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THE SECOND EIGENCURVE FOR NEUMANN p-LAPLACIAN WITH VANISHING WEIGHTS

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Abstract. In this paper, we study the existence and uniqueness of the second eigencurve for a homogeneous Neumann problem involving a p-Laplacian-type differential equation with two parameters and two weight functions, set on a bounded open interval I of the $\mathbb{R}$. We then establish several properties of this eigencurve, in particular its continuity, variational characterization, asymptotic behavior, and differentiability.

Keywords. p-Laplacian, second eigencurve, one-dimensional, Neumann boundary conditions.
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1. Introduction

Stationary problems involving the p-Laplacian operator have attracted considerable attention in recent years, particularly in the presence of indefinite weight functions. These problems can be encountered in various fields, such as physics, engineering and mathematics..., where they model complex phenomena involving nonlinearities see [4, 9, 10]. The notion of eigencurves, which characterizes the evolution of eigenvalues with respect to a varying parameter, offers a useful framework for investigating the spectral properties and solution behavior of such problems. In this paper, we investigate the second eigencurve associated with a one-dimensional elliptic problem involving the p-Laplacian operator with indefinite weights under Neumann boundary conditions. We establish its existence and uniqueness and establish several of its mathematical properties. The main difficulty encountered in this work stems from the interplay between the nonlinear character of the problem and the presence of the indefinite weights m_1 and m_2, particularly under the condition $(\operatorname{ess\,inf}_I m_2 = 0)$.

Let $1 < p < \infty$ and a, b $(a < b)$ are real numbers. We are interested in values of

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the pair $(\alpha, \beta) \in \mathbb{R}^2$ such that the following problem admits a nontrivial solution

$$\begin{cases} -\Delta_p u(x) = (\alpha\, m_1(x) + \beta\, m_2(x))\, |u(x)|^{p-2}u(x) \quad \text{in } I =]a, b[\\ u'(a) = u'(b) = 0, \end{cases} \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$, m_1, m_2 are two weight functions belonging to $L^\infty(I)$, $-\Delta_p u(x) = (|u'(x)|^{p-2}u'(x))' = (\phi_p(u'(x)))'$, with $u'(x)$ denotes the usual derivative of the function u at x and $\phi_p : \mathbb{R} \to \mathbb{R}$ is defined by

$$\phi_p(s) = |s|^{p-2}s.$$

Precisely, our study provides the existence and uniqueness of second eigencurve associated with (1.1). Let us first recall what is called a second eigencurve, let be $\alpha \in \mathbb{R}$, we define the second eigencurve of p-Laplacian with weight as the graph of map $\beta_2 : \alpha \to \beta_2(\alpha) = \beta$ with β is the unique real such that $\lambda_2(\alpha\, m_1 + \beta\, m_2) = 1$ where $\lambda_2(m)$ is the second eigenvalue of the p-Laplacian with weight $m \in L^\infty(I)$ under Neumann boundary conditions. This definition was first introduced by A. Dakkak and M. Hadda in [7].
Throughout this paper, we assume that, m_1 and m_2 the two weighting functions that appear in problem (1.1), satisfy the following assumptions:

(A1) $\quad m_1 \in M^+(I)$ and $-m_1 \in M^+(I)$,

(A2) $\quad m_2 \in L^\infty(I)$ such that $m_2 \gneqq 0$ in I and $\operatorname{ess\,inf}_I m_2 = 0$,

where $M^+(I) = \{m \in L^\infty(I) : \text{meas}(\{x \in I : m(x) > 0\}) \neq 0\}$.
For the problem in dimension N (where N is an integer > 1), we can find in the literature several works that have addressed the study of the eingencurves, for example, we cite here for the problems involving the p-Laplacian differential operator under different boundary conditions. For a Dirichlet problem, under the hypothesis $\operatorname{ess\,inf}_\Omega m_2 > 0$ (where $\Omega \subset \mathbb{R}^N$ is a bounded domain) in [5] P.A. Binding, Y.X. Huang studied the existence and some properties of the principal eigencurve for the p-Laplacian, an existence result, variational formulation and the asymptotic behavior of the n-th eigencurve are proved in [7]. The second eigencurve of the one-dimensional p-Laplacian, along with its properties for a nonnegative weight $m_2 \geq 0$, was studied in [14]. The authors of articles [3] and [6] studied the eigencurves of an elliptic problem with respect to the Sturm-Liouville boundary conditions. Concerning the existence and uniqueness of the first eigencurve associated with p-Laplacian for a Neumann-type problem, with the assumption $\operatorname{ess\,inf}_\Omega m_2 > 0$, is demonstrated in [11] by A. Sanhaji and A. Dakkak. Moreover, the existence, variational characterization, differentiability, and asymptotic behavior of the n-th eigencurve for the one-dimensional p-Laplacian with an indefinite weight were established in [12]. The same authors in a recent article [13] have studied under the hypothesis

$$m_2 \gneqq 0 \text{ in } \Omega \text{ and } \Omega^*_{m_1} \subset \Omega^*_{m_2},$$

where $\Omega_m^* = \{x \in \Omega : \ m(x) \neq 0\}$ for a given $m \in L^\infty(\Omega)$, showed the existence of the first eingencurve and studied some main properties of this eingencurve.
The main purpose of this work is not only to extend some of the aforementioned results to the case of a weight m_2 satisfying $\operatorname{ess\,inf}_I m_2 = 0$, but more importantly, to prove the existence of the second eigencurve through an approximation approach. Furthermore, we establish several useful properties of the second eigencurve. In particular, we provide a detailed analysis of its variational characterization, asymptotic behavior, and differentiability..
The structure of this work is as follows. Section 2 we introduce some well-known results on the spectrum of the p-Laplacian with weights. Finally, Section 3 is devoted to statement and demonstration our main results.

2. Preliminaries and auxiliary results

In this work I will be a bounded open interval of $\mathbb{R}$, and we denote by $W^{1,p}(I)$ the standard Sobolev space with norm

$$\|u\|_{1,p} = (\|u\|_p^p + \|u'\|_p^p)^{\frac{1}{p}},$$

where $\|.\|_p$ denotes the standard Lebesgue norm of $L^p(I)$. For more details on the theory of Sobolev spaces, we refer the reader to [1]. In the rest of this section, we present some results on the spectrum of the one-dimensional p-Laplacian that will be used throughout this article. Let us consider the following nonlinear Neumann boundary value problem

$$\begin{cases} -\Delta_p u = \lambda\, m|u|^{p-2}u \quad \text{in} \quad I =]a,b[\\ \\ u'(a) = u'(b) = 0, \end{cases} \tag{2.1}$$

where $m \in L^\infty(I)$, λ is a real parameter and a, $b \in \mathbb{R}$ with $a < b$.
A function $u \in W^{1,p}(I)$ is called a weak solution of problem (2.1) if the following equality is satisfied for every test function $v \in W^{1,p}(I)$

$$\int_I |u'|^{p-2}u'\, v' dx = \lambda \int_I m\, |u|^{p-2}u.v\, dx. \tag{2.2}$$

We will say that a real number λ is an eigenvalue of p-Laplacian with weight m, if there exists $u \in W^{1,p}(I) \setminus \{0\}$ verifying (2.2). In this case u is called an corresponding to the eigenvalue λ. We will denote $\sigma_p^+(-\Delta_p, m, I)$ the set of all positive eigenvalues, and by $\sigma_p^-(-\Delta_p, m, I)$ the set of negative eigenvalues. It is well known that $\sigma_p^+(-\Delta_p, m, I)$ the positive spectrum of the p-Laplacian ($p \neq 2$) with weights in dimension $N \geq 1$ [2] contains an increasing sequence of positive eigenvalues $(\lambda_n(m))_n$ defined as follows

$$\frac{1}{\lambda_n(m)} = \sup_{K \in \Gamma_n} \min_{u \in K} \frac{\int_I m|u|^p dx}{\int_I |u'|^p dx}, \text{ with } \lambda_n(m) \to +\infty, \text{ as } n \to +\infty, \tag{2.3}$$

where $\Gamma_n = \big\{K \subset S : \ \text{K is symmetric, compact and } \gamma(K) \geq n\big\}$, S is the unit sphere of $W^{1,p}(I)$ and γ is the Krasnoselskii genus, also in the same work [2], the

authors showed that if the weight m changes its sign in I and $\int_I m(x)dx < 0$, then the first eigenvalue defined by

$$\lambda_1(m) = \inf_{u\in\mathcal{A}(m)} \|u'\|_p^p, \text{ where } \mathcal{A}(m) = \left\{ u \in W^{1,p}(I); \int_I m|u|^p dx = 1 \right\} \quad (2.4)$$

is strictly positive, simple, isolated and any eigenfunction associated to $\lambda_1(m)$ is of constant sign in I.
In the case $N = 1$, A. Dakkak, S. El Habib and N. Tsouli [8] obtained a sequence of eigenvalues and proved that its corresponding eigenfunctions have properties (i), (ii) and (iii) given below, using a standard min-max formula of the Lusternik-Schnirelman type (see [15])

(i) Any eigenfunction corresponding to an eigenvalue $\lambda_n(m)$ has exactly $(n-1)$ zeros in I,
(ii) eigenvalues in the spectrum are simple,
(iii) $\lambda_n(m)$, $n \geq 1$ satisfies the strict monotonicity property with respect to the weight.

When $-m \in M^+(\Omega)$, we can also define the negative spectrum by $\sigma_p^-(-\Delta_p, m, I) = -\sigma_p^+(-\Delta_p, -m, I)$ which admits a decreasing sequence $(\lambda_{-n}(m))_{n\geq 1}$ of negative eigenvalues such that $\lambda_{-n}(m) \to -\infty$ as $n \to +\infty$, and

$$\lambda_{-n}(m) = -\lambda_n(-m).$$

Proposition 2.1. *(See, for example [7]) The mapping $\lambda_n : m \to \lambda_n(m)$ is continuous in $M^+(I)$ for the distance associated with the infinity norm $\|.\|_\infty$.*

Proposition 2.2. *Let $(m_k)_k$ be a sequence in $M^+(I)$ such that $m_k \to m$ in $L^\infty(I)$, then*

$$\lim_{k\to\infty} \lambda_n(m_k) = +\infty \text{ if and only if } m \leq 0 \text{ almost everywhere in } I.$$

Proof. Let $(m_k)_k$ be a sequence in $M^+(I)$ such that $m_k \to m$ in $L^\infty(I)$. Assume first that $\lim_{k\to\infty} \lambda_n(m_k) = +\infty$, we prove that $m \leq 0$ almost everywhere in I. Indeed, we assume by contradiction that $meas(\{x \in I : m(x) > 0\}) \neq 0$, by using the continuity of λ_n with respect to the weight (cf. proposition 2.1), we have $\lim_{k\to\infty} \lambda_n(m_k) = \lambda_n(m)$, is a finite quantity, it results in a contradiction.
Conversely, if $m \leq 0$ almost everywhere in I, Assume, to the contrary that there exists $\lambda > 0$ and a subsequence of the $(m_k)_k$ still denoted $(m_k)_k$ such that

$$\lambda_n(m_k) \leq \lambda.$$

We put $r = \frac{2\lambda}{\lambda_n(2)}$, since $m_k \to m$ in $L^\infty(I)$, then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, we have

$$\|m_k - m\|_\infty \leq \frac{2}{r},$$

hence

$$m_k \leq m + \frac{2}{r} \quad \text{a.e. } x \in I.$$

Therefore, since $m \leq 0$ a.e. in I, we deduce that

$$m_k \leq \frac{2}{r} \quad \text{a.e. } x \in I.$$

By Proposition 2.1, we have

$$\lambda_n(m_k) \geq \lambda_n(\frac{2}{r}) = r\lambda_n(2) = 2\lambda,$$

this leads to a contradiction. Consequently, $\lim_{k\to\infty} \lambda_n(m_k) = +\infty$. □

Proposition 2.3. *(See, [14]) We can characterize the second eigenvalue $\lambda_2(m)$ as follows*

$$\lambda_2(m) = \inf\Big\{ \lambda : \ \lambda \text{ positive eigenvalue of (2.1) and } \lambda > \lambda_1(m) \Big\}.$$

3. Main results and their proofs

Before stating our main results, we introduce the following notation, for given $m \in L^\infty(I)$ we set

$$I_m^+ = \{x \in I : \ m(x) > 0\} \text{ and } I_m^- = \{x \in I : \ m(x) < 0\}.$$

Our first main result in this paper is devoted to existence and uniqueness of the second eigencurve associated with problem (1.1).

Theorem 3.1. *Assume m_1, m_2 satisfy (A1) and (A2). Then,*

(i) *For $\lambda_{-2}(m_1) \leq \alpha \leq \lambda_2(m_1)$, there exists a unique real number $\beta_2 = \beta_2(\alpha) \geq 0$ such that*

$$\lambda_2(\alpha\, m_1 + \beta_2\, m_2) = 1. \tag{3.1}$$

(ii) *For each $\beta < 0$, there exists a unique real number $\alpha_2^+ = \alpha^+(\beta) > \lambda_2(m_1)$ such that*

$$\lambda_2(\alpha_2^+\, m_1 + \beta\, m_2) = 1. \tag{3.2}$$

(iii) *For each $\beta < 0$ and $\operatorname{meas}(I_{m_1}^-) > 0$, there exists a unique real number $\alpha_2^- = \alpha^-(\beta) < \lambda_{-2}(m_1)$ such that*

$$\lambda_2(\alpha_2^-\, m_1 + \beta\, m_2) = 1. \tag{3.3}$$

Proof. (i) Let $\alpha \in [\lambda_{-2}(m_1), \lambda_2(m_1)]$ and consider the sequence of weight functions $(m_{2,k})_{k>0} \subset M^+(I)$ defined by $m_{2,k} = m_2 + \frac{1}{k}$. Since for all $k \in \mathbb{N}^*$ $\operatorname{ess\,inf}_I m_{2,k} > 0$, then according to the results established in [12], we have for all $\alpha \in [\lambda_{-2}(m_1), \lambda_2(m_1)]$ there exists a unique real $\beta_{2,k}(\alpha)$ verifying:

$$\lambda_2(\alpha\, m_1 + \beta_{2,k}(\alpha)\, m_{2,k}) = 1. \tag{3.4}$$

In order to complete and simplify the exposition of the rest of the proof, we will divide it into several steps.

First, we show that $(\beta_{2,k}(\alpha))_k$ is a bounded sequence.

We assume by contradiction that $\beta_{2,k}(\alpha) \to +\infty$ when $k \to +\infty$. Indeed, let $\alpha \in [\lambda_{-2}(m_1), \lambda_2(m_1)]$ there exists a unique real number $\beta_{2,k}(\alpha) \in \mathbb{R}$, such that

$$\lambda_2(\alpha\, m_1 + \beta_{2,k}(\alpha)\, m_{2,k}) = 1.$$

On the one hand by the homogeneity of λ_2, we obtain

$$\lambda_2(\frac{\alpha}{\beta_{2,k}(\alpha)}\, m_1 + m_{2,k}) = \beta_{2,k}(\alpha),$$

and by hypothesis we have

$$\lambda_2(\frac{\alpha}{\beta_{2,k}(\alpha)}\, m_1 + m_{2,k}) \to +\infty \ \text{ as } \ k \to +\infty. \tag{3.5}$$

On the other hand, we have

$$\frac{\alpha}{\beta_{2,k}(\alpha)}\, m_1 + m_{2,k} \to m_2 \ \text{ in } L^\infty(I) \text{ as } k \to +\infty,$$

Using the continuity of λ_2 with respect to the weight, we obtain that as $k \to +\infty$

$$\lambda_2(\frac{\alpha}{\beta_{2,k}(\alpha)}\, m_1 + m_{2,k}) \to \lambda_2(m_2) \ \text{ which is a finite quantity.}$$

This is a contradiction with (3.5). Consequently, $(\beta_{2,k})_k$ is a bounded sequence of functions.

Second, we show the monotonicity of the sequence $(\beta_{2,k}(\alpha))_k$.

To establish the monotony of $(\beta_{2,k}(\alpha))_k$, we will consider two cases:

Case $\lambda_{-2}(m_1) < \alpha < \lambda_2(m_1)$. In this case $(\beta_{2,k}(\alpha))_k$ is increasing. Indeed, we argue by contradiction, assuming that there exists $k \in \mathbb{N}^*$ such that, $\beta_{2,k}(\alpha) > \beta_{2,k+1}(\alpha)$.

We know that for $\lambda_{-2}(m_1) < \alpha < \lambda_2(m_1)$, we have $\beta_{2,k}(\alpha) > 0$, (see [12]). Since for all $k \in \mathbb{N}^*$, we have $m_{2,k} > m_{2,k+1}$, then

$$\beta_{2,k}(\alpha)\, m_{2,k} > \beta_{2,k+1}(\alpha)\, m_{2,k+1},$$

which yields

$$\alpha\, m_1 + \beta_{2,k}(\alpha)\, m_{2,k} \gneqq \alpha\, m_1 + \beta_{2,k+1}(\alpha)\, m_{2,k+1}.$$

Thanks to the strict monotonicity of λ_2 with respect to the weight, we obtain

$$1 = \lambda_2(\alpha\, m_1 + \beta_{2,k}(\alpha)\, m_{2,k}) < \lambda_2(\alpha\, m_1 + \beta_{1,k+1}(\alpha)\, m_{2,k+1}) = 1. \tag{3.6}$$

This is a contradiction.

Thus for all $k \in \mathbb{N}^*$, we have $\beta_{2,k}(\alpha) \leq \beta_{2,k+1}(\alpha)$ i.e., the sequence of functions $(\beta_{2,k})_k$ is increasing.

Case $\alpha = \lambda_{-2}(m_1)$ or $\alpha = \lambda_2(m_1)$. In this case for all $k \in \mathbb{N}^*$, we have $\beta_{2,k}(\alpha) = 0$. Consequently, the sequence $(\beta_{2,k}(\alpha))_k$ is stationary.

Third, we pass to the limit.

According to the results obtained on the monotony of $\beta_{2,k}(\alpha)$ in the previous step, we will distinguish the following two cases:

- If $\lambda_{-2}(m_1) < \alpha < \lambda_2(m_1)$, then for each fixed $\alpha \in]\lambda_{-2}(m_1), \lambda_2(m_1)[$, we have that $(\beta_{2,k}(\alpha))_k$ is an increasing and bounded sequence of real numbers, then $(\beta_{2,k}(\alpha))_k$ is convergent (for a subsequence). This yields $(\beta_{2,k})_k$ is pointwise convergent on $]\lambda_{-2}(m_1),\ \lambda_2(m_1)[$. We then set $\beta_2^+(\alpha) = \lim\limits_{k\to\infty} \beta_{2,k}(\alpha)$.
- If $\alpha = \lambda_{-2}(m_1)$ or $\alpha = \lambda_2(m_1)$, in this case, we have $(\beta_{2,k}(\alpha))_k$ is a stationary numerical sequence, then $(\beta_{2,k})_k$ converges pointwise to zero function.

Consequently, the sequence $(\beta_{2,k})_k$ converges pointwise to β_2 on the interval $]\lambda_{-2}(m_1), \lambda_2(m_1)[$, with β_2 defined as follows

$$\beta_2(\alpha) = \begin{cases} \beta_2^+(\alpha) & \text{if } \lambda_{-2}(m_1) < \alpha < \lambda_2(m_1) \\ \\ 0 & \text{if } \alpha = \lambda_{-2}(m_1) \text{ or } \alpha = \lambda_2(m_1). \end{cases}$$

Finally, the equation (3.1) will be reached after passing to the limit as $k \to \infty$ in the equation (3.4).

Fourth, we show the uniqueness of the $\beta_2(\alpha)$.

Indeed, for a fixed $\alpha \in]\lambda_{-2}(m_1), \lambda_2(m_1)[$ assume by contradiction that there exists two distinct real numbers $\beta_2(\alpha)$ and $\beta_2'(\alpha)$ that

$$\lambda_2(\alpha\, m_1 + \beta_2(\alpha)\, m_2) = 1 \text{ and } \lambda_2(\alpha\, m_1 + \beta_2'(\alpha)\, m_2) = 1,$$

suppose for example that $\beta_2(\alpha) > \beta_2'(\alpha)$. Then, since $\operatorname{ess\,inf}_I m_2 \geq 0$, we have

$$\alpha\, m_1 + \beta_2(\alpha)\, m_2 \gneqq \alpha\, m_1 + \beta_2'(\alpha)\, m_2.$$

Thus, we can conclude from the strict monotonicity of λ_2 with respect to the weight that

$$1 = \lambda_2(\alpha\, m_1 + \beta_2(\alpha) m_2) < \lambda_2(\alpha\, m_1 + \beta_2'(\alpha) m_2) = 1.$$

This yields a contradiction, hence it is necessary that $\beta_2(\alpha) = \beta_2'(\alpha)$.

(ii) For $\beta < 0$, as in the proof of (i) we introduce a sequence of weight functions $(m_{2,k})_{k>0} \subset M^+(I)$ defined by $m_{2,k} = m_2 + \frac{1}{k}$. Since for all $k \in \mathbb{N}^*$ $\operatorname{ess\,inf}_I m_{2,k} > 0$ then, according to the results established in [14], we have for all $\beta < 0$ there are two reals $\alpha_{2,k}^+(\beta)$ and $\alpha_{2,k}^-(\beta)$ verifying:

$$\lambda_2(\alpha_{2,k}^+(\beta)\, m_1 + \beta\, m_{2,k}) = 1 \text{ and } \lambda_2(\alpha_{2,k}^-(\beta)\, m_1 + \beta\, m_{2,k}) = 1. \tag{3.7}$$

We show that $(\alpha_{2,k}^+(\beta))_k$ is a bounded sequence.

We assume by contradiction that $\alpha_{2,k}^+(\beta) \to +\infty$ when $k \to +\infty$. Indeed, let $\beta < 0$ then there exists a real $\alpha_{2,k}^+(\beta)$ such that

$$\lambda_2(\alpha_{2,k}^+(\beta)\, m_1 + \beta\, m_{2,k}) = 1.$$

On the one hand by the homogeneity of λ_2, we obtain

$$\lambda_2(m_1 + \frac{\beta}{\alpha_{2,k}^+(\beta)} m_{2,k}) = \alpha_{2,k}^+(\beta),$$

and by hypothesis we have

$$\lambda_2(m_1 + \frac{\beta}{\alpha_{2,k}^+(\beta)} m_{2,k}) \to +\infty \text{ as } k \to +\infty. \tag{3.8}$$

Moreover, we have

$$m_1 + \frac{\beta}{\alpha_{2,k}^+(\beta)} m_{2,k} \to m_1 \text{ in } L^\infty(I) \text{ as } k \to +\infty.$$

By the continuity of λ_2 with respect to the weight, we obtain that as $k \to +\infty$

$$\lambda_2(m_1 + \frac{\beta}{\alpha^+_{2,k}(\beta)} m_{2,k}) \to \lambda_2(m_1) \text{ which is a finite quantity.}$$

This is a contradiction with (3.8). Consequently, $(\alpha^+_{2,k})_k$ is a bounded sequence of functions.

We prove the monotony of the sequence $(\alpha^+_{2,k}(\beta))_k$.

To establish the monotony of $(\alpha^+_{2,k}(\beta))_k$, Suppose, by contradiction, that there exists $k \in \mathbb{N}^*$ such that, $\alpha^+_{2,k}(\beta) > \alpha^+_{2,k+1}(\beta)$. So we distinguish two cases

- if $x \in I^+_{m_1}$, then $m_1 > 0$ and since for all $k \in \mathbb{N}^*$, $\alpha^+_{2,k}(\beta) > 0$, we have

  $$\alpha^+_{2,k}(\beta)\, m_1 > \alpha^+_{2,k+1}(\beta)\, m_1,$$

  which gives

  $$\alpha^+_{2,k}\, m_1 + \beta\, m_{2,k} \gneqq \alpha^+_{2,k+1}\, m_1 + \beta\, m_{2,k+1}.$$

  Using the strict monotonicity of λ_2 with respect to the weight, we obtain that

  $$1 = \lambda_2(\alpha^+_{2,k}\, m_1 + \beta\, m_{2,k}) < \lambda_2(\alpha^+_{2,k+1}\, m_1 + \beta\, m_{2,k+1}) = 1. \tag{3.9}$$

  This is a contradiction.
- If $x \in I^-_{m_1}$, in this case $m_1 < 0$ and for all $k \in \mathbb{N}^*$, $\alpha^+_{2,k}(\beta) > 0$, then

  $$\alpha^+_{2,k}(\beta)\, m_1 < \alpha^+_{2,k+1}(\beta)\, m_1,$$

  which yields

  $$\alpha^+_{2,k}\, m_1 + \beta\, m_{2,k} \lneqq \alpha^+_{2,k+1}\, m_1 + \beta\, m_{2,k+1}.$$

  Similarly as in the previous case, by the strict monotonicity of λ_2 with respect to the weight, we obtain that

  $$1 = \lambda_2(\alpha^+_{2,k}\, m_1 + \beta\, m_{2,k}) > \lambda_2(\alpha^+_{2,k+1}\, m_1 + \beta\, m_{2,k+1}) = 1. \tag{3.10}$$

Thus, in two cases, for all $k \in \mathbb{N}^*$ we have $\alpha^+_{2,k}(\beta) \leq \alpha^+_{2,k+1}(\beta)$ i.e., the sequence of functions $(\alpha^+_{2,k})_k$ is increasing.

We pass to the limit.

According to the results obtained on the monotony of $\alpha^+_{2,k}(\beta)$ in the previous step, for each fixed $\beta < 0$, we have that $(\alpha^+_{2,k}(\beta))_k$ is an increasing and bounded sequence of real numbers, then $(\alpha^+_{2,k}(\beta))_k$ is convergent (for a subsequence). This yields $(\alpha^+_{2,k})_k$ is pointwise convergent on $]-\infty, 0[$. We then set $\alpha^+(\beta) = \lim\limits_{k\to\infty} \alpha^+_{2,k}(\beta)$.

Consequently, the sequence of functions $(\alpha^+_{2,k})_k$ is pointwise convergent on $]-\infty, 0[$ to the function α^+.

Finally, the equation (3.2) will be reached after passing to the limit as $k \to \infty$ in the equation (3.7).

We establish the uniqueness of $\alpha^+_2(\beta)$**.**

Indeed, for a fixed $\beta < 0$ assume by contradiction that there exists two distinct real numbers $\alpha^+(\beta)$ and $\delta^+(\beta)$ that

$$\lambda_2(\alpha^+(\beta)\, m_1 + \beta\, m_2) = 1 \text{ and } \lambda_2(\delta^+(\beta)\, m_1 + \beta\, m_2) = 1,$$

suppose for example that $\alpha^+(\beta) > \delta^+(\beta)$ and $x \in I_1^+$ (i.e. $m_1 > 0$), we have

$$\alpha^+(\beta)\, m_1 + \beta\, m_2 \gneqq \delta^+(\beta)\, m_1 + \beta\, m_2.$$

Thus, we deduce from the strict monotonicity property of λ_2 with respect to the weight that

$$1 = \lambda_2(\alpha^+(\beta)\, m_1 + \beta\, m_2) < \lambda_2(\delta^+(\beta)\, m_1 + \beta\, m_2) = 1.$$

This yields a contradiction, hence it's necessary that $\alpha^+(\beta) = \delta^+(\beta)$.
(iii) Suppose that $meas(I_{m_1}^-) > 0$, we show the third result in a similar way. □

Furthermore, in the present theorem by employing the min-max arguments of Lusternik-Schnirelman type, to give a variational characterization of the second eigencurve.

Theorem 3.2. *Under the hypotheses of of Theorem 3.1*

(i) *For $\lambda_{-2}(m_1) \le \alpha \le \lambda_2(m_1)$, we have*

$$\beta_2(\alpha) = \inf_{K\in\Gamma_2} \max_{u\in K\setminus\mathbb{R}} \frac{\int_I |u'|^p dx - \alpha \int_I m_1|u|^p dx}{\int_I m_2|u|^p dx}, \tag{3.11}$$

(ii) *For each $\beta < 0$, we have*

$$\alpha_2^+(\beta) = \inf_{K\in\Gamma_2} \max_{u\in K\setminus\mathbb{R}} \frac{\int_I |u'|^p dx - \beta \int_I m_2|u|^p dx}{\int_I m_1|u|^p dx}, \tag{3.12}$$

(iii) *For each $\beta < 0$ and $\mathrm{meas}(I_{m_1}^-) > 0$, we have*

$$\alpha_2^-(\beta) = \sup_{K\in\Gamma_2} \min_{u\in K\setminus\mathbb{R}} \frac{\int_I |u'|^p dx - \beta \int_I m_2|u|^p dx}{\int_I m_1|u|^p dx}. \tag{3.13}$$

Proof. (i) If $\lambda_{-2}(m_1) \le \alpha \le \lambda_2(m_1)$ we pose

$$\theta(\alpha) = \inf_{K\in\Gamma_2} \max_{u\in K} \frac{\int_I |u'|^p dx - \alpha \int_I m_1|u|^p dx}{\int_I m_2|u|^p dx}.$$

We have for all $K \in \Gamma_2$, there is $u_K \in K$ such that

$$\theta(\alpha) \le \max_{u\in K} \frac{\int_I |u'|^p dx - \alpha \int_I m_1|u|^p dx}{\int_I m_2|u|^p dx} = \frac{\int_I |u_K'|^p dx - \alpha \int_I m_1|u_K|^p dx}{\int_I m_2|u_K|^p dx}$$

since, $\int_I m_2|u_K|^p dx > 0$ we get

$$\alpha \int_I m_1|u_K|^p dx + \theta(\alpha) \int_I m_2|u_K|^p dx \le \int_I |u_K'|^p dx = 1.$$

So that

$$\min_{u\in K} \int_I (\alpha\, m_1 + \theta(\alpha)\, m_2)|u|^p dx \le \alpha \int_I m_1|u_K|^p dx + \theta(\alpha) \int_I m_2|u_K|^p dx \le 1$$

this for all $K \in \Gamma_2$ which gives

$$\sup_{K\in\Gamma_2} \min_{u\in K} \int_I (\alpha\, m_1 + \theta(\alpha)\, m_2)|u|^p dx \le 1$$

we know that

$$\frac{1}{\lambda_2(\alpha\, m_1+\theta(\alpha)\, m_2)}=\sup_{K\in\Gamma_2}\min_{u\in K}\int_I(\alpha\, m_1+\theta(\alpha)\, m_2)|u|^p dx,$$

it follows that

$$\lambda_2(\alpha\, m_1+\theta(\alpha)\, m_2)\geq 1. \tag{3.14}$$

On the other hand, according to the definition of $\beta_2(\alpha)$ we have

$$\lambda_2(\alpha\ m_1+\beta_2(\alpha)\ m_2)=1$$

that means

$$\sup_{K\in\Gamma_2}\min_{u\in K}\int_I(\alpha\ m_1+\beta_2(\alpha)\ m_2)|u|^p dx=1.$$

Moreover, for every $K\in\Gamma_n$, there exists $u_K\in K$ such that

$$\int_I(\alpha\, m_1+\beta_2(\alpha)\, m_2)|u_K|^p dx=\min_{u\in K}\int_I(\alpha\, m_1+\beta_2(\alpha)\, m_2)|u|^p dx\leq\frac{1}{\lambda_2(\alpha\, m_1+\beta_2(\alpha)\, m_2)}=1$$

which implies

$$\alpha\int_I m_1|u_K|^p dx+\beta_2(\alpha)\int_I m_2|u_K|^p dx\leq\int_I|u_K'|^p dx,$$

it follows that

$$\beta_2(\alpha)\int_I m_2|u_K|^p dx\leq\int_I|u_K'|^p dx-\alpha\int_I m_1|u_K|^p dx.$$

Since $\int_I m_2|u_K|^p dx>0$, because if not we have $u_K\equiv 0$ or $m_2\equiv 0$, this contradicts the hypothesis on m_2 and the fact that $u_K\in K$, then

$$\beta_2(\alpha)\leq\frac{\int_I|u_K'|^p dx-\alpha\int_I m_1|u_K|^p dx}{\int_I m_2|u_K|^p dx}\leq\max_{u\in K}\frac{\int_I|u'|^p dx-\alpha\int_I m_1|u|^p dx}{\int_I m_2|u|^p dx}$$

this is true for all $K\in\Gamma_2$, so

$$\beta_2(\alpha)\leq\inf_{K\in\Gamma_2}\max_{u\in K}\frac{\int_I|u'|^p dx-\alpha\int_I m_1|u|^p dx}{\int_I m_2|u|^p dx}=\theta(\alpha). \tag{3.15}$$

Finally, by combining (3.14) and (3.15), we obtain

$$\beta_2(\alpha)=\theta(\alpha).$$

(ii) If $\beta<0$, we pose $\delta^+(\beta)=\inf_{K\in\Gamma_2}\max_{u\in K}\dfrac{1-\beta\int_I m_2|u|^p}{\int_I m_1|u|^p}$.

By definition of $\delta^+(\beta)$ and the fact that for any u eigenfunction associated to $\lambda_2(m_1)$ changes sign in I, we obtain

$$\int_I|u'|^{p-2}u'.v'x-\beta\int_I m_2|u|^{p-2}u.v\,dx=\int_I\delta^+(\beta)\, m_1|u|^{p-2}u.v\,dx,\ \forall v\in W^{1,p}(I), \tag{3.16}$$

we deduce also that, if $w \in W^{1,p}(I)$ is an eigenfunction of the operator $-\Delta_p(.) - \beta\, m_2\phi_p(.)$ that changes sign in I associated with the corresponding eigenvalue $\lambda > 0$ then $\lambda > \delta^+(\beta)$. Then from (3.16) we have

$$\int_I |u'|^{p-2}u'.v'dx = \int_I (\delta^+(\beta)\, m_1 \, + \beta m_2)|u'|^{p-2}u'.v\, dx \;\; \forall v \in W^{1,p}(I);$$

Hence, 1 is an eigenvalue of the p-Laplacian with weight $(\delta^+(\beta)m_1 + \beta m_2)$, since the corresponding eigenfunction u changes sign in I, we conclude that

$$\lambda_2(\delta^+(\beta)\, m_1 + \beta\, m_2) \leq 1. \tag{3.17}$$

On the other hand, by definition of $\delta^+(\beta)$, for every $K \in \Gamma_2$, there exists $u_K \in K$ such that

$$\max_{u\in K} \frac{1 - \beta \displaystyle\int_I m_2|u|^p dx}{\displaystyle\int_I m_1|u|^p dx} = \frac{1 - \beta \displaystyle\int_I m_2|u_K|^p dx}{\displaystyle\int_I m_1|u_K|^p dx} \geq \delta^+(\beta),$$

so we deduce

$$\max_{u\in K} \frac{1}{\displaystyle\int_I (\delta^+(\beta)\, m_1 + \beta\, m_2)|u|^p dx} \geq 1, \;\text{ for all }\; K \in \Gamma_2$$

then

$$\inf_{K\in\Gamma_2} \max_{u\in K} \frac{1}{\displaystyle\int_I (\delta^+(\beta)\, m_1 + \beta\, m_2)|u|^p dx} \geq 1,$$

thus

$$\lambda_2(\delta^+(\beta)\, m_1 + \beta\, m_2) \geq 1, \tag{3.18}$$

therefore we infer (3.17) and by (3.18) we get

$$\lambda_2(\delta^+(\beta)\, m_1 + \beta\, m_2) = 1.$$

Finally, by uniqueness of $\alpha^+(\beta)$ we obtain

$$\alpha^+(\beta) = \delta^+(\beta) = \inf_{K\in\Gamma_2} \max_{u\in K} \frac{1 - \beta \int_I m_2|u|^p dx}{\int_I m_1|u|^p dx}$$

.

For the proof of (iii) we follow the same approach as in the proof of (ii). □

Concerning the asymptotic behavior of the second eigencurve of the p-Laplacian with weights, we state the following result.

Theorem 3.3. *Suppose that assumptions* $(A1)$ *and* $(A2)$ *are satisfied. Then*

(i) $\displaystyle\lim_{\beta\to-\infty} \frac{\alpha_2^+(\beta)}{\beta} = -\operatorname*{ess\,inf}_{I^+_{m_1}} \frac{m_2}{m_1}$,

(ii) *if* $\operatorname{meas}(I^-_{m_1}) > 0$, *then* $\displaystyle\lim_{\alpha\to-\infty} \frac{\alpha_2^-(\beta)}{\beta} = -\operatorname*{ess\,sup}_{I^-_{m_1}} \frac{m_2}{m_1}$.

Proof. Let us verify assertion (i). We consider $\beta < 0$, then by Theorem 3.1 there exist $\alpha_2^+(\beta) \geq \lambda_2(m_1)$ such that

$$\lambda_2(\alpha_2^+(\beta)\, m_1 + \beta\, m_2) = 1.$$

thus, there exists $I_\beta \subset I$ such that

$$\alpha_2^+(\beta)\, m_1 + \beta\, m_2 > 0 \text{ a.e. in } I_\beta \text{ and } meas(I_\beta) \neq 0.$$

Since $m_2 \geq 0$, we have $m_1 > 0$ a.e. in I_β, then $I_\beta \subset I_{m_1}^+$, so we can write

$$\frac{-\alpha_2^+(\beta)}{\beta} > \frac{m_2}{m_1} \text{ a.e. in } I_\beta \subset I_{m_1}^+,$$

thus

$$\liminf_{\beta \to -\infty} \frac{-\alpha_2^+(\beta)}{\beta} \geq \operatorname*{ess\,inf}_{I_{m_1}^+} \frac{m_2}{m_1}. \tag{3.19}$$

We set $l = \liminf\limits_{\beta \to -\infty} \dfrac{-\alpha_2^+(\beta)}{\beta}$, then for a sequence $(\beta_k)_k$ such that $\beta_k \to +\infty$, we have

$$\lim_{k \to +\infty} \frac{-\alpha_2^+(\beta_k)}{\beta_k} = l \text{ and } \lambda_2\Big(\frac{-\alpha_2^+(\beta_k)}{\beta_k} m_1 - m_2\Big) = -\beta_k.$$

Since

$$\frac{-\alpha_2^+(\beta_k)}{\beta_k} m_1 - m_2 \to l\, m_1 - m_2 \text{ in } L^\infty(I) \text{ and } -\beta_k \to +\infty \text{ as } k \to +\infty,$$

by Proposition 2.2 that $l\, m_1 - m_2 \leq 0$ a.e. in I, and hence

$$l \leq \operatorname*{ess\,inf}_{I_{m_1}^+} \frac{m_2}{m_1}. \tag{3.20}$$

Finally, combining (3.19) and (3.20) we obtain the equality of the first assertion. The assertion in (ii) can be established by following the same arguments as in. This completes the proof of 3.3. □

Our final result in this work is related to the differentiability of the function $\alpha \to \beta_1(\alpha)$ for all $\alpha \in [\lambda_{-2}(m_1), \lambda_2(m_1)]$.

Theorem 3.4. *For $\lambda_{-2}(m_1) \leq \alpha \leq \lambda_{-2}(m_1)$, assume that hypotheses (A_1) and (A_2) are satisfied. Then*

(i) *the function $\alpha \to \varphi_\alpha$ is continuous from $\mathbb{R}$ into $L^p(I)$, where φ_α an eigenfunction associated to $\lambda_2(\alpha\, m_1 + \beta_1(\alpha)\, m_2) = 1$.*

(ii) *The function $\alpha \mapsto \beta_2(\alpha)$ is differentiable on $]\lambda_{-2}(m_1), \lambda_2(m_1)[$. Moreover, for every $\alpha_0 \in]\lambda_{-2}(m_1), \lambda_2(m_1)[$, we have*

$$\beta_2'(\alpha_0) = -\frac{\int_I m_1 |\varphi_{\alpha_0}|^p dx}{\int_I m_2 |\varphi_{\alpha_0}|^p dx}. \tag{3.21}$$

Proof. (i) Let $\alpha_0 \in [\lambda_{-2}(m_1), \lambda_2(m_1)]$ be a fixed real scalar and denote by $\varphi_{\alpha_0} \in W^{1,p}(I)$ a fixed eigenfunction associated to the first eigenvalue $\lambda_2(\alpha_0\, m_1 + \beta_1(\alpha_0)\, m_2) = 1$, i.e.,

$$\int_I |\varphi_{\alpha_0}'|^{p-2} \varphi_{\alpha_0}' \theta' dx = \int_I (\alpha_0\, m_1 + \beta_2(\alpha_0)\, m_2)\, |\varphi_{\alpha_0}|^{p-2} \varphi_{\alpha_0}\, \theta\, dx,\ \forall \theta \in W^{1,p}(I).$$

For $\alpha \in \mathbb{R}$ such that $\alpha \neq \alpha_0$, we consider $0 \neq \varphi_\alpha \in W^{1,p}(I)$ an eigenfunction associated to $\lambda_2(\alpha\, m_1 + \beta_1(\alpha)\, m_2) = 1$, since $\varphi_\alpha \in K \subset S$ thus (φ_α) is bounded in $W^{1,p}(I)$, we have for a subsequence

$$\varphi_\alpha \rightharpoonup u_0 \text{ weakly in } W^{1,p}(I) \text{ as } \alpha \to \alpha_0,$$

and

$$\varphi_\alpha \to u_0 \text{ strongly in } L^p(I) \text{ as } \alpha \to \alpha_0.$$

Let us first show that $u_0 \neq 0$. Indeed, as φ_α is an eigenfunction associated to $\lambda_2(\alpha\, m_1 + \beta_2(\alpha)\, m_2) = 1$, then

$$\int_I |\varphi'_\alpha|^{p-2}\varphi'_\alpha \theta' dx = \int_I (\alpha\, m_1 + \beta_2(\alpha)\, m_2)\, |\varphi_\alpha|^{p-2}\varphi_\alpha \theta\, dx,\ \forall \theta \in W^{1,p}(I). \quad (3.22)$$

Taking $\theta = \varphi_\alpha$ as testing function in (3.22) gives

$$\|\varphi'_\alpha\|_p^p = \int_I (\alpha\, m_1 + \beta_2(\alpha)\, m_2)\, |\varphi_\alpha|^p dx,$$

and since $\|\varphi'_\alpha\|_p^p = 1$ ($\varphi_\alpha \in K \subset S$), then

$$1 = \int_I (\alpha\, m_1 + \beta_2(\alpha)\, m_2)\, |\varphi_\alpha|^p dx.$$

Moreover, since $m_1,\ m_2 \in L^\infty(I)$ then there exists a positive constant $c > 0$ such that

$$1 \leq c\, \|\varphi_\alpha\|_p^p.$$

Therefore, going to the limit in the last inequality as $\alpha \to \alpha_0$, we obtain

$$\|u_0\|_p^p \geq \frac{1}{c} > 0,$$

which shows that $u_0 \neq 0$.
Now, using the continuity of $\beta_2(\alpha)$ and taking the limit as $\alpha \to \alpha_0$ in (3.22), we obtain

$$\int_I |u'_0|^{p-2}u'_0 \theta' dx = \int_I (\alpha_0\, m_1 + \beta_2(\alpha_0)\, m_2)\, |u_0|^{p-2}u_0 \theta\, dx,$$

for all $\theta \in W^{1,p}(I)$. So, from the last equality, we deduce that u_0 is an eigenfunction associated to $\lambda(\alpha_0\, m_1 + \beta_1(\alpha_0)\, m_2) = 1 = \lambda_2(\alpha_0\, m_1 + \beta_2(\alpha_0)\, m_2)$. Since the second eigenvalue of the p-Laplacian with weights is simple, we deduce without loss of generality that $u_0 = \varphi_{\alpha_0}$. Consequently, (i) is proved.

(ii) For any $\alpha, \alpha_0 \in]\lambda_{-2}(m_1), \lambda_2(m_1)[$ such that $\alpha \neq \alpha_0$, by the variational characterization of $\beta_2(\alpha)$ and $\beta_2(\alpha_0)$, we have

$$\beta_2(\alpha_0) = \frac{\int_I |\varphi'_{\alpha_0}|^p dx - \alpha_0 \int_I m_1 |\varphi_{\alpha_0}|^p dx}{\int_I m_2 |\varphi_{\alpha_0}|^p dx} \leq \frac{\int_I |\varphi'_\alpha|^p dx - \alpha_0 \int_I m_1 |\varphi_\alpha|^p dx}{\int_I m_2 |\varphi_\alpha|^p dx},$$

and

$$\beta_2(\alpha) = \frac{\int_I |\varphi'_\alpha|^p dx - \alpha \int_I m_1 |\varphi_\alpha|^p dx}{\int_I m_2 |\varphi_\alpha|^p dx} \leq \frac{\int_I |\varphi'_{\alpha_0}|^p dx - \alpha \int_I m_1 |\varphi_{\alpha_0}|^p dx}{\int_I m_2 |\varphi_{\alpha_0}|^p dx},$$

then

$$(\alpha_0-\alpha)\frac{\int_I m_1|\varphi_\alpha|^p dx}{\int_I m_2|\varphi_\alpha|^p dx} \le \beta_2(\alpha)-\beta_2(\alpha_0) \le (\alpha_0-\alpha)\frac{\int_I m_1|\varphi_{\alpha_0}|^p dx}{\int_I m_2|\varphi_{\alpha_0}|^p dx}.$$

Finally, the desired result follows by dividing by $\alpha_0-\alpha$ and letting $\alpha \to \alpha_0$. □

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