

SUCCESSIVE APPROXIMATION OF NEUTRAL STOCHASTIC PARTIAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY AND POISSON JUMPS

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ABSTRACT. This work presents the result on existence, uniqueness of mild solutions of neutral stochastic partial integro-differential equations with infinite delay and Poisson jumps in real separable Hilbert spaces. We study the continuous dependence of solution on the initial value. The nonlinear term in our equations are not assumed to Lipschitz continuous. To obtain the results, we use the theory of resolvent operator developed in R. Grimmer (1982)[31] and by using the method of successive approximation under sufficient condition.

1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction. In this paper, we consider the following neutral stochastic integrodifferential equation with infinite delay and poisson jumps:

$$\left\{ \begin{array}{l} d[x(t) + g(t, x_t)] = \left[A[x(t) + g(t, x_t)] + \int_0^t B(t-s)[x(s) + g(s, x_s)]ds \right] dt \\ + f(t, x_t)dt + \sigma(t, x_t)dw(t) + \int_Z h(t, x(t-), y)\tilde{N}(dt, dy), \quad t \in I := [0, T], \\ x_0 = \varphi \in \mathcal{B}. \end{array} \right. \quad (1.1)$$

Here, the state $x(\cdot)$ takes values in a separable real Hilbert spaces $\mathbb{H}$ with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t), t \geq 0$ on $\mathbb{H}$, with $D(A) \subset \mathbb{H}$, and $B(t), t \in J$ is a closed linear operator on $\mathbb{H}$. The history $x_t :] - \infty, 0] \rightarrow \mathbb{H}, x_t(\theta) = x(t + \theta)$, for $t \geq 0$, belongs to some abstract phase space $\mathcal{B}$ which will be described axiomatically in Section 2. Let $\mathbb{K}$ be another separable Hilbert spaces with inner product $\langle \cdot, \cdot \rangle_{\mathbb{K}}$ and norm $\|\cdot\|_{\mathbb{K}}$. Suppose $\{w(s) : 0 \leq s \leq t\}$ is a given $\mathbb{K}$ -valued Wiener process with increment

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covariance given by the nuclear operator $Q \geq 0$ and $\tilde{N}(ds, dz)$ is a compensating martingale measure induced by a Poisson point process $k(\cdot)$, which is independent of the Wiener process w and takes values in a measurable space $(Z, \mathcal{B}(Z))$ defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$ which is generated by the Wiener process w . We are also employing the same notation $\|\cdot\|$ for the norm $\mathcal{L}(\mathbb{H}; \mathbb{K})$, where $\mathcal{L}(\mathbb{H}; \mathbb{K})$ denotes the space of all bounded linear operator from $\mathbb{K}$ into $\mathbb{H}$. Assume that $g, f : I \times \mathcal{B} \rightarrow \mathbb{H}$, $\sigma : I \times \mathbb{H} \rightarrow \mathcal{L}_2^0$ and $h : I \times \mathbb{H} \times (Z - \{0\}) \rightarrow \mathbb{H}$, are appropriate mappings specified later. Here $\mathcal{L}_2^0 = \mathcal{L}_2(\mathbb{K}_0; \mathbb{H})$ denotes the space of all Q -Hilbert-Schmidt operators from $\mathbb{K}_0$ to $\mathbb{H}$ which will be defined in Section 2. The initial data $\varphi = \{\varphi(t) : -\infty < t \leq 0\}$ is an $\mathcal{F}_0$ -adapted, $\mathcal{B}$ -valued random variable independent of the Wiener process w with finite second moment.

The nonlinear integrodifferential equation with resolvent operator served as abstract formulation of partial integrodifferential equations which arises in many physical phenomena [31, 32, 33]. The resolvent operator is similar to the semigroup operator for abstract differential equations in Banach spaces. However, the resolvent operator does not satisfy the semigroup properties, see for instance [23, 26].

Stochastic differential equations are well known to model problems from many areas of science and engineering, wherein, quite often the future state of such systems depends not only on the present state but also on its past history (delay) leading to stochastic functional differential equations (SFDEs) and it has played an important role in many ways such as the model of the systems in physics, chemistry, biology, economics and finance from various points of the view (see, e.g. and the references therein).

On the other hand, the study of neutral stochastic partial differential equations with Poisson jumps processes also have begun to gain attention and strong growth in recent years. For example, they have been used to develop models for neuronal activity that account for synaptic impulses occurring randomly, both in time and at different locations of a spatially extended neuron. Other applications arise in chemical reaction-diffusion systems and stochastic turbulence models. To be more precise, in 2009, Luo and Taniguchi [10] considered the existence and uniqueness of mild solutions to stochastic evolution equations with finite delay and Poisson jumps by the Banach fixed point theorem, in 2010, Boufoussi and Hajji [11] proved the existence and uniqueness result for a class of neutral SFDEs driven both by the cylindrical Brownian motion and by the Poisson processes.

To the best of our knowledge, so far no work has been reported in the literature about neutral stochastic integrodifferential equation with infinite delay and Poisson jumps and the aim of this paper is to close this gap. We aim to establish the existence and uniqueness of mild solutions to neutral stochastic integrodifferential equation with infinite delay and Poisson jumps under non-Lipschitz condition. We should point out that this kind of non-Lipschitz condition has been used in Mahmudov and McKibben [24] and Mao [25]. In the process of the existence and uniqueness result, we have mainly adopted the ideas appeared in Mahmudov and McKibben [24] and Mao [25]. Furthermore, we give the continuous dependence of

solutions on the initial data by means of a corollary of the Bihari inequality. An application to the stochastic nonlinear equation with infinite delay and Poisson jumps is given to illustrate the theory.

The format of this work is organized as follows. In Subsection 2, we recall some preliminaries which are used throughout this paper. In Section 2, the existence and uniqueness of mild solution for neutral stochastic integrodifferential equation with infinite delay and poisson jump is presented. Some sufficient conditions about the stability through the continuous dependence on the initial values are given in Section 3. Finally, an example is provided to demonstrate the obtained results.

1.2. Preliminaries.

1.2.1. Wiener process. For more details, we refer the reader to Da Prato and Zabczyk [20] and [21]. Let $(\mathbb{H}, |\cdot|_{\mathbb{H}}, \langle \cdot, \cdot \rangle_{\mathbb{H}})$ and $(\mathbb{K}, |\cdot|_{\mathbb{K}}, \langle \cdot, \cdot \rangle_{\mathbb{K}})$ be two separable Hilbert spaces. $\mathcal{L}(\mathbb{K}, \mathbb{H})$ stands for the set of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operators norm $\|\cdot\|$. Throughout this paper, we use the symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces involved when no confusion possibility arises.

Let $(\Omega, \mathcal{F}, P, \mathbf{F})$ with $(\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0})$ be a complete filtered probability space satisfying the usual condition, which means that the filtration is right continuous increasing family and $\mathcal{F}_0$ contains all P -null sets of $\mathcal{F}$. Suppose that $\{p(t), t \geq 0\}$ is a σ -finite stationary $\mathcal{F}_t$ -adapted Poisson point process taking values in a measurable space $(Z, \mathcal{B}(Z))$. The random measure N_p defined by $N_p((t_1, t_2] \times \Lambda) := \sum_{s \in (t_1, t_2]} 1_{\Lambda}(p(s))$ for $\Lambda \in \mathcal{B}(Z - 0)$ is called the Poisson random measure induced by $p(\cdot)$. Thus, we can define the measure $\tilde{N}$ by $\tilde{N}(dt, dy) = N_p(dt, dy) - \nu(dy)dt$, where ν is the characteristic measure of N_p , which is called the compensated Poisson random measure. Let $W = (W_t)_{t \geq 0}$, independent of the Poisson point process, be a $\mathbb{K}$ -valued Wiener process defined on $(\Omega, \mathcal{F}, P, \mathbf{F})$ with covariance operator Q that is,

$$\mathbb{E} \langle W(t), x \rangle_{\mathbb{K}} \langle W(t), y \rangle_{\mathbb{K}} = (t \wedge s) \langle Qx, y \rangle_{\mathbb{K}}, \quad x, y \in \mathbb{K},$$

where Q is positive, self-adjoint, trace class operator on $\mathbb{K}$. Let $\mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ denote the space of all Q -Hilbert-Schmidt operators from $\mathbb{K}$ to $\mathbb{H}$ with the norm

$$|\xi|_{\mathcal{L}_2^0}^2 := \text{tr}(\xi Q \xi^*) < \infty, \quad \xi \in \mathcal{L}(\mathbb{K}, \mathbb{H})$$

1.2.2. Partial integrodifferential equations in Banach spaces. In what follows, $\mathbb{H}$ is a Banach space, A and $B(t)$ are closed linear operators on $\mathbb{H}$. Y represents the Banach space $D(A)$ equipped with the graph norm defined by

$$|y|_Y := |Ay| + |y| \quad \text{for } y \in Y.$$

The notations $C([0, +\infty); Y)$, $\mathcal{L}(Y, \mathbb{H})$ stand for the space of all continuous functions from $[0, +\infty)$ into Y , the set of all bounded linear operators from Y into $\mathbb{H}$,

respectively. We consider the following Cauchy problem

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds & \text{for } t \geq 0 \\ v(0) = v_0 \in \mathbb{H}. \end{cases} \quad (1.2)$$

Definition 1.1. ([31]) A resolvent operator for Eq.(1.2) is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathbb{H})$ for $t \geq 0$, having the following properties:

- (i) $R(0) = I$ and $|R(t)| \leq \tilde{C}e^{\beta t}$ for some constants $\tilde{C}$ and β .
- (ii) For each $x \in \mathbb{H}$, $R(t)x$ is strongly continuous for $t \geq 0$.
- (iii) For $x \in Y$, $R(\cdot)x \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \quad \text{for } t \geq 0. \end{aligned}$$

For additional details on resolvent operators, we refer the reader to [31, 33]. The resolvent operators plays an important role to study the existence of solutions and to give a variation of constants formula for non linear systems. We need to know when the linear system (1.2) has a resolvent operator. Theorem 2.2 gives a satisfactory answer to this problem.

In what follows we suppose the following assumptions:

(H1) A is the generator of a strongly continuous semigroup on $\mathbb{H}$.

(H2) For $t \geq 0$, $B(t)$ is a closed linear operator from $D(A)$ to $\mathbb{H}$, and $B(t) \in \mathcal{L}(Y, \mathbb{H})$.

For any $y \in Y$, the map $t \rightarrow B(t)y$ is bounded, differentiable and the derivative $t \rightarrow B'(t)y$ is bounded uniformly continuous on $\mathbb{R}^+$

Theorem 1.2. Assume that the assumptions **(H1)** and **(H2)** hold. Then there exists a unique resolvent operator of the Cauchy problem Eq.(1.2).

In the following, we give some results on the existence of solutions for the following integrodifferential equation

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds + q(t) & \text{for } t \geq 0 \\ v(0) = v_0 \in \mathbb{H}, \end{cases} \quad (1.3)$$

where $q : [0, +\infty[\rightarrow \mathbb{H}$ is a continuous function.

Definition 1.3. ([31]) A continuous function $v : [0, +\infty) \rightarrow \mathbb{H}$ is said to be a strict solution of Eq.(1.3) if

- (i) $v \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); Y)$,
- (ii) v satisfies Eq.(1.3) for $t \geq 0$.

Remark 1.4. From this definition, we deduce that $v(t) \in D(A)$, the function $B(t-s)v(s)$ is integrable, for all $t > 0$ and $s \in [0, +\infty)$.

Theorem 1.5. ([31]) Assume that **(H1)**-**(H2)** hold. If v is a strict solution of Eq.(1.3), then the following variation of constants formula holds

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \quad \text{for } t \geq 0. \quad (1.4)$$

Accordingly, we make the following definition.

Definition 1.6. ([31]) For $v_0 \in \mathbb{H}$, a function $v : [0, +\infty) \rightarrow \mathbb{H}$ is called a mild solution of (1.3) if v satisfies the variation of constants formula (1.4).

The next theorem provides sufficient conditions for the regularity of solutions of Eq.(1.3).

Theorem 1.7. ([31]) Let $q \in C^1([0, +\infty); \mathbb{H})$ and v be defined by (1.4). If $v_0 \in D(A)$, then v is a strict solution of Eq.(1.3).

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato[17].

Definition 1.8. The phase space $\mathcal{B}((-\infty, 0], \mathbb{H})$ ((denoted by $\mathcal{B}$ simply) is the space of $\mathcal{F}_0$ -measurable functions from $(-\infty, 0]$ to $\mathbb{H}$ endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. The axioms on $\mathcal{B}$ are as follows:

(A1) If $x : (-\infty, T] \rightarrow \mathbb{H}$, $T > 0$, is such that $x_0 \in \mathcal{B}$, then, for every $t \in I := [0, T]$, the following conditions hold:

- (1) $x_t \in \mathcal{B}$;
- (2) $\|x(t)\| \leq C \|x_t\|_{\mathcal{B}}$;
- (3) $\|x_t\|_{\mathcal{B}} \leq M(t) \sup_{0 \leq s \leq t} |x(s)| + N(t) \|x_0\|_{\mathcal{B}}$,

where $C > 0$ is a constant, $M, N : [0, +\infty) \rightarrow [1, +\infty)$, M is continuous, N is locally bounded, M, N are independent of $x(\cdot)$

(A2) The space $\mathcal{B}$ is complete.

The $\mathcal{B}$ -valued stochastic process $x_t : \Omega \rightarrow \mathcal{B}, t \in I$, is defined by setting $x_t = \{x(t + \theta)(\omega); \theta \in (-\infty, 0]\}$.

Remark 1.9. From the condition (3) in Definition 1.8 we have

$$(3)' \quad \|x_t\|_{\mathcal{B}} \leq \hat{M} \sup_{0 \leq s \leq T} |x(s)| + N \|x_0\|_{\mathcal{B}},$$

where $\hat{M} = \sup_{0 \leq s \leq T} M(s)$, $N = \sup_{0 \leq s \leq T} N(s)$.

Remark 1.10. In retarded functional differential equations without jumps, the axioms of the phase space $\mathcal{B}$ include the continuity of the function $t \rightarrow x_t$, due to the jumping effect, this property is not satisfied in neutral stochastic integrodifferential functional equations with Poisson jumps and infinite delay, for this reason, this condition has been eliminated in our abstract description of $\mathcal{B}$ (also see [12, 13])

Now we present the definition of the mild solution for (1.1).

Definition 1.11. An $\mathcal{F}_t$ -adapted, $\mathbb{H}$ -valued stochastic process $x(t)$ defined on $-\infty < t \leq T, T > 0$, is called a mild solution for (1.1) if

- (a) $x(t)$ is càdlàg, and $\{x_t : 0 \leq t \leq T\}$ is a $\mathcal{B}$ -valued stochastic process;

- (b) $\int_0^T |x(u)|^2 du < \infty$ almost surely;
(c) For arbitrary $0 \leq t \leq T$, $x(t)$ satisfies the following integral equation:

$$\begin{cases} x(t) = R(t)[\varphi(0) + g(0, \varphi) - g(t, x_t) + \int_0^t R(t-s)f(s, x_s)ds \\ + \int_0^t R(t-s)\sigma(s, x_s)dw(s) + \int_0^t R(t-s) \int_Z h(s, x(s-), y)\tilde{N}(ds, dy), \\ x_0 = \varphi \in \mathcal{B}. \end{cases} \quad (1.5)$$

We denote by $\mathcal{M}^2((-\infty, T], \mathbb{H})$ the space of all $\mathbb{H}$ -valued $\mathcal{F}_t$ -adapted process $x = \{x(t), -\infty < t \leq T\}$ such that

- (1) $x_0 = \varphi \in \mathcal{B}$ and $x(t)$ is càdlàg on $[0, T]$,
(2) for all $x \in \mathcal{M}^2((-\infty, T], \mathbb{H})$,

$$\|x\|_{\mathcal{M}^2}^2 := \mathbb{E} \|\varphi\|_{\mathcal{B}}^2 + \mathbb{E} \int_0^T |x(t)|^2 dt < \infty. \quad (1.6)$$

Lemma 1.12. *The space $\mathcal{M}^2((-\infty, T], \mathbb{H})$ is a Banach space with the norm defined by (1.6)*

The proof is routine, we omit it here.

In order to obtain the existence and uniqueness, we shall impose the following assumptions:

(H3) The measurable mapping $f(\cdot)$, $\sigma(\cdot)$ and $h(\cdot)$ satisfy the following conditions:

- (2a) for all $t \in I$, $\varphi_1, \varphi_2 \in \mathcal{B}$,

$$|f(t, \varphi_1) - f(t, \varphi_2)|^2 \vee |\sigma(t, \varphi_1) - \sigma(t, \varphi_2)|_{\mathcal{L}_0^2}^2 \leq k(\|\varphi_1 - \varphi_2\|_{\mathcal{B}}^2)$$

- (2b) for any $\mathbb{H}$ -valued processes $x(t), y(t)$, $t \in I$,

$$\begin{aligned} & \int_0^t \int_Z |h(s, x(s-), z) - h(s, y(s-), z)|^2 \nu(dz) ds \vee \left(\int_0^t \int_Z |h(s, x(s-), z) - h(s, y(s-), z)|^4 \nu(dz) ds \right)^{\frac{1}{2}} \\ & \leq \int_0^t k(|x(s) - y(s)|^2) ds. \end{aligned}$$

$$\left(\int_0^t \int_Z |h(s, x(s-), z)|^4 \nu(dz) ds \right)^{\frac{1}{2}} \leq \int_0^t k(|x(s)|^2) ds,$$

where $k(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $k(0) = 0$, $k(u) > 0$ for $u > 0$ and $\int_{0+} \frac{du}{k(u)} = \infty$.

(H4) For all $t \in I$, there exists a constant $M > 0$ such that

$$|f(t, 0)|^2 \vee |\sigma(t, 0)|^2 \vee \int_Z |h(t, 0, z)|^2 \nu(dz) \leq M.$$

(H5) There exist a constant $K_g > 0$ such that for any $\varphi_1, \varphi_2 \in \mathcal{B}, t \geq 0$,

$$|g(t, \varphi_1) - g(t, \varphi_2)| \leq K \|\varphi_1 - \varphi_2\|_{\mathcal{B}}$$

and we further assume that $g(t, 0) \equiv 0$ for all $t \geq 0$.

2. MAIN RESULTS

In this section, we prove the existence and uniqueness theorem. Let us start with a lemma, and our main result is proved with the help of this lemma. We first introduce the sequence of successive approximations defined as follows

$$\begin{aligned} x^0(t) &= R(t)\varphi(0) \quad t \in [0, T] \\ x^n(t) &= \varphi(t), \quad t \in (-\infty, 0], \quad n = 0, 1, 2, \dots, \\ x^n(t) &= R(t)[\varphi(0) + g(0, \varphi)] - g(t, x_t^n) + \int_0^t R(t-s)f(s, x_s^{n-1})ds + \int_0^t R(t-s)\sigma(s, x_s^{n-1})dw(s) \\ &\quad + \int_0^t R(t-s) \int_Z h(s, x_s^{n-1}(s-), z) \tilde{N}(ds, dz), \quad t \in I, \quad n = 1, 2, \dots, \end{aligned}$$

In order to obtain the uniqueness of mild solutions, we need the following Bihari inequality which is introduced in [4]

Lemma 2.1. (*Bihari inequality*) ([4]) *Let $T > 0$ and $u_0 \geq 0$, $u(t)$, $v(t)$ be continuous functions on $[0, T]$. Let $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a concave, and nondecreasing continuous function such that $K(r) > 0$ for $r > 0$. If*

$$u(t) \leq u_0 + \int_0^t v(s)K(u(s))ds,$$

then

$$u(t) \leq G^{-1} \left(G(u_0) + \int_0^t v(s)ds \right)$$

for all $t \in [0, T]$ and $G(u_0) + \int_0^t v(s)ds \in \text{Dom}(G^{-1})$, where $G(r) = \int_1^r \frac{ds}{K(s)}$ ($r \geq 0$) and G^{-1} is the inverse function of the G . In particular, if moreover, $u_0 = 0$ and $\int_{0+} \frac{ds}{K(s)} = +\infty$, then $u(t) = 0$ for all $t \in [0, T]$.

Lemma 2.2. *Assume that **(H1)** – **(H5)** hold, together with $K^2 \tilde{M}^2 < \frac{1}{2}$ hold, then for all $t \in (-\infty, T], n \geq 0$, there exists a positive constant K_1 such that*

$$\|x^n\|_{\mathcal{M}^2}^2 \leq K_1.$$

Proof. It is obvious that $x^0 \in \mathcal{M}^2((-\infty, 0], T)$. For $n \geq 1$, by an elementary inequality we have

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} |x^n(t)|^2 &\leq 5\mathbb{E} \sup_{0 \leq s \leq t} |R(s)[\varphi(0) + g(0, \varphi)]|^2 + 5\mathbb{E} \sup_{0 \leq s \leq t} |g(s, x_s^n)|^2 \\
&+ 5\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s R(s-r)f(r, x_r^{n-1})dr \right|^2 + 5\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s R(s-r)\sigma(r, x_r^{n-1})dw(r) \right|^2 \\
&+ 5\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \int_Z R(s-r)h(r, x_r^{n-1}(r-), z)\tilde{N}(dr, dz) \right|^2 \\
&= 5 \sum_{i=1}^5 I_i.
\end{aligned} \tag{2.1}$$

Now, we estimate I_i , $i = 1, 2, \dots, 5$. By assumption **(H5)** we have

$$I_1 \leq 2 \left[\mathbb{E} \sup_{0 \leq s \leq t} |R(s)\varphi(0)|^2 + \mathbb{E} \sup_{0 \leq s \leq t} |R(s)g(0, \varphi)|^2 \right] \leq 2(1 + K^2)\mathbb{E} \|\varphi\|_{\mathcal{B}}^2. \tag{2.2}$$

Applying Hölder's inequality, Lemma 1.12 and assumption **(H5)** it follows that

$$I_2 \leq K^2 \mathbb{E} \sup_{0 \leq s \leq t} \|x_s^n\|_{\mathcal{B}}^2. \tag{2.3}$$

Applying assumption **(H3)** and Hölder's inequality, we derive that

$$I_3 \leq T \mathbb{E} \sup_{0 \leq s \leq t} \int_0^s |R(s-r)f(r, x_r^{n-1})|^2 dr \leq 2T[MT + \int_0^t \mathbb{E} k(\|x_s^{n-1}\|_{\mathcal{B}}^2) ds]. \tag{2.4}$$

On the other hand, by assumption **(H3)** and Burkholder-type inequality (see Lemma 7.2 in [20]) in Hilbert space it follows that there exist some positive constants C_1 and C_2 such that

$$I_4 \leq C_1 \int_0^t \mathbb{E} |\sigma(s, x_s^{n-1})|_{\mathcal{L}_2^0}^2 ds \leq 2C_1 \left[MT + \int_0^t \mathbb{E} k(\|x_s^{n-1}\|_{\mathcal{B}}^2) ds \right] \tag{2.5}$$

and

$$\begin{aligned}
I_5 &\leq C_2 \left[\mathbb{E} \int_0^t \int_Z |h(s, x^{n-1}(s-), z)|^2 \nu(dz) ds + \mathbb{E} \left(\int_0^t \int_Z |h(s, x^{n-1}(s-), z)|^4 \nu(dz) ds \right)^{\frac{1}{2}} \right] \\
&\leq C_2 \left[\mathbb{E} \int_0^t \int_Z |h(s, x^{n-1}(s-), z) - h(s, 0, z) + h(s, 0, z)|^2 \nu(dz) ds \right. \\
&\quad \left. + \mathbb{E} \left(\int_0^t \int_Z |h(s, x^{n-1}(s-), z)|^4 \nu(dz) ds \right)^{\frac{1}{2}} \right] \\
&\leq 2C_2 \left[MT + \int_0^t \mathbb{E} k(|x^{n-1}(s)|^2) ds \right] + C_2 \int_0^t \mathbb{E} k(|x^{n-1}(s)|^2) ds
\end{aligned}$$

$$= 2C_2MT + 3C_2 \int_0^t \mathbb{E}k(|x^{n-1}(s)|^2)ds \quad (2.6)$$

Recalling (2.1), from (2.2)-(2.6), it follows that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |x^n(s)|^2 &\leq S_1 + 5K^2 \mathbb{E} \sup_{0 \leq s \leq t} \|x_s^n\|_{\mathcal{B}}^2 + 10T \int_0^t \mathbb{E}k(\|x_s^{n-1}\|_{\mathcal{B}}^2)ds \\ &\quad + 10C_1 \int_0^t \mathbb{E}k(\|x_s^{n-1}\|_{\mathcal{B}}^2)ds + 15C_2 \int_0^t \mathbb{E}k(|x^{n-1}(s)|_{\mathcal{B}}^2)ds, \end{aligned} \quad (2.7)$$

where $S_1 = 10(1 + K^2)\mathbb{E}\|\varphi\|_{\mathcal{B}}^2 + 10MTC_1 + 10MTC_2$. Since $k(\cdot)$ is concave and $k(0) = 0$, we can find a pair of positive constants a and b so that $k(u) \leq a + bu$ for $u \geq 0$, we get that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |x^n(s)|^2 &\leq S_2 + 5K^2 \mathbb{E} \sup_{0 \leq s \leq t} \|x_s^n\|_{\mathcal{B}}^2 + 10b(T + C_1) \int_0^t \mathbb{E}\|x_s^{n-1}\|_{\mathcal{B}}^2 ds \\ &\quad + 15C_2b \int_0^t \mathbb{E}|x^{n-1}(s)|_{\mathcal{B}}^2 ds, \end{aligned} \quad (2.8)$$

where $S_2 = S_1 + 10aT^2 + 10aTC_1 + 15aTC_2$. Nothing that

$$\|x_s^{n-1}\|_{\mathcal{B}}^2 \leq \tilde{M} \sup_{0 \leq r \leq s} |x^{n-1}(r)| + N \|x_0^{n-1}\|_{\mathcal{B}}^2,$$

we get

$$\mathbb{E} \sup_{0 \leq s \leq t} |x^n(s)|^2 \leq \frac{1}{1 - 10K^2\tilde{M}^2} \left[S_3 + 5b(4T\tilde{M}^2 + 4C_1\tilde{M}^2 + 3C_2) \times \int_0^t \mathbb{E} \sup_{0 \leq r \leq s} |x^n(r)|^2 ds \right] \quad (2.9)$$

where $S_3 = S_2 + [10K^2N^2 + 10b(T + C_1)NT]\mathbb{E}\|\varphi\|_{\mathcal{B}}^2$.

On the other hand, for any $k \geq 1$,

$$\max_{1 \leq n \leq k} \mathbb{E} \sup_{0 \leq s \leq t} |x^{n-1}(s)|^2 \leq \mathbb{E}|x^0(s)|^2 + \max_{1 \leq n \leq k} \mathbb{E} \sup_{0 \leq s \leq t} |x^n(s)|^2,$$

it follows that for some constants S_4, S_5

$$\max_{1 \leq n \leq k} \mathbb{E} \sup_{0 \leq s \leq t} |x^n(s)|^2 \leq S_4 + S_5 \int_0^t \max_{1 \leq n \leq k} \mathbb{E} \sup_{0 \leq r \leq s} |x^n(r)|^2 ds,$$

where

$$\begin{aligned} S_4 &= \frac{1}{1 - 10K^2\tilde{M}^2} \left[S_3 + (20bT^2\tilde{M}^2 + 20bTC_1\tilde{M}^2 + 15bTC_2)\mathbb{E}\|\varphi\|_{\mathcal{B}}^2 \right], \\ S_5 &= \frac{1}{1 - 10K^2\tilde{M}^2} \left[(10T^2\tilde{M}^2 + 20b\tilde{M}^2(T + C_1) + 15bC_2) \right]. \end{aligned}$$

Since k is arbitrary, the Gronwall inequality yields

$$\mathbb{E} \sup_{0 \leq s \leq t} |x^n(s)|^2 \leq S_4 e^{S_5 T}.$$

Moreover,

$$\|x^n\|_{\mathcal{M}^2}^2 = \mathbb{E} \|x_0^n\|_{\mathcal{B}}^2 + \mathbb{E} \int_0^T |x^n(s)|^2 ds \leq \mathbb{E} \|\varphi\|_{\mathcal{B}}^2 + TS_4 e^{S_5 T} < \infty$$

which implies $x^n(\cdot) \in \mathcal{M}^2((\infty, T], \mathbb{H})$ and $\|x^n\|_{\mathcal{M}^2}^2 \leq K_1$ with $K_1 = \mathbb{E} \|\varphi\|_{\mathcal{B}}^2 + TS_4 e^{S_5 T}$.

Lemma 2.3. *Assume that the conditions of Lemma 2.2 hold. We further assume that*

$$4K^2 \tilde{M}^2 + 5K^2 M^2 \tilde{M}^2 < 1. \quad (2.10)$$

There exist positive constants K_2, K_3 such that for all $t \in [0, T]$, $m, n \geq 1$,

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |x^{m+n}(s) - x^n(s)|^2 \right) \leq K_2 \int_0^t k \left(\tilde{M}^2 \mathbb{E} \sup_{0 \leq r \leq s} |x^{m+n-1}(r) - x^{n-1}(r)|^2 \right) ds \quad (2.11)$$

and

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |x^{m+n}(s) - x^n(s)|^2 \right) \leq K_3 t. \quad (2.12)$$

Proof. By the definition of x^n we can derive that for any $m, n \geq 1$ and $t \in [0, T]$

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq t} |x^{m+n}(s) - x^n(s)|^2 \right) &\leq 4\mathbb{E} \left(\sup_{0 \leq s \leq t} |g(s, x^{m+n}(s)) - g(s, x^n(s))|^2 \right) \\ &+ 4\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s R(s-r) [f(r, x_r^{m+n-1}) - f(r, x_r^{n-1})] dr \right|^2 \\ &+ 4\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s R(s-r) [\sigma(r, x_r^{m+n-1}) - \sigma(r, x_r^{n-1})] dw(r) \right|^2 \\ &+ 4\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \int_Z R(s-r) [h(r, x_r^{m+n-1}(r-), z) \right. \\ &\quad \left. - h(r, x_r^{n-1}(r-), z)] \tilde{N}(dr, dz) \right|^2 \\ &= 4 \sum_{i=1}^4 I_i. \end{aligned} \quad (2.13)$$

We now estimate each term I_i , Recalling that

$$\|x_s^n - x_s^m\|_{\mathcal{B}} \leq \tilde{M} \sup_{0 \leq r \leq s} |x^n(r) - x^m(r)|,$$

we get

$$I_1 \leq K^2 \mathbb{E} \sup_{0 \leq s \leq t} \|x_s^{m+n} - x_s^n\|_{\mathcal{B}}^2 \leq K^2 \tilde{M}^2 \mathbb{E} \sup_{0 \leq s \leq t} |x^{m+n}(s) - x^n(s)|^2, \quad (2.14)$$

Next, by a similar argument as that in Lemma 2.2 we can show that there exists a positive constant S_6 such that

$$4(I_2 + I_3 + I_4) \leq S_6 \int_0^t k \left(\tilde{M}^2 \mathbb{E} \sup_{0 \leq r \leq s} |x^{m+n-1}(r) - x^{n-1}(r)|^2 \right) ds. \quad (2.15)$$

Hence, from (2.14) and (2.16) we get the desired inequality (2.11) with

$$K_2 = \frac{S_6}{1 - 4K^2\tilde{M}^2}$$

On the other hand, by Lemma 2.1 and (2.11) it follows that

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq t} |x^{m+n}(s) - x^n(s)|^2 \right) &\leq K_2 \int_0^t k \left(\tilde{M}^2 \mathbb{E} \sup_{0 \leq r \leq s} |x^{m+n-1}(r) - x^{n-1}(r)|^2 \right) ds \\ &\leq K_2 \int_0^t k(2\tilde{M}^2 K_1) ds = K_3 t, \end{aligned}$$

which is the desired inequality (2.12) with $K_3 = K_2 k(2\tilde{M}^2 K_1)$.

Theorem 2.4. *Under the conditions of Lemma 2.2 and 2.3, the system (1.1) admits a unique mild solution $x(t) \in \mathcal{M}^2((-\infty, T], \mathbb{H})$.*

Proof. We shall divide the proof into two steps

Step 1. Let us show that $\{x^n(t), t \in [0, T]\}$ is a Cauchy sequence. Let $k_1(u) = K_2 k(u)$. Choose $T_1 \in [0, T]$ such that $k_1(\tilde{M}^2 K_3 u) \leq K_3$ for $u \in [0, T_1]$.

We first introduce two sequences of functions $\{\phi_n(t)\}_{n \in \mathbb{N}}$ and $\{\phi_{m,n}(t)\}_{n,m \in \mathbb{N}}$ by

$$\phi_1(t) = K_3 t, \quad \phi_{n+1}(t) = \int_0^t k_1(\tilde{M}^2 \phi_n(u)) du.$$

$$\phi_{n,m}(t) = \mathbb{E} \left(\sup_{0 \leq u \leq t} |x^{m+n}(u) - x^n(u)|^2 \right).$$

Then $\{\phi_n(t)\}_{n \in \mathbb{N}}$ is monotonically decreasing when $n \rightarrow \infty$ and $0 \leq \phi_{m,n}(t) \leq \phi_n(t)$ for all $m, n \geq 1, t \in [0, T_1]$. In fact, it is obvious that $\phi_{1,m}(t) \leq \phi_1(t)$ and

$$\begin{aligned} \phi_{2,m}(t) &= \mathbb{E} \left(\sup_{0 \leq u \leq t} |x^{m+2}(u) - x^2(u)|^2 \right) \\ &\leq \int_0^t k_1 \left(\tilde{M}^2 \mathbb{E} \left(\sup_{0 \leq u \leq t} |x^{m+1}(u) - x^1(u)|^2 \right) \right) ds \\ &\leq \int_0^t k_1 \left(\tilde{M}^2 \phi_1(s) \right) ds \\ &\leq \int_0^t k_1 \left(\tilde{M}^2 K_3 s \right) ds = \phi_2(t) \leq K_3 t = \phi_1(t), \end{aligned} \quad (2.16)$$

which implies that $\phi_{2,m}(t) \leq \phi_2(t) \leq \phi_1(t)$. Now we assume the result holds for n , then

$$\begin{aligned}\phi_{n+1,m}(t) &= \mathbb{E} \left(\sup_{0 \leq u \leq t} |x^{m+n+1}(u) - x^{n+1}(u)|^2 \right) \\ &\leq \int_0^t k_1 \left(\tilde{M}^2 \phi_{n,m}(s) \right) ds \\ &\leq \int_0^t k_1 \left(\tilde{M}^2 \phi_n(s) \right) ds \\ &= \phi_{n+1}(t) \leq \int_0^t k_1 \left(\tilde{M}^2 \phi_{n-1}(s) \right) ds = \phi_n(t).\end{aligned}$$

This shows that $\phi_n(t)$ is a nonnegative and decreasing continuous function on $[0, T_1]$ by induction on n , so we can define a function $\phi(t)$ by $\phi_k(t) \downarrow \phi(t)$, and it is easy to verify that $\phi(0) = 0$ and $\phi(t)$ is continuous function on $[0, T_1]$.

Consequently,

$$\phi(t) = \lim_{n \rightarrow \infty} \phi_n(t) = \lim_{n \rightarrow \infty} \int_0^t k_1(\phi_{n-1}(s)) ds = \int_0^t k_1(\phi(s)) ds$$

From $\phi(0) = 0$, $\int_{0+} \frac{du}{k_1(u)} = +\infty$ together with Bihari inequality we get $\phi(t) \equiv 0$.

Thus, $0 \leq \phi_{n,n}(t) \leq \phi_n(T_1) \rightarrow 0$ as $n \rightarrow \infty$.

This shows that $\{x^n(t), t \in [0, T_1]\}$ is a Cauchy sequence in $\mathcal{M}^2((-\infty, T], \mathbb{H})$, so there exists an $x(t) \in \mathcal{M}^2((-\infty, T], \mathbb{H})$ such that $x^n(t)$ converge to $x(t)$ in $\mathcal{M}^2((-\infty, T], \mathbb{H})$ as $n \rightarrow \infty$. By Borel-Cantelli lemma, there exists a subsequence $\{x^{n_k}(t)\} \subseteq \{x^n(t)\}$ such that $x^{n_k}(t) \rightarrow x(t)$ uniformly on $[0, T_1]$ as $n_k \rightarrow \infty$. Using standard method, it follows that $x(t)$ really satisfies (1.1). By iteration, the existence of mild solution can be obtained.

Step 2. We shall show the uniqueness. Let $x(t)$ and $y(t)$ be two mild solutions to Eq (1.1). For $t \in [0, T]$, by the same way as that in Lemma 2.1 we can show that

$$\mathbb{E} \sup_{0 \leq s \leq t} |x(s) - y(s)|^2 \leq K_2 \int_0^t k \left(\tilde{M}^2 \mathbb{E} \sup_{0 \leq r \leq s} |x(r) - y(r)|^2 \right) ds.$$

From Lemma 2.1 it follows that

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |x(s) - y(s)|^2 \right) = 0 \quad \text{on } t \in [0, T].$$

On the other hand, it is obvious that $x(t) = y(t)$ for $t \in (-\infty, 0]$, thus, $x(t) = y(t)$ almost surely for $t \in (-\infty, 0]$. This Proof is complete.

Remark 2.5. 1. In special case, when the delay is finite, and $h \equiv 0$ or $\nu \equiv 0$, (1.1) reduces to

$$\begin{aligned} d[x(t) - g(t, x_t)] &= A[x(t) - g(t, x_t)] dt + \left[\int_0^t B(t-s) [x(s) - g(s, x_s)] ds + f(t, x_t) \right] dt \\ &\quad + \sigma(t, x_t) dw(t) \end{aligned}$$

with the initial value $x(t) = \xi(t)$, where $\xi(\cdot)$ is a bounded continuous measurable function mapping $[-\tau, 0]$ into $\mathbb{H}$, which was recently studied.

2. We obtain the existence and uniqueness of mild solution to (1.1) under non-Lipschitz conditions makes it more feasible that the conditions of solution can be satisfied.

3. CONTINUOUS DEPENDENCE OF INITIAL DATA

In this section, we are interested in the continuous dependence on the initial data. From now on, we will use $x^\varphi(t)$ to represent the mild solution of (1.1) to emphasize that the solution depends on the initial value φ . We need the following assumptions :

(H6)

For all $t \in I$, $\varphi_1, \varphi_2 \in \mathcal{B}$, $x, y \in \mathbb{H}$, there exist a pair of positive constants L, L_1 such that

$$|f(t, \varphi_1) - f(t, \varphi_2)|^2 \vee |\sigma(t, \varphi_1) - \sigma(t, \varphi_2)|_{\mathcal{L}_2^0}^2 \leq L \|\varphi_1 - \varphi_2\|_{\mathcal{B}}^2,$$

$$\int_Z |h(t, x, z) - h(t, y, z)|^k \nu(dz) \leq L_1 |x - y|^k, \quad k = 2, 4.$$

Theorem 3.1. *Let assumption (H3), (H5) and (H6) be satisfied and*

$K^2 \tilde{M}^2 + L_1^{\frac{1}{2}} T^{\frac{1}{2}} < \frac{1}{5}$. Then the mild solution of (1.1) is continuous in the initial value φ (with respect to the strong topology on $\mathbb{H}$).

Proof. Let $x^{\varphi_1}(t)$ and $x^{\varphi_2}(t)$ be two mild solutions with initial values φ_1, φ_2 respectively. Then we have

$$\begin{aligned} x^{\varphi_1}(s) - x^{\varphi_2}(s) &= R(s)[\varphi_1(0) - \varphi_2(0) + g(0, \varphi_1) - g(0, \varphi_2)] + g(s, x^{\varphi_1}(s)) - g(s, x^{\varphi_2}(s)) \\ &\quad + \int_0^s R(s-r)[f(r, x^{\varphi_1}(r)) - f(r, x^{\varphi_2}(r))]dr + \int_0^s R(s-r)[\sigma(r, x^{\varphi_1}(r)) - \sigma(r, x^{\varphi_2}(r))]dw(r) \\ &\quad + \int_0^s \int_Z R(s-r)[h(r, x^{\varphi_1}(r-), z) - h(r, x^{\varphi_2}(r-), z)]\tilde{N}(dr, dz). \end{aligned}$$

Further,

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |x^{\varphi_1}(s) - x^{\varphi_2}(s)|^2 &\leq 5\mathbb{E} \sup_{0 \leq s \leq t} |R(s)[\varphi_1(0) - \varphi_2(0) + g(0, \varphi_1) - g(0, \varphi_2)]|^2 \\ &\quad + 5\mathbb{E} \sup_{0 \leq s \leq t} |g(s, x^{\varphi_1}(s)) - g(s, x^{\varphi_2}(s))|^2 + 5\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s R(s-r)[f(r, x^{\varphi_1}(r)) - f(r, x^{\varphi_2}(r))]dr \right|^2 \\ &\quad + 5\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \int_Z R(s-r)[h(r, x^{\varphi_1}(r-), z) - h(r, x^{\varphi_2}(r-), z)]\tilde{N}(dr, dz) \right|^2 \end{aligned}$$

$$\begin{aligned}
& +5\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^t R(s-r)[\sigma(r, x^{\varphi_1}(r)) - \sigma(r, x^{\varphi_2}(r))]dw(r) \right|^2 \\
& +5\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^t \int_Z R(s-r)[h(r, x^{\varphi_1}(r-), z) - h(r, x^{\varphi_2}(r-), z)]\tilde{N}(dr, dz) \right|^2,
\end{aligned}$$

a similar arguments as before it follows that

$$\mathbb{E} \sup_{0 \leq s \leq t} |x^{\varphi_1}(s) - x^{\varphi_2}(s)|^2 \leq K_1 \mathbb{E} \|\varphi_1 - \varphi_2\|_{\mathcal{B}}^2 + K_2 \int_0^t E \sup_{0 \leq r \leq s} |x^{\varphi_1}(r) - x^{\varphi_2}(r)|^2 ds,$$

where

$$\begin{aligned}
K_1 &= \frac{10(1 + K^2)}{1 - 5\tilde{M}^2 K^2 - 5L_1^{\frac{1}{2}} T^{\frac{1}{2}}}, \\
K_2 &= \frac{5(K^2 \tilde{M}^2 + LT \tilde{M}^2 + LC_1 \tilde{M}^2 + L_1 C_2)}{1 - 5\tilde{M}^2 K^2 - 5L_1^{\frac{1}{2}} T^{\frac{1}{2}}},
\end{aligned}$$

C_1, C_2 are the positive constants in Lemma 2.2. Applying Gronwall's inequality, we have

$$\mathbb{E} \sup_{0 \leq s \leq t} |x^{\varphi_1}(s) - x^{\varphi_2}(s)|^2 \leq K_1 e^{K_2 t} \mathbb{E} \|\varphi_1 - \varphi_2\|_{\mathcal{B}}^2.$$

Which means the mild solution is continuous in the initial value.

4. EXAMPLE

In this section, we provide an example to illustrate the obtained results above. We consider the following neutral stochastic partial integrodifferential equations with infinite delay and Poisson jumps:

$$\left\{ \begin{aligned}
& \frac{\partial}{\partial t} [u(t, \xi) + g(t, u(t - \tau, \xi))] = \frac{\partial^2}{\partial \xi^2} [u(t, \xi) + g(t, u(t - \tau, \xi))] \\
& \quad + \int_0^t b(t-s) \frac{\partial^2}{\partial \xi^2} [u(s, \xi) + g(s, u(s - \tau, \xi))] ds + f(t, u(t - \tau, \xi)) dt \\
& \quad + \sigma(t, u(t - \tau, \xi)) dw(t) + \int_Z l(t, x(t-, \xi), z) \tilde{N}(dt, dy), \quad t \geq 0. \\
& u(t, 0) + g(t, u(t - \tau, 0)) = 0 \quad \text{for } t \geq 0, \\
& u(t, \pi) + g(t, u(t - \tau, \pi)) = 0 \quad \text{for } t \geq 0, \\
& u(\theta, \xi) = u_0(\theta, \xi) \quad \text{for } \theta \in]-\infty, 0] \quad \text{and } \xi \in [0, \pi],
\end{aligned} \right. \tag{4.1}$$

where $w(t)$ denotes the standard cylindrical Wiener process and $Z = \{z \in \mathbb{R} : 0 < |z| \leq c, c > 0\}$, $b(t)$, $t \in \mathbb{R}$ is continuous and there exists a positive constant M_b such that $|b(t-s)|^2 \leq M_b$.

Let $\mathbb{H} = L^2(0, \pi)$ with the norm $\|\cdot\|$ and

$e_n := \sqrt{\frac{2}{\pi}} \sin(nx)$, $(n = 1, 2, 3, \dots)$ denote the completed orthonormal basis in $\mathbb{H}$.

Let $w(t) := \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n$ ($\lambda_n > 0$), where $\beta_n(t)$ are one dimensional standard Brownian motion mutually independent on a usual complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

Define $A : \mathbb{H} \rightarrow \mathbb{H}$ by $A = \frac{\partial^2}{\partial y^2}$, with domain $D(A) = H^2(0, \pi) \cap H_0^1(0, \pi)$.

Then $A\tilde{h} = -\sum_{n=1}^{\infty} n^2 \langle \tilde{h}, e_n \rangle e_n$, $\tilde{h} \in D(A)$, where e_n , $n = 1, 2, 3, \dots$, is also the orthonormal set of eigenvectors of A . It is well-known that A is the infinitesimal generator of a strongly continuous semigroup on $\mathbb{H}$, thus **(H1)** is true.

Let $B : D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ be the operator defined by

$B(t)(y) = b(t)Ay$ for $t \geq 0$ and $y \in D(A)$.

Let $\gamma > 0$, define the phase space

$\mathcal{B} = \{\varphi \in C((-\infty, 0], \mathbb{H}) : \lim_{\theta \rightarrow -\infty} e^{\theta\gamma} \varphi(\theta) \text{ exists in } \mathbb{H}\}$

and let $\|\varphi\|_{\mathcal{B}} = \sup_{\theta \in (-\infty, 0]} \{e^{\gamma\theta} \|\varphi\|_{L_2}\}$. Then $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space and satisfies (A1)-(A2) with $L = 1$, $u(t) = e^{-\gamma t}$, $v(t) = \max\{1, e^{-\gamma t}\}$. Therefore, for $(t, \psi) \in J \times \mathcal{B}$, where $\psi(\theta)(\xi) = \psi(\theta, \xi)$, $(\theta, \xi) \in (-\infty, 0] \times [0, \pi]$, let $x(t)(\theta) = u(t, \xi)$ and define the functions

$g, f : J \times \mathcal{B} \rightarrow \mathbb{H}$ $\sigma : J \times \mathcal{B} \rightarrow \mathcal{L}_2^0(\mathbb{H}, \mathbb{H})$ and

$h : [0, \infty[\times \mathbb{H} \times (Z - \{0\}) \rightarrow \mathbb{H}$ for the infinite delay as follows:

$$\begin{aligned} g(t, \psi)(\xi) &= \int_{-\infty}^0 k_1(\theta) \psi(\theta)(\xi) d\theta, \\ f(t, \psi)(\xi) &= \int_{-\infty}^0 k_2(t, \xi, \theta) G_1(\psi(\theta)) d\theta, \\ \sigma(t, \psi)(\xi) &= \int_{-\infty}^0 k_3(t, \xi, \theta) G_2(\psi(\theta)) d\theta, \\ h(t, \psi(\xi), z) &= l(t, \psi(\xi), z). \end{aligned}$$

where

(I) the functions $k_1(\theta) \geq 0$ is continuous in $(-\infty, 0]$ and satisfies

$$\int_{-\infty}^0 k_1^2(\theta) d\theta < \infty, \quad L_g = \left(-\frac{1}{2\gamma} \int_{-\infty}^0 k_1^2(\theta) d\theta\right);$$

(II) the functions $k_2(t, \xi, \theta), k_3(t, \xi, \theta)$ are continuous in $J \times [0, \pi] \times (-\infty, 0]$ and satisfy $\int_{-\infty}^0 k_2^2(t, \xi, \theta) d\theta = p_2(t, \xi) < \infty$, $(\int_0^\pi p_2^2(t, \xi) d\xi) < 1$,

$$\int_{-\infty}^0 k_3^2(t, \xi, \theta) d\theta = p_3(t, \xi) < \infty, \quad (\int_0^\pi p_3^2(t, \xi) d\xi) < 1,$$

(III) the functions G_i , $i = 1, 2$ is continuous in $J \times [0, \pi] \times (-\infty, 0)$ and satisfies

$$0 \leq G_1(x_1(\theta, \xi)) - G_1(x_2(\theta, \xi)) \leq k^{\frac{1}{2}}(\|x_1(\theta, \cdot) - x_2(\theta, \cdot)\|_{L^2}^2),$$

$$0 \leq G_2(x_1(\theta, \xi)) - G_2(x_2(\theta, \xi)) \leq k^{\frac{1}{2}}(\|x_1(\theta, \cdot) - x_2(\theta, \cdot)\|_{L^2}^2),$$

for $(\theta, \xi) \in (-\infty, 0] \times [0, \pi]$, where $k(\cdot)$ is defined as **(H3)**.

(IV) the function $l(t, \psi, z)$ satisfies

$$\int_Z |l(t, \psi_1, z) - l(t, \psi_2, z)|^2 \nu(dz) \leq k(|\psi_1 - \psi_2|^2).$$

Under the above assumptions, we can rewritten Eq. (4.1) as the abstract form of Eq.(1.1).

$$\left\{ \begin{array}{l} d[x(t) + g(t, x_t)] = \left[A[x(t) + g(t, x_t)] + \int_0^t B(t-s)[x(s) + g(s, x_s)]ds \right] dt \\ + f(t, x_t)dt + \sigma(t, x_t)dw(t) + \int_Z h(t, x(t-), y)\tilde{N}(dt, dy), \quad t \in I := [0, T], \\ x_0 = \varphi \in \mathcal{B}. \end{array} \right. \quad (4.2)$$

We suppose b is bounded and C^1 function such that b' is bounded and uniformly continuous, then **(H1)** and **(H2)** are satisfied and hence, by Theorem 2.2, Eq. (1.1) has a resolvent operator $(R(t))_{t \geq 0}$ on $\mathbb{H}$.

By assumption (I),(II), (III) and (IV) we have

$$\begin{aligned} \|g(t, \psi_1) - g(t, \psi_2)\|_{\mathbb{H}}^2 &= \int_0^\pi \left(\int_{-\infty}^0 k_1(\theta) (\psi_1(\theta)(\xi) - \psi_2(\theta)(\xi)) d\theta \right)^2 d\xi \\ &\leq \left(\int_{-\infty}^0 k_1^2(\theta) d\theta \right) \left(\int_0^\pi \int_{-\infty}^0 (\psi_1(\theta)(\xi) - \psi_2(\theta)(\xi))^2 d\theta d\xi \right) \\ &\leq \left(-\frac{1}{2\gamma} \int_{-\infty}^0 k_1^2(\theta) d\theta \right) \sup_{\theta \in]-\infty, 0]} \{e^{2\gamma\theta} \|\psi\|_{L^2}^2\} = L_g(\|\psi_1 - \psi_2\|_{\mathcal{B}}^2). \end{aligned}$$

$$\begin{aligned}
\|f(t, \psi_1) - f(t, \psi_2)\|_{\mathbb{H}}^2 &= \int_0^\pi \left(\int_{-\infty}^0 k_2(t, \xi, \theta) (G_1(\psi_1(\theta)) - G_1(\psi_2(\theta))) d\theta \right)^2 d\xi \\
&\leq \int_0^\pi \left(\int_{-\infty}^0 k_2(t, \xi, \theta) k^{\frac{1}{2}}(\|\psi_1(\theta, \cdot) - \psi_2(\theta, \cdot)\|_{L^2}^2) d\theta \right)^2 d\xi \\
&\leq \int_0^\pi \left(\int_{-\infty}^0 k_2(t, \xi, \theta) k^{\frac{1}{2}}(e^{2\alpha\theta} \|\psi_1(\theta, \cdot) - \psi_2(\theta, \cdot)\|_{L^2}^2) d\theta \right)^2 d\xi \\
&\leq \left(\int_0^\pi p_2^2(t, \xi) d\xi \right) k(\|\psi_1 - \psi_2\|_{\mathcal{B}}^2).
\end{aligned}$$

In the same way we obtain the following estimation

$$\|\sigma(t, \psi_1) - \sigma(t, \psi_2)\|_{\mathbb{H}}^2 \leq \left(\int_0^\pi p_3^2(t, q) dq \right) k(\|\psi_1 - \psi_2\|_{\mathcal{B}}^2).$$

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