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NONLOCAL FRANKL PROBLEM FOR SINGULAR MIXED-TYPE EQUATIONS

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Abstract. We investigate a boundary value problem with nonlocal Frankl-type conditions for a three-dimensional mixed-type equation whose coefficient is singular on one face of the domain. The domain is a cylinder whose base is the union of a quarter disk, where the equation is elliptic, and four triangles, where it is hyperbolic. Separating the variables, we construct eigenvalues and eigenfunctions of the associated planar spectral problem and use their completeness to prove that a regular solution is unique. The solution is then written as a double Fourier–Bessel series whose uniform convergence, with second derivatives, follows from asymptotic bounds for Bessel functions.

Keywords. Frankl conditions, mixed-type equation, singular coefficient, Bessel functions, eigenfunctions.
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1. Introduction

Over roughly the last five decades, boundary value problems with nonlocal conditions have become one of the most active branches of the theory of partial differential equations. Several distinct families have emerged: problems in which the boundary data are linked through a shift operator [14], problems of Frankl type [5, 15, 6], and problems involving integral conditions [21]. Interest in such problems is motivated both theoretically, since they contain many classical local problems as special cases, and practically, since they arise in models of transonic gas flow, heat transfer and population dynamics.

Frankl-type conditions occupy a special place among nonlocal conditions. They appeared originally in F. I. Frankl's study of transonic flow past an airfoil, where the equation changes type across the sonic line, and they relate the values of the solution at points symmetric with respect to the line of type change. A. V. Bitsadze was the first to treat the Frankl problem both for the Lavrent'ev–Bitsadze equation $u_{xx}+(\operatorname{sgn} y)\,u_{yy}=0$ and for Chaplygin-type equations $K(y)u_{xx}+u_{yy}=0$ with $K(0)=0$, $K'(y)>0$ [1, 2]. Devingtal' then studied the Chaplygin equation

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under the assumptions $K \in C^1(\bar{D})$ and $K(-y) = -K(y)$ [3, 4]. Moiseev applied spectral analysis to the equation $u_{xx} + \operatorname{sgn} y\, u_{yy} - \mu^2 \operatorname{sgn}(x+y)\, u = 0$, $\mu = \text{const}$ [9], and a series of papers by Moiseev and his coauthors established completeness and the basis property of eigenfunctions for modified Frankl problems with nonlocal oddness and parity conditions [11, 10, 12, 13]. A detailed survey of results on the Frankl problem for the Lavrent'ev–Bitsadze and Chaplygin equations may be found in [16].

Most of the works cited above deal with planar problems. Spectral methods for three-dimensional mixed-type equations containing Bessel operators were developed in [19, 20], and the local Frankl problem for a three-dimensional mixed-type equation with a singular coefficient was recently studied in [7]. In the present paper we go further and replace the local conditions by *nonlocal* Frankl conditions imposed on both planes of type change, with a parameter $k \in \{-1, 1\}$. The hyperbolic part of the domain now consists of four prisms, which leads to a non-self-adjoint spectral problem in the angular variable and to a new family of eigenvalues. We prove uniqueness and existence of a regular solution by means of the Fourier method.

2. Statement of the problem

In the domain $D = \Omega \times (0, c)$ we consider the equation

$$\operatorname{sgn}(x+y)\big[\operatorname{sgn} x \cdot U_{xx} + \operatorname{sgn} y \cdot U_{yy}\big] + U_{zz} + \frac{2\gamma}{z} U_z = 0, \tag{2.1}$$

where $\gamma \in (-\infty, 1/2)$ and Ω is the bounded simply connected plane region enclosed by the arc $\bar{\sigma}_0 = \{(x, y) : x^2 + y^2 = 1,\ x \geq 0,\ y \geq 0\}$ and the segments

$$MM^* = \{y - x = 1,\ 0 \leq y \leq 1\}, \quad M^*O = \{y = 0,\ -1 \leq x \leq 0\},$$
$$OP^* = \{x = 0,\ -1 \leq y \leq 0\}, \quad PP^* = \{x - y = 1,\ 0 \leq x \leq 1\}.$$

Here $O(0,0)$, $M(0,1)$, $M^*(-1,0)$, $P(1,0)$, $P^*(0,-1)$, $K(-1/2, 1/2)$, $Q(1/2, -1/2)$. We split Ω into the parts

$$\Omega_0 = \Omega \cap \{x > 0,\ y > 0\}, \quad \Omega_1 = \Omega \cap \{y < 0,\ x + y > 0\},$$
$$\Omega_2 = \Omega \cap \{y < 0,\ x + y < 0\}, \quad \Omega_3 = \Omega \cap \{x < 0,\ x + y > 0\},$$
$$\Omega_4 = \Omega \cap \{x < 0,\ x + y < 0\},$$

and put $D_j = \Omega_j \times (0, c)$, $j = \overline{0,4}$, $OP = \bar{\Omega}_0 \cap \bar{\Omega}_1$, $OQ = \bar{\Omega}_1 \cap \bar{\Omega}_2$, $OM = \bar{\Omega}_0 \cap \bar{\Omega}_3$, $OK = \bar{\Omega}_3 \cap \bar{\Omega}_4$.

Equation (2.1) is elliptic in D_0 (a quarter of a cylinder) and hyperbolic in each of the triangular prisms D_j, $j = \overline{1,4}$. It changes type across the rectangles $OP \times (0, c)$ and $OM \times (0, c)$, its coefficient is singular on the plane $z = 0$, and its coefficients have jump discontinuities across $OQ \times (0, c)$ and $OK \times (0, c)$.

Definition 2.1. A function $U(x,y,z)$ is called a *regular solution* of (2.1) in D if it satisfies (2.1) in every D_j, $j=\overline{0,4}$, and belongs to the class

$$C(\bar D)\cap C^1\Big(\big(D\cup(M^*O\times(0,c))\cup(OP^*\times(0,c))\big)\setminus\big((\bar D_1\cap\bar D_2)\cup(\bar D_3\cap\bar D_4)\big)\Big)\cap C^{2,2,2}_{x,y,z}\Big(\bigcup\nolimits_{j=0}^{4}D_j\Big). \quad (2.2)$$

Problem F**.** Find a regular solution of (2.1) in D satisfying

$$U(x,y,z)=0,\quad (x,y)\in\bar\sigma_0,\ z\in(0,c), \quad (2.3)$$

$$U_x(0,y,z)=0,\ y\in(-1,0),\qquad U_y(x,0,z)=0,\ x\in(-1,0),\qquad z\in(0,c), \quad (2.4)$$

$$U(0,y,z)+kU(0,-y,z)=0,\quad y\in[-1,0],\ z\in[0,c], \quad (2.5)$$

$$U(x,0,z)+kU(-x,0,z)=0,\quad x\in[-1,0],\ z\in[0,c], \quad (2.6)$$

$$U(x,y,0)=f_1(x,y),\qquad U(x,y,c)=f_2(x,y),\qquad (x,y)\in\bar\Omega, \quad (2.7)$$

where f_1, f_2 are given functions and k takes one of the values $k=1$ or $k=-1$.

3. Auxiliary Cauchy–Goursat problems

Separation of variables reduces the hyperbolic parts of Problem F to planar problems for the equation

$$\operatorname{sgn}x\,u_{xx}+\operatorname{sgn}y\,u_{yy}+\lambda\operatorname{sgn}(x+y)\,u=0, \quad (3.1)$$

which in Ω_1 takes the form $u_{xx}-u_{yy}+\lambda u=0$. A regular solution of this equation in Ω_1 is a function $u\in C^2(\Omega_1)\cap C(\bar\Omega_1)$ such that $u(x,0)\in C[0,1]\cap C^2(0,1)$ and $\lim_{y\to-0}u_y(x,y)\in C^2(0,1)$, the limit being allowed to have an integrable singularity (of order less than one) as $x\to0$ and $x\to1$; regular solutions in Ω_j, $j=\overline{2,4}$, are defined in the same way. We use the notation

$$\left.\begin{aligned}&\tau_1(x)=u(x,0),\ -1\le x\le1;\qquad &&\nu_1(x)=u_y(x,0),\ -1<x<1;\\&\tau_2(y)=u(0,y),\ -1\le y\le1;\qquad &&\nu_2(y)=u_x(0,y),\ -1<y<1.\end{aligned}\right\} \quad (3.2)$$

Consider the Cauchy–Goursat problem of finding a regular solution of $u_{xx}-u_{yy}+\lambda u=0$ in Ω_1 with $u_y(x,0)=\nu_1(x)$, $x\in(0,1)$, and $u(x,-x)=\psi_1(2x)$, $x\in[0,1/2]$. By the results of [18], this problem has a unique solution, given by

$$u(x,y)=\int_0^\xi\nu_1(t)J_0\Big[\sqrt{\lambda(\xi-t)(\eta-t)}\Big]dt+\psi_1(\xi/2)+\psi_1(\eta/2)-\psi_1(0)J_0\big(\sqrt{\lambda\xi\eta}\big)-\int_0^\eta\psi_1(z/2)\,B_z(0,z;\xi,\eta)\,dz, \quad (3.3)$$

where $\xi=x+y$, $\eta=x-y$, J_l is the Bessel function of the first kind of order l [22], $B_z=\partial B/\partial z$, and

$$B(t,z;\xi,\eta)=\begin{cases}J_0\Big[\sqrt{\lambda(\xi-t)(\eta-t)}\Big], & z>\xi,\\ J_0\Big[\sqrt{\lambda(\xi-t)(\eta-z)}\Big]+J_0\Big[\sqrt{\lambda(\xi-z)(\eta-t)}\Big], & z<\xi.\end{cases}$$

The analogous problem in Ω_3 for $-u_{xx}+u_{yy}+\lambda u=0$ with $u_x(0,y)=\nu_2(y)$, $y\in(0,1)$, and $u(-y,y)=\psi_2(2y)$, $y\in[0,1/2]$, is transformed into the previous one by interchanging x and y. Consequently its solution is given by (3.3) with ν_1,ψ_1 replaced by ν_2,ψ_2 and with $\xi=x+y$, $\eta=y-x$.

4. The spectral problem

Seeking non-trivial solutions of (2.1) of the form $U(x,y,z)=u(x,y)Z(z)$ that satisfy (2.2)–(2.6), we obtain for Z the equation

$$Z''(z)+\frac{2\gamma}{z}Z'(z)-\lambda Z(z)=0,\qquad 0<z<c, \tag{4.1}$$

and for u the spectral problem F_λ:

$$u\in C(\bar\Omega)\cap C^1\big((\Omega\cup M^*O\cup OP^*)\setminus KQ\big)\cap C^{2,2}_{x,y}\big(\Omega\setminus(OP\cup OM\cup KQ)\big), \tag{4.2}$$

$$\operatorname{sgn}x\,u_{xx}+\operatorname{sgn}y\,u_{yy}+\lambda\operatorname{sgn}(x+y)\,u=0,\quad (x,y)\in\Omega\setminus(OP\cup OM\cup KQ), \tag{4.3}$$

$$u(x,y)=0,\quad (x,y)\in\bar\sigma_0, \tag{4.4}$$

$$u_x(0,y)=0,\ y\in(-1,0),\qquad u(0,y)+ku(0,-y)=0,\ y\in[-1,0], \tag{4.5}$$

$$u_y(x,0)=0,\ x\in(-1,0),\qquad u(x,0)+ku(-x,0)=0,\ x\in[-1,0]. \tag{4.6}$$

4.1. **Reduction to the elliptic part.** For $(x,y)\in\Omega_1$ let $V(x,y)=u(x,y)-u(-y,-x)$; note that $(-y,-x)\in\Omega_2$. Since u is continuous across OQ, i.e. $u(x,-x-0)=u(x,-x+0)$ for $0\le x\le 1/2$, the function V is a regular solution of (4.3) in Ω_1 that vanishes on the characteristic $y=-x$. Formula (3.3) with $\psi_1\equiv 0$ therefore yields

$$V(x,y)=\int_0^{x+y}V_y(t,0)\,J_0\Big[\sqrt{\lambda(\xi-t)(\eta-t)}\Big]dt.$$

Setting $y=0$ and using (3.2), we get

$$\tau_1(x)-\tau_2(-x)=\int_0^x\big[\nu_1(t)+\nu_2(-t)\big]J_0\big[\sqrt{\lambda}(x-t)\big]dt,\qquad 0\le x\le 1,$$

which, combined with (4.5), becomes

$$\tau_1(x)+k\tau_2(x)=\int_0^x\nu_1(t)J_0\big[\sqrt{\lambda}(x-t)\big]dt,\qquad 0\le x\le 1. \tag{4.7}$$

Repeating the argument in $\Omega_3\cup\Omega_4$ with the help of (4.6) and the continuity of u across OK gives $\tau_2(x)+k\tau_1(x)=\int_0^x\nu_2(t)J_0[\sqrt{\lambda}(x-t)]\,dt$, $0\le x\le 1$. Multiplying this identity by k, comparing it with (4.7) and using $k^2=1$, we find

$$\nu_1(x)=k\,\nu_2(x),\qquad x\in(0,1). \tag{4.8}$$

Thus Problem F_λ is equivalent to the following Problem C_λ posed in the elliptic part: *find the values of λ for which there exist non-trivial functions $u\in C^2(\Omega_0)\cap C(\bar\Omega_0)$ solving (4.3) in Ω_0 and satisfying (4.4), (4.7) and (4.8).*

4.2. **Eigenvalues and eigenfunctions in Ω_0.** We introduce polar coordinates $r=\sqrt{x^2+y^2}$, $\varphi=\operatorname{arctg}(y/x)$ and look for solutions of the form $u=R(r)\,\Phi(\varphi)$. Then Problem C_λ splits into the problems

$$r^2R''(r)+rR'(r)+(\lambda r^2-\omega^2)R(r)=0,\ 0<r<1,\quad |R(0)|<\infty,\quad R(1)=0, \tag{4.9}$$

$$\Phi''(\varphi)+\omega^2\Phi(\varphi)=0,\ \varphi\in(0,\pi/2),\quad \Phi'(0)+k\Phi'(\pi/2)=0, \tag{4.10}$$

$$R(x)\big[\Phi(0)+k\Phi(\pi/2)\big]=\Phi'(0)\int_0^x t^{-1}R(t)J_0\big[\sqrt{\lambda}(x-t)\big]dt,\ 0\le x\le 1, \tag{4.11}$$

where ω^2 is the separation constant. The function $R(r)=J_\omega(\sqrt{\lambda}\,r)$, $\operatorname{Re}\omega\ge 0$, solves the equation in (4.9) and is bounded at $r=0$. Substituting it into (4.11) and evaluating the integral by the identity [10]

$$\int_0^u J_p(c\xi)J_q(cu-c\xi)\frac{d\xi}{\xi}=\frac{1}{p}J_{p+q}(cu),\qquad \operatorname{Re}p>0,\ \operatorname{Re}q>-1,$$

with $q=0$, we arrive at a second boundary condition for Φ:

$$\omega\big[\Phi(0)+k\Phi(\pi/2)\big]-\Phi'(0)=0. \tag{4.12}$$

The cases $k=-1$ and $k=1$ are treated separately; we give the details for $k=-1$.

Inserting the general solution $\Phi(\varphi)=c_1\cos(\omega\varphi)+c_2\sin(\omega\varphi)$ into the conditions of (4.10) and (4.12), we obtain the linear system

$$\begin{cases} c_1\sin(\omega\pi/2)+c_2\big[1-\cos(\omega\pi/2)\big]=0,\\ c_1\big[1-\cos(\omega\pi/2)\big]-c_2\big[1+\sin(\omega\pi/2)\big]=0. \end{cases} \tag{4.13}$$

Its determinant vanishes if and only if $2\cos(\omega\pi/2)-\sin(\omega\pi/2)=2$. Writing $t=\operatorname{tg}(\omega\pi/4)$, this equation becomes $t(2t+1)=0$, so the problem (4.10), (4.12) has the two sequences of eigenvalues

$$\omega_n^{(1)}=4(n-1),\qquad \omega_n^{(2)}=4n-\frac{4}{\pi}\operatorname{arcctg}2,\qquad n\in\mathbb{N},$$

which may be combined into the single formula

$$\omega_n=2(n-1)+\big[1+(-1)^n\big]\Big(1-\frac{2}{\pi}\operatorname{arcctg}2\Big),\qquad n\in\mathbb{N}. \tag{4.14}$$

Taking $\omega=\omega_n$ and using the second equation of (4.13), we obtain the eigenfunctions

$$\Phi_n(\varphi)=\sqrt{\frac{2}{\pi}}\,\sin\big[\omega_n\varphi+\varphi_{0n}\big],\qquad \varphi_{0n}=\operatorname{arcctg}\big(1+(-1)^n\big),\quad n\in\mathbb{N}. \tag{4.15}$$

The system (4.15) is complete in $L_2(0,\pi/2)$ [18].

With $\omega=\omega_n$, the condition $R(1)=0$ gives $J_{\omega_n}(\sqrt{\lambda})=0$. Since $\omega_n\ge 0>-1$, each of these equations has a countable set of positive roots [22]. Denoting by α_{nm} the m-th positive root, we obtain the eigenvalues

$$\lambda_{nm}=\alpha_{nm}^2,\qquad n,m\in\mathbb{N}, \tag{4.16}$$

and the corresponding eigenfunctions of (4.9), normalized with weight r,

$$R_{nm}(r) = \sqrt{2}\, J_{\omega_n}(\alpha_{nm} r)/J_{\omega_n+1}(\alpha_{nm}), \qquad n, m \in \mathbb{N}. \tag{4.17}$$

For each fixed n the system (4.17) is orthonormal and complete in $L_2(0,1)$ with weight r [18]. Hence the eigenfunctions of Problem C_λ are

$$u^{(0)}_{nm}(x,y) = d_{nm} \sin\left[\omega_n \varphi + \varphi_{0n}\right] J_{\omega_n}(\alpha_{nm} r), \quad (x,y) \in \Omega_0, \tag{4.18}$$

where $d_{nm} = 2/\left(\sqrt{\pi}\, J_{\omega_n+1}(\alpha_{nm})\right)$.

4.3. **Continuation into the hyperbolic parts.** We now show that every λ_{nm} is an eigenvalue of F_λ by continuing (4.18) into Ω_j, $j = \overline{1,4}$. Following [17], we introduce in Ω_1 the variables $\xi = \sqrt{x^2 - y^2}$, $\eta = x^2/\xi^2$. Solving (4.3) in these variables and returning to x, y gives the family

$$u(x,y) = J_\rho\Big(\sqrt{\lambda(x^2-y^2)}\Big)\left[c_3\Big(\frac{x-y}{x+y}\Big)^{\rho/2} + c_4\Big(\frac{x+y}{x-y}\Big)^{\rho/2}\right], \quad (x,y) \in \Omega_1, \tag{4.19}$$

where $\operatorname{Re}\rho \geq 0$ and c_3, c_4 are arbitrary constants. From (4.18) we compute, with $l_n = \sqrt{1 + [1 + (-1)^n]^2}$,

$$u^{(0)}_{nm}(x,0) = \frac{d_{nm}}{l_n} J_{\omega_n}(\alpha_{nm} x), \quad \frac{\partial u^{(0)}_{nm}}{\partial y}(x,0) = \frac{[1+(-1)^n]\, d_{nm}}{l_n}\, \omega_n x^{-1} J_{\omega_n}(\alpha_{nm} x). \tag{4.20}$$

Choosing $\rho = \omega_n$, $\lambda = \lambda_{nm}$ and the constants c_3, c_4 so that (4.19) matches the data (4.20) on OP, we obtain

$$\begin{aligned} u^{(1)}_{nm}(x,y) = &-\frac{d_{nm}}{2l_n} J_{\omega_n}\left[\alpha_{nm}\sqrt{x^2-y^2}\right] \\ &\times \left[(-1)^n\Big(\frac{x-y}{x+y}\Big)^{\omega_n/2} - \left[2 + (-1)^n\right]\Big(\frac{x+y}{x-y}\Big)^{\omega_n/2}\right], \quad (x,y) \in \Omega_1. \end{aligned} \tag{4.21}$$

In the same way, (4.18) gives on OM

$$\begin{aligned} u^{(0)}_{nm}(0,y) &= -\frac{(-1)^n d_{nm}}{l_n} J_{\omega_n}(\alpha_{nm} y), \\ \frac{\partial u^{(0)}_{nm}}{\partial x}(0,y) &= -\frac{[1+(-1)^n]\, d_{nm}}{l_n}\, \omega_n y^{-1} J_{\omega_n}(\alpha_{nm} y). \end{aligned} \tag{4.22}$$

Interchanging x and y in (4.19) produces solutions of (4.3) in Ω_3; with $\rho = \omega_n$, $\lambda = \lambda_{nm}$, $c_5 = d_{nm}/(2l_n)$, $c_6 = -[1 + 2(-1)^n] d_{nm}/(2l_n)$, the data (4.22) are matched by

$$\begin{aligned} u^{(3)}_{nm}(x,y) = &\frac{d_{nm}}{2l_n} J_{\omega_n}\left[\alpha_{nm}\sqrt{y^2-x^2}\right] \\ &\times \left[\Big(\frac{y-x}{y+x}\Big)^{\omega_n/2} - \left[1 + 2(-1)^n\right]\Big(\frac{y+x}{y-x}\Big)^{\omega_n/2}\right], \quad (x,y) \in \Omega_3. \end{aligned} \tag{4.23}$$

By (4.5) and (4.22), on the segment OP^* we have

$$u(0,y) = -\frac{(-1)^n d_{nm}}{l_n} J_{\omega_n}(-\alpha_{nm} y), \qquad u_x(0,y) = 0, \qquad y \in (-1,0).$$

Replacing x, y by $-y, -x$ in (4.19) and taking $c_3 = c_4 = -(-1)^n d_{nm}/(2l_n)$, we obtain the solution in Ω_2:

$$u^{(2)}_{nm}(x,y) = -\frac{(-1)^n d_{nm}}{2l_n} J_{\omega_n}\left[\alpha_{nm}\sqrt{y^2-x^2}\right] \times \left[\left(\frac{y-x}{y+x}\right)^{\omega_n/2} + \left(\frac{y+x}{y-x}\right)^{\omega_n/2}\right], \quad (x,y) \in \Omega_2. \quad (4.24)$$

Similarly, (4.6) and (4.20) give $u(x,0) = d_{nm} l_n^{-1} J_{\omega_n}(-\alpha_{nm} x)$ and $u_y(x,0) = 0$ on M^*O; replacing x, y by $-x, -y$ in (4.19) and setting $c_3 = c_4 = d_{nm}/(2l_n)$ yields

$$u^{(4)}_{nm}(x,y) = \frac{d_{nm}}{2l_n} J_{\omega_n}\left[\alpha_{nm}\sqrt{x^2-y^2}\right] \times \left[\left(\frac{x-y}{x+y}\right)^{\omega_n/2} + \left(\frac{x+y}{x-y}\right)^{\omega_n/2}\right], \quad (x,y) \in \Omega_4. \quad (4.25)$$

Finally, for $\lambda = \lambda_{nm}$ the general solution of (4.1) is [22, 8]

$$Z_{nm}(z) = c_7 z^{1/2-\gamma} I_{1/2-\gamma}(\alpha_{nm} z) + c_8 z^{1/2-\gamma} K_{1/2-\gamma}(\alpha_{nm} z), \qquad z \in [0,c], \quad (4.26)$$

where c_7, c_8 are arbitrary constants, and I_l and K_l denote the modified Bessel function of the first kind and the Macdonald function of order l, respectively [22].

5. Uniqueness

In D_0 we pass to cylindrical coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$. The domain D_0 becomes $\tilde{D}_0 = \{(r,\varphi,z) : 0 < r < 1,\ 0 < \varphi < \pi/2,\ 0 < z < c\}$, and for $V(r,\varphi,z) = U(r\cos\varphi, r\sin\varphi, z)$ equation (2.1) and conditions (2.2), (2.3) take the form

$$V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\varphi\varphi} + V_{zz} + \frac{2\gamma}{z}V_z = 0, \quad (r,\varphi,z) \in \tilde{D}_0, \quad (5.1)$$

$$V \in C(\bar{\tilde{D}}_0) \cap C^1\big(\tilde{D}_0 \cup \{\varphi = 0\} \cup \{r = 1\} \cup \{\varphi = \pi/2\}\big) \cap C^{2,2,2}_{r,\varphi,z}(\tilde{D}_0), \quad (5.2)$$

$$V(1,\varphi,z) = 0, \quad \varphi \in [0,\pi/2],\ z \in [0,c]. \quad (5.3)$$

Theorem 5.1. *Suppose that Problem F has a solution and that this solution satisfies $V_\varphi(r,\pi/2,z) = -V_\varphi(r,0,z)$ and $V(r,\pi/2,z) = V(r,0,z)$. Then the solution is unique.*

Proof. Let U (equivalently V) be a solution of Problem F. For $n, m \in \mathbb{N}$ put

$$\vartheta_{nm}(z) = d_{nm}\int_0^1\int_0^{\pi/2} V(r,\varphi,z)\, r J_{\omega_n}(\alpha_{nm} r)\sin(\omega_n\varphi + \varphi_{0n})\, d\varphi\, dr, \quad (5.4)$$

and let $\vartheta^{\varepsilon_1\varepsilon_2}_{nm}(z)$ denote the same integral taken over $r \in [\varepsilon_2, 1-\varepsilon_2]$, $\varphi \in [\varepsilon_1, \pi/2 - \varepsilon_1]$, where $\varepsilon_1, \varepsilon_2 > 0$ are small. Clearly $\vartheta^{\varepsilon_1\varepsilon_2}_{nm}(z) \to \vartheta_{nm}(z)$ as $\varepsilon_1, \varepsilon_2 \to 0$. Denote by $B^z_q = \partial^2/\partial z^2 + \frac{2q+1}{z}\,\partial/\partial z$ the Bessel operator; for $q > -1/2$ it coincides with

the operator studied in [8]. Applying $B^z_{\gamma-1/2}$ to $\vartheta^{\varepsilon_1\varepsilon_2}_{nm}$ and replacing $B^z_{\gamma-1/2}V$ by means of (5.1), we obtain

$$\begin{aligned} &B^z_{\gamma-1/2}\vartheta^{\varepsilon_1\varepsilon_2}_{nm}(z) \\ &\quad = -d_{nm}\int_{\varepsilon_1}^{\pi/2-\varepsilon_1}\sin(\omega_n\varphi+\varphi_{0n})\int_{\varepsilon_2}^{1-\varepsilon_2}\left(rV_r\right)_r J_{\omega_n}(\alpha_{nm}r)\,dr\,d\varphi \\ &\qquad - d_{nm}\int_{\varepsilon_2}^{1-\varepsilon_2}\frac{1}{r}J_{\omega_n}(\alpha_{nm}r)\int_{\varepsilon_1}^{\pi/2-\varepsilon_1}V_{\varphi\varphi}\sin(\omega_n\varphi+\varphi_{0n})\,d\varphi\,dr. \end{aligned}$$

We integrate by parts twice in r and in φ and use the Bessel equation satisfied by $J_{\omega_n}(\alpha_{nm}r)$. This gives

$$\begin{aligned} &B^z_{\gamma-1/2}\vartheta^{\varepsilon_1\varepsilon_2}_{nm}(z)-\lambda_{nm}\vartheta^{\varepsilon_1\varepsilon_2}_{nm}(z) \\ &\quad = -d_{nm}\int_{\varepsilon_1}^{\pi/2-\varepsilon_1}\Big[r\Big(V_rJ_{\omega_n}(\alpha_{nm}r)-V\frac{d}{dr}J_{\omega_n}(\alpha_{nm}r)\Big)\Big]_{r=\varepsilon_2}^{r=1-\varepsilon_2} \\ &\qquad\qquad \times\sin(\omega_n\varphi+\varphi_{0n})\,d\varphi \\ &\qquad - d_{nm}\int_{\varepsilon_2}^{1-\varepsilon_2}\Big[V_\varphi\sin(\omega_n\varphi+\varphi_{0n})-\omega_nV\cos(\omega_n\varphi+\varphi_{0n})\Big]_{\varphi=\varepsilon_1}^{\varphi=\pi/2-\varepsilon_1} \\ &\qquad\qquad \times\frac{1}{r}J_{\omega_n}(\alpha_{nm}r)\,dr. \end{aligned}$$

Letting $\varepsilon_1,\varepsilon_2\to 0$ and using (5.2), (5.3), $J_{\omega_n}(\alpha_{nm})=0$ and the hypotheses of the theorem, we conclude that ϑ_{nm} solves (4.1) with $\lambda=\lambda_{nm}$. By (2.7),

$$\vartheta_{nm}(0)=f_{1nm},\qquad \vartheta_{nm}(c)=f_{2nm}, \tag{5.5}$$

where, with $\tilde f_l(r,\varphi)=f_l(r\cos\varphi,r\sin\varphi)$,

$$f_{lnm}=d_{nm}\int_0^1\int_0^{\pi/2}\tilde f_l(r,\varphi)\,rJ_{\omega_n}(\alpha_{nm}r)\sin(\omega_n\varphi+\varphi_{0n})\,d\varphi\,dr,\qquad l=1,2. \tag{5.6}$$

Determining c_7 and c_8 in (4.26) from (5.5), we get

$$c_8=\frac{2^{1/2+\gamma}\alpha_{nm}^{1/2-\gamma}f_{1nm}}{\Gamma(1/2-\gamma)},\qquad c_7=\frac{f_{2nm}-\bar K_{1/2-\gamma}(\alpha_{nm}c)f_{1nm}}{c^{1/2-\gamma}I_{1/2-\gamma}(\alpha_{nm}c)},$$

where $\bar K_\nu(x)=2^{1-\nu}x^\nu K_\nu(x)/\Gamma(\nu)$, $\nu>0$. Hence ϑ_{nm} is uniquely determined:

$$\vartheta_{nm}(z)=P_{nm}(z)f_{2nm}+\big[\bar K_{1/2-\gamma}(\alpha_{nm}z)-P_{nm}(z)\bar K_{1/2-\gamma}(\alpha_{nm}c)\big]f_{1nm}, \tag{5.7}$$

with $P_{nm}(z)=(z/c)^{1/2-\gamma}I_{1/2-\gamma}(\alpha_{nm}z)/I_{1/2-\gamma}(\alpha_{nm}c)$.

It remains to check that the homogeneous problem has only the trivial solution. If $f_1\equiv f_2\equiv 0$, then $f_{lnm}=0$ for all n,m, and (5.7) together with (5.4) gives

$$\int_0^1\int_0^{\pi/2}V(r,\varphi,z)\,rJ_{\omega_n}(\alpha_{nm}r)\sin(\omega_n\varphi+\varphi_{0n})\,d\varphi\,dr=0,\qquad z\in[0,c].$$

The completeness of (4.17) in $L_2(0,1)$ with weight r and the continuity of V in $\bar{\bar D}_0$ imply $\int_0^{\pi/2}V(r,\varphi,z)\sin(\omega_n\varphi+\varphi_{0n})\,d\varphi=0$ for $r\in[0,1]$, $z\in[0,c]$, and

the completeness of (4.15) in $L_2(0,\pi/2)$ then yields $U\equiv 0$ in $\bar D_0$. In particular, $U(x,+0,z)\equiv 0$ and $U_y(x,+0,z)\equiv 0$ for $x\in(0,1)$, $z\in(0,c)$, and by (2.2)

$$U(x,-0,z)\equiv 0,\ x\in[0,1],\ z\in[0,c],\qquad U_y(x,-0,z)\equiv 0,\ x\in(0,1),\ z\in(0,c). \tag{5.8}$$

According to [23], the solution of (2.1) in D_1 satisfying (5.8) and the homogeneous conditions (2.7) vanishes identically in $\bar D_1$. The substitutions $(x,y,z)\mapsto(-y,-x,z)$, (y,x,z), $(-x,-y,z)$ reduce the problems in D_2, D_3, D_4 to the same Cauchy-type problem with zero data, so $U\equiv 0$ in these prisms as well. Thus $U\equiv 0$ in $\bar D$. □

6. Existence

We seek the solution of Problem F as the Fourier–Bessel series

$$U(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{nm}^{(j)}(x,y)\,\vartheta_{nm}(z),\qquad (x,y,z)\in\bar D_j,\ j=\overline{0,4}, \tag{6.1}$$

where $u_{nm}^{(j)}$, $j=\overline{0,4}$, are given by (4.18), (4.21), (4.24), (4.23), (4.25), and ϑ_{nm} by (5.7).

Theorem 6.1. *Let $\gamma\in(-\infty,1/2)$ and let $f_l(x,y)=\tilde f_l(r,\varphi)$, $l=1,2$, satisfy:*

(i) $\tilde f_l\in C^{4,7}_{r,\varphi}(\bar\Pi)$, *where* $\Pi=\{(r,\varphi):0<r<1,\ 0<\varphi<\pi/2\}$;

(ii) $\partial_\varphi^{2j}\tilde f_l(r,\pi/2)=0$, $j=\overline{0,3}$, *and* $\partial_\varphi^{k}\tilde f_l(r,0)=0$, $k=\overline{0,6}$;

(iii) $\partial_r^{j}\tilde f_l(1,\varphi)=0$, $j=\overline{0,2}$, *and* $\partial_r^{k}\tilde f_l(0,\varphi)=0$, $k=\overline{0,3}$.

Then Problem F has a solution, and it is given by (6.1).

Proof. It suffices to show that (6.1) converges uniformly in $\bar D$ and that the series obtained by term-by-term application of $r\,\partial_r(r\,\partial_r)$, ∂_φ^2, $B^z_{\gamma-1/2}$, ∂_x^2 and ∂_y^2 converge uniformly on every compact set $K\subset D_j$, $j=\overline{0,4}$.

From the asymptotic behaviour of Bessel functions [19] we have

$$\begin{aligned}|R_{nm}(r)|&\le\begin{cases}c_9\alpha_{nm}^{1/2}, & \alpha_{nm}r<1,\\ c_{10}, & \alpha_{nm}r>1,\end{cases}\\ \bigl|r(rR'_{nm}(r))'\bigr|&\le\begin{cases}c_9\omega_n^2\alpha_{nm}^{1/2}, & \alpha_{nm}r<1,\\ c_{10}|\omega_n^2-\alpha_{nm}^2r^2|, & \alpha_{nm}r>1,\end{cases}\end{aligned} \tag{6.2}$$

and, directly from (4.15), $|\Phi_n|\le 2/\sqrt{\pi}$, $|\Phi_n'|\le 2\omega_n/\sqrt{\pi}$, $|\Phi_n''|\le 2\omega_n^2/\sqrt{\pi}$. Under conditions (i)–(iii), repeated integration by parts in (5.6), carried out as in [20], leads to the bounds

$$\begin{aligned}|\vartheta_{nm}(z)|&\le\begin{cases}c_{11}\,\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-7/2-\varepsilon_4}, & \alpha_{nm}r<1,\\ c_{12}\,\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-4-\varepsilon_4}, & \alpha_{nm}r>1,\end{cases}\\ \bigl|B^z_{\gamma-1/2}\vartheta_{nm}(z)\bigr|&\le\begin{cases}c_{11}\,\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-3/2-\varepsilon_4}, & \alpha_{nm}r<1,\\ c_{12}\,\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-2-\varepsilon_4}, & \alpha_{nm}r>1,\end{cases}\end{aligned} \tag{6.3}$$

where $c_9,\dots,c_{12}$ are positive constants and $\varepsilon_3,\varepsilon_4>0$ are small.

Convergence in $\bar{D}_0$. Since $\omega_n \approx 2n$ and $\alpha_{nm} \approx \pi(m+n) > \pi m$, estimates (6.2)–(6.3) show that the series (6.1) is majorized in $\bar{\bar{D}}_0$ by $c_{13}\sum_n n^{-3-\varepsilon_3}\sum_m m^{-3-\varepsilon_4}$, while the series for $r(rV_r)_r$, $V_{\varphi\varphi}$ and $B^z_{\gamma-1/2}V$ are majorized by

$$c_{14}\sum_{n=1}^{\infty} n^{-1-\varepsilon_3}\sum_{m=1}^{\infty} m^{-3-\varepsilon_4}, \qquad c_{15}\sum_{n=1}^{\infty} n^{-1-\varepsilon_3}\sum_{m=1}^{\infty} m^{-3-\varepsilon_4},$$

$$c_{16}\sum_{n=1}^{\infty} n^{-3-\varepsilon_3}\sum_{m=1}^{\infty} m^{-1-\varepsilon_4},$$

respectively. All these numerical series converge; hence (6.1) converges absolutely and uniformly in $\bar{\bar{D}}_0$, and the differentiated series converge uniformly on compact subsets of $\tilde{D}_0$.

Convergence in $\bar{D}_1$. Let $\xi \in (0,1)$ be arbitrary. If $\alpha_{nm}\xi < 1$, the small-argument behaviour $J_{\omega_n}(s) \approx s^{\omega_n}/\bigl(2^{\omega_n}\Gamma(\omega_n+1)\bigr)$ allows us to write

$$u^{(1)}_{nm}(x,y)\vartheta_{nm}(z) = -\frac{d_{nm}\alpha_{nm}^{\omega_n}\vartheta_{nm}(z)}{2^{\omega_n+1}\Gamma(\omega_n+1)\,l_n}\Bigl[(-1)^n(x-y)^{\omega_n} - \bigl[2+(-1)^n\bigr](x+y)^{\omega_n}\Bigr].$$

In $\bar{\Omega}_1$ we have $0 \le x \pm y \le 1$, so the expression in square brackets does not exceed 4 in absolute value. Together with $|d_{nm}| \le c_{17}\alpha_{nm}^{1/2}$ and (6.3), this gives $|u^{(1)}_{nm}\vartheta_{nm}| \le c_{18}\,\omega_n^{-3-\varepsilon_3}\alpha_{nm}^{-3-\varepsilon_4}$. If $\alpha_{nm}\xi > 1$, the large-argument asymptotics $J_{\omega_n}(s) \approx \sqrt{2/(\pi s)}\cos\bigl(s-\omega_n\pi/2-\pi/4\bigr)$ leads in the same way to $|u^{(1)}_{nm}\vartheta_{nm}| \le c_{19}\,n^{-3-\varepsilon_3}m^{-4-\varepsilon_4}$. In both cases the Weierstrass test shows that (6.1) converges uniformly in $\bar{D}_1$. The remaining series in $D_1,\dots,D_4$ are treated in exactly the same manner.

The case $k=1$ is handled analogously, with only the eigenvalues ω_n and the phases φ_{0n} changed. This completes the proof. □

7. Conclusion

We have studied a three-dimensional mixed-type equation with a singular coefficient in a domain consisting of a quarter cylinder and four triangular prisms, subject to nonlocal Frankl conditions on both planes of type change. Reducing the problem to a spectral problem in the elliptic part, we found its eigenvalues and eigenfunctions explicitly, proved uniqueness of the regular solution by means of the completeness of the eigenfunction system, and established existence by justifying the uniform convergence of the corresponding Fourier–Bessel series.

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References

[1] Bitsadze, A. V. (1956). On a certain Frankl problem. *Doklady Akademii Nauk SSSR*, 109(6), 1091–1094.

[2] Bitsadze, A. V. (1957). On the uniqueness of the solution of the Frankl problem for the Chaplygin equation. *Doklady Akademii Nauk SSSR*, 112(3), 375–376.
[3] Devingtal', Yu. V. (1958). On the existence of a solution to a problem of F. I. Frankl. *Doklady Akademii Nauk SSSR*, 119(1), 15–18.
[4] Devingtal', Yu. V. (1959). On the question of the existence and uniqueness of the solution to the Frankl problem. *Uspekhi Matematicheskikh Nauk*, 14(1(85)), 177–182.
[5] Kal'menov, T. Sh., & Sadybekov, M. A. (2017). On a Frankl-type problem for a mixed parabolic-hyperbolic equation. *Siberian Mathematical Journal*, 58(2), 227–231. https://doi.org/10.1134/S0037446617020057
[6] Karimov, E. T. (2020). Frankl-type problem for a mixed type equation with the Caputo fractional derivative. *Lobachevskii Journal of Mathematics*, 41(9), 1829–1836. https://doi.org/10.1134/S1995080220090152
[7] Karimov, K. T. (2026). Frankl's problem for a three-dimensional equation of mixed type with a singular coefficient. *Journal of Contemporary Applied Mathematics*, 16(1). https://doi.org/10.62476/jcam.161.1
[8] Kipriyanov, I. A. (1997). *Singular Elliptic Boundary Value Problems*. Moscow: Nauka.
[9] Moiseev, E. I. (1992). On the solution of the Frankl problem in a special domain. *Differential Equations*, 28(4), 721–723.
[10] Moiseev, E. I., & Abbasi, N. (2008). On the basis property of eigenfunctions of the Frankl problem with a nonlocal parity condition. *Differential Equations*, 44(6), 855–860. https://doi.org/10.1134/S0012266108060098
[11] Moiseev, E. I., & Abbasi, N. (2008). On the completeness of eigenfunctions of the Frankl problem with the oddness condition. *Differential Equations*, 44(4), 408–412. https://doi.org/10.1134/S0012266108030117
[12] Moiseev, E. I., & Abbasi, N. (2009). Basis property of the eigenfunctions of a generalized gasdynamic Frankl problem with a nonlocal evenness condition and with a discontinuity in the solution gradient. *Differential Equations*, 45(10), 1485–1490. https://doi.org/10.1134/S0012266109100115
[13] Moiseev, E. I., & Ambartsumyan, V. E. (2009). On the basis property of eigenfunctions of the Frankl problem with a nonlocal parity condition of the second kind. *Differential Equations*, 45(12), 1769–1774. https://doi.org/10.1134/S0012266109120076
[14] Nakhushev, A. M. (1969). Certain boundary value problems for hyperbolic equations and equations of mixed type. *Differentsial'nye Uravneniya*, 5(1), 44–59. (In Russian.)
[15] Pskhu, A. V. (2003). The Frankl problem for a hyperbolic-parabolic equation. *Differential Equations*, 39(1), 105–112. https://doi.org/10.1023/A:1025180210792
[16] Sabitov, K. B. (2017). On the theory of the Frankl problem for equations of mixed type. *Izvestiya: Mathematics*, 81(1), 99–136. https://doi.org/10.4213/im8401
[17] Sabitov, K. B., & Tikhomirov, V. V. (1990). On the construction of eigenvalues and eigenfunctions for a certain gas-dynamic Frankl problem. *Matematicheskoe Modelirovanie*, 2(10), 100–109.
[18] Salakhitdinov, M. S., & Urinov, A. K. (2010). *On the Spectral Theory of Mixed-Type Equations*. Tashkent: Mumtoz So'z.
[19] Urinov, A. K., & Karimov, K. T. (2020). The Dirichlet problem for an elliptic equation with singular coefficients in a semi-cylindrical domain. *Lobachevskii Journal of Mathematics*, 41(9), 1891–1902. https://doi.org/10.1134/S1995080220090292
[20] Urinov, A. K., & Karimov, K. T. (2023). The third boundary problem for a mixed-type equation with three singular coefficients. *Lobachevskii Journal of Mathematics*, 44(8), 3582–3593. https://doi.org/10.1134/S1995080223080541
[21] Urinov, A. K., & Nishonova, S. T. (2017). A problem with integral conditions for an elliptic-parabolic equation. *Mathematical Notes*, 102(1–2), 60–69. https://doi.org/10.1134/S0001434617070082
[22] Watson, G. N. (1944). *A Treatise on the Theory of Bessel Functions* (2nd ed.). Cambridge: Cambridge University Press.

[23] Young, E. C. (1969). On a generalized Euler–Poisson–Darboux equation. *Journal of Mathematics and Mechanics*, 18(12), 1167–1175.

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