09.007

[14] Chen, C., & Kang, Y. (2014). Dynamics of a stochastic SIS epidemic model with saturated incidence. *Abstract and Applied Analysis*, *2014*, Article 723825. https://doi.org/10.1155/2014/723825

[15] Hattaf, K. (2022). On the stability and numerical scheme of fractional differential equations with application to biology. *Computation*, *10*(6), Article 97. https://doi.org/10.3390/computation10060097

[16] Tinde, H., Sawadogo, W. O., Ouedraogo, P. O. F., & Kiemtore, A. (2026). Mathematical modeling of the 2023 dengue outbreak in the Centre Region of Burkina Faso: Parameter estimation and assessment of control strategies. *Mathematical Biosciences and Engineering*, *23*(1), 40–75. https://doi.org/10.3934/mbe.2026003

[17] Chen, C., & Kang, Y. (2017). Dynamics of a stochastic multi-strain SIS epidemic model driven by Lévy noise. *Communications in Nonlinear Science and Numerical Simulation*, *42*, 379–395.

[18] Guo, M., Hu, L., & Nie, L.-F. (2021). Stochastic dynamics of the transmission of dengue fever virus between mosquitoes and humans. *International Journal of Biomathematics*, *14*(7), Article 2150062.

[19] Kiouach, D., Sabbar, Y., & El Azami El-idrissi, S. (2021). New results on the asymptotic behavior of an SIS epidemiological model with quarantine strategy, stochastic transmission, and Lévy disturbance. *Mathematical Methods in the Applied Sciences*, *44*(17), 13468–13492.

[20] Konané, F. V., & Sawadogo, R. (2023). Mathematical analysis of a deterministic and a stochastic epidemic models of dengue. *European Journal of Mathematics and Applications*, *3*, Article 3.
[21] Liu, Q. (2023). Dynamical behavior of a stochastic dengue model with Ornstein–Uhlenbeck process. *Journal of Mathematical Physics*, *64*(9).
[22] Liu, Q., & Chen, Q. (2016). Dynamics of a stochastic SIR epidemic model with saturated incidence. *Applied Mathematics and Computation*, *282*, 155–166.
[23] Liu, Q., Jiang, D., Hayat, T., & Alsaedi, A. (2018). Stationary distribution and extinction of a stochastic dengue epidemic model. *Journal of the Franklin Institute*, *355*(17), 8891–8914.
[24] Liu, Q., Jiang, D., Shi, N., Hayat, T., & Alsaedi, A. (2017). Stochastic mutualism model with Lévy jumps. *Communications in Nonlinear Science and Numerical Simulation*, *43*, 78–90.
[25] N'zi, E., & N'zi, M. (2019). Extinction and stationary distribution of a stochastic SIRS epidemic model incorporating media coverage. *Journal of Advances in Mathematics and Computer Science*, *34*(3), 1–19.
[26] Wang, L., & Zhao, H. (2019). Dynamics analysis of a Zika–dengue co-infection model with dengue vaccine and antibody-dependent enhancement. *Physica A: Statistical Mechanics and Its Applications*, *522*, 248–273.
[27] Zhang, G., Li, Z., & Din, A. (2022). A stochastic SIQR epidemic model with Lévy jumps and three-time delays. *Applied Mathematics and Computation*, *431*, Article 127329.
[28] Zhou, Y., & Zhang, W. (2016). Threshold of a stochastic SIR epidemic model with Lévy jumps. *Physica A: Statistical Mechanics and Its Applications*, *446*, 204–216.

[1] Laboratory of Mathematics, Computer Science and Applications, Faculty of Exact and Applied Sciences, Nazi BONI University, Bobo-Dioulasso, Burkina Faso.
Email address: sragnimwende@gmail.com

[2] Laboratory of Mathematics and Computer Science (LAMI), Faculty of Exact and Applied Sciences, Department of Mathematics, Joseph KI-ZERBO University, PO Box 7021, Ouagadougou, Burkina Faso.
Email address: :oharounao@gmail.com
Email address: :konanevictorien@yahoo.fr