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ON SOME CONGRUENCES FOR (j, k)-REGULAR OVERPARTITIONS

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Abstract. Let $\overline{p}_{j,k}(n)$ denote the number of (j, k)-regular overpartitions of n in which none of the parts congruent to $j \pmod{k}$. In this paper, we obtain many infinite families of congruences modulo powers of 2 for $\overline{p}_{3,6}(n)$, $\overline{p}_{5,10}(n)$ and $\overline{p}_{9,18}(n)$. For example, for all $n \geq 0$ and $\alpha, \beta \geq 0$,

$$\overline{p}_{9,18}\left(3^{4\alpha+1} \cdot 5^{2\beta+1}(24(5n+i)+23)\right) \equiv 0 \pmod{64},$$

where $i = 0, 1, 2, 4$.

1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

An overpartitions introduced by Corteel and Lovejoy [2]. Overpartition of a non-negative integer n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. For example, the 8 overpartitions of 3 are

$$3, \ \overline{3}, \ 2+1, \ \overline{2}+1, \ 2+\overline{1}, \ \overline{2}+\overline{1}, \ 1+1+1, \ \overline{1}+1+1.$$

Let $\overline{p}(n)$ denote the number of overpartitions of n with $\overline{p}(0) = 1$. The generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}\,(n)\, q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{1.1}$$

An overpartition of a positive integer n is an (j, k)-regular overpartition if none of the parts congruent to $j \pmod{k}$. Let $\overline{p}_{j,k}(n)$ denote the number of such partitions of n with $\overline{p}_{j,k}(0) = 1$. The generating function for $\overline{p}_{j,k}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}_{j,k}\,(n)\, q^n = \frac{(-q;q)_\infty (q^j;q^k)_\infty}{(q;q)_\infty (-q^j;q^k)_\infty}. \tag{1.2}$$

For example, the $(3, 6)$-regular overpartitions of 3 are

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$$2+1, \quad \overline{2}+1, \quad 2+\overline{1}, \quad \overline{2}+\overline{1}, \quad 1+1+1, \quad \overline{1}+1+1.$$

In this paper, we considered the case when $j \neq k$ and established many infinite families of congruences modulo powers of 2 for $\overline{p}_{3,6}(n)$, $\overline{p}_{5,10}(n)$ and $\overline{p}_{9,18}(n)$. For example, for all $n \geq 0$ and $\alpha, \beta \geq 0$,

$$\overline{p}_{9,18}\left(3^{4\alpha+1} \cdot 5^{2\beta+1}(24(5n+i)+23)\right) \equiv 0 \pmod{64},$$

where $i = 0, 1, 2, 4$.

2. Preliminaries

In this section, we record several identities which are useful in proving our main results.

Lemma 2.1. *The following 2-dissections hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.1}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2.2}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}} \tag{2.3}$$

and

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}. \tag{2.4}$$

The identity (2.1) is the 2-dissection of $\phi(q)$ [3, (1.9.4)]. The identity (2.3) is the 2-dissection of $\phi(q)^2$ [3, (1.10.1)]. The equations (2.2) and (2.4) can be obtained from the equations (2.1) and (2.3) by replacing q by $-q$ respectively. Also, one can see [1, p.40].

Lemma 2.2. *The following 2-dissections hold:*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \tag{2.5}$$

and

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}. \tag{2.6}$$

The equation (2.5) was proved by Hirschhorn and Sellers [4]; see also [7]. The identity (2.6) was obtained by Xia and Yao [8].

Lemma 2.3. *The following 2-dissections hold:*

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}, \tag{2.7}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{2.8}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{2.9}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}. \tag{2.10}$$

The identities (2.7) and (2.8) are essentially (30.10.1) and (30.10.4) respectively in [3]. The equation (2.9) is the same as (22.1.14) in [3] (after using 22.1.6 and 22.1.7). The equation (2.10) can be obtained from the equation (22.1.13) in [3] (after using 22.1.6 and 22.1.7) by replacing q by $-q$.

Lemma 2.4. *The following 3-dissections hold:*

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \tag{2.11}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}. \tag{2.12}$$

Lemma 2.4 was proved by Hirschhorn and Sellers [5].

Lemma 2.5. *The following 3-dissections hold:*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3\frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}, \tag{2.13}$$

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{2.14}$$

The equation (2.13) is the same as (14.8.5) in [3]. See also [1, p.345]. The equation (2.14) was obtained by Hirschhorn and Sellers [6].

Lemma 2.6. *We require the 5-dissection due to Ramanujan,*

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2 R(q^5)), \tag{2.15}$$

where

$$R(q) = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}.$$

The identity (2.15) is the same as (8.1.1) in [3].

Lemma 2.7. *We require the 7-dissection due to Ramanujan,*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.16}$$

where $A(q) = f(-q^3,-q^4), B(q) = f(-q^2,-q^5)$ *and* $C(q) = f(-q,-q^6)$.

Lemma 2.7 is an exercise in [3], see [3, (10.5.1)]. Also, one can see [1, p.303, Entry 17(v)].

3. Congruences for $\overline{p}_{3,6}(n)$

Theorem 3.1. *For all $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(24 \cdot 7^{2\alpha} n + 17 \cdot 7^{2\alpha}\right) q^n \equiv 32 f_1 f_{16} \pmod{64}, \tag{3.1}$$

$$\overline{p}_{3,6}\left(24 \cdot 7^{2\alpha+1}(7n+i) + 23 \cdot 7^{2\alpha+1}\right) \equiv 0 \pmod{64}, \tag{3.2}$$

where $i = 0, 1, 2, 3, 5, 6$.

Proof. Setting $j = 3$ and $k = 6$ in (1.2), we see that

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}(n) q^n = \frac{f_2 f_3^2 f_{12}}{f_1^2 f_6^3}. \tag{3.3}$$

Employing (2.11) in (3.3) and then collecting the coefficients of q^{3n}, q^{3n+1} and q^{3n+2} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}(3n) q^n = \frac{f_2 f_3^6 f_4}{f_1^6 f_6^3}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}(3n+1) q^n = 2\frac{f_3^3 f_4}{f_1^5} \tag{3.5}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}(3n+2) q^n = 4\frac{f_4 f_6^3}{f_1^4 f_2}. \tag{3.6}$$

From the binomial theorem, it is easy to prove that for any positive integers k and m,

$$f_k^{2m} \equiv f_{2k}^m \pmod{2}, \tag{3.7}$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod{4}, \tag{3.8}$$

$$f_k^{8m} \equiv f_{2k}^{4m} \pmod{8}, \tag{3.9}$$

$$f_k^{16m} \equiv f_{2k}^{8m} \pmod{16}. \tag{3.10}$$

Substituting (2.3) in (3.6) and invoking (3.8) and (3.9), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}(6n+2) q^n \equiv 4\frac{f_1 f_3^3}{f_2} \pmod{32} \tag{3.11}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}(6n+5) q^n \equiv 16\frac{f_1 f_2^5 f_6^2}{f_3} \pmod{64}. \tag{3.12}$$

Using (2.7) in (3.12), we get

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}(12n+5) q^n \equiv 16\frac{f_4 f_8 f_{12}^2}{f_1^2 f_{24}} \pmod{64} \tag{3.13}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(12n+11\right)q^n \equiv 48\frac{f_1^6 f_4^2 f_6 f_{24}}{f_2 f_8 f_{12}} \pmod{64}. \tag{3.14}$$

Using (2.1) in (3.13), we get

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(24n+5\right)q^n \equiv 16\frac{f_2^3 f_6^2}{f_1 f_{12}} \pmod{64} \tag{3.15}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(24n+17\right)q^n \equiv 32 f_1 f_{16} \pmod{64}, \tag{3.16}$$

which is $\alpha = 0$ case of (3.1). Suppose that the congruence (3.1) is true for $\alpha \geq 0$. Using (2.16) in (3.1), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(24\cdot 7^{2\alpha+1}n+23\cdot 7^{2\alpha+1}\right)q^n \equiv 32q^4 f_7 f_{112} \pmod{64}, \tag{3.17}$$

which implies (3.2) and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(24\cdot 7^{2\alpha+2}n+17\cdot 7^{2\alpha+2}\right)q^n \equiv 32 f_1 f_{16} \pmod{64}, \tag{3.18}$$

which proves that the congruence (3.1) is true for $\alpha + 1$. By induction, the congruence (3.1) holds for all integer $\alpha \geq 0$. □

Theorem 3.2. *For any $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+2}n+2^{2\alpha+3}\right)q^n \equiv 4\frac{f_4^5 f_6^3}{f_2^3 f_8^2} + 24\frac{f_2^3 f_3^2 f_4^3}{f_1^2 f_6} \pmod{32}, \tag{3.19}$$

$$\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}n+17\cdot 2^{2\alpha+2}\right) \equiv 0 \pmod{32}, \tag{3.20}$$

$$\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}n+23\cdot 2^{2\alpha+2}\right) \equiv 0 \pmod{32}, \tag{3.21}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}\cdot 7^{2\gamma}n+5\cdot 2^{2\alpha+2}\cdot 7^{2\gamma}\right)q^n \equiv 16 f_1^5 \pmod{32}, \tag{3.22}$$

$$\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}\cdot 7^{2\gamma+1}(7n+i)+11\cdot 2^{2\alpha+2}\cdot 7^{2\gamma+1}\right) \equiv 0 \pmod{32}, \tag{3.23}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}\cdot 5^{2\beta}n+11\cdot 2^{2\alpha+2}\cdot 5^{2\beta}\right)q^n \equiv 16 f_2 f_3^3 \pmod{32}, \tag{3.24}$$

$$\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}\cdot 5^{2\beta+1}(5n+j)+7\cdot 2^{2\alpha+2}\cdot 5^{2\beta+1}\right) \equiv 0 \pmod{32}, \tag{3.25}$$

where $i = 0, 2, 3, 4, 5, 6$ and $j = 0, 1, 3, 4$.

Proof. Substituting (2.2) and (2.9) in (3.11), we get

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(12n+2\right)q^n \equiv 4\frac{f_2 f_3^2 f_4^5}{f_1^2 f_6 f_8^2} + 24q\frac{f_6^3 f_8^2}{f_2 f_4} \pmod{32} \tag{3.26}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(12n+8\right)q^n \equiv 4\frac{f_4^5 f_6^3}{f_2^3 f_8^2} + 24\frac{f_2^3 f_3^2 f_4^3}{f_1^2 f_6} \pmod{32}, \tag{3.27}$$

which is $\alpha = 0$ case of (3.19). Let us consider the congruence (3.19) is true for $\alpha \geq 0$. Employing (2.8) in (3.19), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+3}n + 2^{2\alpha+3}\right)q^n \equiv 4\frac{f_2^5 f_3^3}{f_1^3 f_4^2} + 24\frac{f_2^3 f_4 f_6^2}{f_1^2 f_{12}} \pmod{32} \tag{3.28}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+3}n + 5\cdot 2^{2\alpha+2}\right)q^n \equiv 16\frac{f_3^3 f_4^3}{f_1} \pmod{32}. \tag{3.29}$$

Employing (2.1) and (2.9) in (3.28) and then comparing the coefficients of q^{2n+1} on both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+4}n + 2^{2\alpha+5}\right)q^n \equiv 4\frac{f_4^5 f_6^3}{f_2^3 f_8^2} + 24\frac{f_2^3 f_3^2 f_4^3}{f_1^2 f_6} \pmod{32}, \tag{3.30}$$

which proves that the congruence (3.19) is true for $\alpha + 1$. By induction, the congruence (3.19) holds for all integer $\alpha \geq 0$.

Using (2.9) in (3.29), we get

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+4}n + 5\cdot 2^{2\alpha+2}\right)q^n \equiv 16 f_2^5 \pmod{32} \tag{3.31}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+4}n + 11\cdot 2^{2\alpha+2}\right)q^n \equiv 16 f_4 f_6^3 \pmod{32}. \tag{3.32}$$

Collecting the coefficients of q^{2n+1} from the equations (3.31) and (3.32), we obtain (3.20) and (3.21) respectively.

The equations (3.31) and (3.32) implies

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}n + 5\cdot 2^{2\alpha+2}\right)q^n \equiv 16 f_1^5 \pmod{32} \tag{3.33}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}n + 11\cdot 2^{2\alpha+2}\right)q^n \equiv 16 f_2 f_3^3 \pmod{32}. \tag{3.34}$$

The equation (3.33) is $\gamma = 0$ case of (3.22). Suppose that the congruence (3.22) is true for $\gamma \geq 0$. Utilizing (2.16) in (3.22), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}\cdot 7^{2\gamma+1}n + 11\cdot 2^{2\alpha+2}\cdot 7^{2\gamma+1}\right)q^n \equiv 16 q f_7^5 \pmod{32}, \tag{3.35}$$

which implies (3.23) and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}\cdot 7^{2\gamma+2}n + 5\cdot 2^{2\alpha+2}\cdot 7^{2\gamma+2}\right)q^n \equiv 16 f_1^5 \pmod{32}, \tag{3.36}$$

which shows that the congruence (3.22) is true for $\gamma+1$. By induction, the congruence (3.22) holds for all integers $\alpha, \gamma \geq 0$.

The equation (3.34) is $\beta=0$ case of (3.24). Suppose that the congruence (3.24) is true for $\beta \geq 0$. Using (2.15) in (3.24), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}\cdot 5^{2\beta+1}n + 7\cdot 2^{2\alpha+2}\cdot 5^{2\beta+1}\right) q^n \equiv 16q^2 f_{10} f_{15}^3 \pmod{32}, \tag{3.37}$$

which implies (3.25) and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}\cdot 5^{2\beta+2}n + 11\cdot 2^{2\alpha+2}\cdot 5^{2\beta+2}\right) q^n \equiv 16 f_2 f_3^3 \pmod{32}, \tag{3.38}$$

which proves that the congruence (3.24) is true for $\beta+1$. By induction, the congruence (3.24) holds for all integers $\alpha, \beta \geq 0$. ☐

Theorem 3.3. *For any $n \geq 0$ and $\alpha, \beta \geq 0$, we have*

$$\overline{p}_{3,6}(24n+23) \equiv 0 \pmod{32}, \tag{3.39}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(24\cdot 7^{2\alpha}n + 5\cdot 7^{2\alpha}\right) q^n \equiv 16 f_1^5 \pmod{32}, \tag{3.40}$$

$$\overline{p}_{3,6}\left(24\cdot 7^{2\alpha+1}(7n+i) + 11\cdot 7^{2\alpha+1}\right) \equiv 0 \pmod{32}, \tag{3.41}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(24\cdot 5^{2\beta}n + 11\cdot 5^{2\beta}\right) q^n \equiv 16 f_2 f_3^3 \pmod{32}, \tag{3.42}$$

$$\overline{p}_{3,6}\left(24\cdot 5^{2\beta+1}(5n+j) + 7\cdot 5^{2\beta+1}\right) \equiv 0 \pmod{32}, \tag{3.43}$$

where $i = 0,2,3,4,5,6$ and $j = 0,1,3,4$.

Proof. The equation (3.14) becomes

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}(12n+11) q^n \equiv 16 f_4 f_6^3 \pmod{32}, \tag{3.44}$$

which implies (3.39) and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}(24n+11) q^n \equiv 16 f_2 f_3^3 \pmod{32}. \tag{3.45}$$

The equation (3.15) becomes

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}(24n+5) q^n \equiv 16 f_1^5 \pmod{32}. \tag{3.46}$$

The rest of the proof is similar to the proofs of the identities (3.22)-(3.25). So, we omit the details. ☐

Theorem 3.4. *For any $n \geq 0$ and $\beta \geq 0$, we have*

$$\overline{p}_{3,6}\left(48n+14\right) \equiv 0 \pmod{32}, \tag{3.47}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(48 \cdot 5^{2\beta} n + 38 \cdot 5^{2\beta}\right) q^n \equiv 16 f_1 f_6^3 \pmod{32}, \tag{3.48}$$

$$\overline{p}_{3,6}\left(48 \cdot 5^{2\beta+1}(5n+i) + 46 \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{32}, \tag{3.49}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(48 \cdot 5^{2\beta} n + 26 \cdot 5^{2\beta}\right) q^n \equiv 16 f_1^{13} \pmod{32}, \tag{3.50}$$

$$\overline{p}_{3,6}\left(48 \cdot 5^{2\beta+1}(5n+j) + 34 \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{32}, \tag{3.51}$$

where $i = 0, 1, 2, 4$ and $j = 0, 1, 3, 4$.

Proof. Employing (2.8) in (3.26), we have

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(24n+2\right) q^n \equiv 4 \frac{f_2 f_4 f_6^2}{f_1^4 f_{12}} \pmod{32} \tag{3.52}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(24n+14\right) q^n \equiv 8 \frac{f_2^2 f_3 f_4 f_{12}}{f_1^3 f_6} + 24 \frac{f_3^3 f_4^2}{f_1 f_2} \pmod{32}. \tag{3.53}$$

Using (2.9) and (2.10) in (3.53), we obtain (3.47) and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(48n+38\right) q^n \equiv 16 f_1 f_6^3 \pmod{32}, \tag{3.54}$$

which is $\beta = 0$ case of (3.48). The rest of the proofs of the identities (3.48) and (3.49) are similar to the proofs of the identities (3.24) and (3.25). So, we omit the details.

Substituting (2.3) in (3.52), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(48n+2\right) q^n \equiv 4 \frac{f_2^3 f_3^2}{f_1^5 f_6} \pmod{32} \tag{3.55}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(48n+26\right) q^n \equiv 16 f_1^{13} \pmod{32}, \tag{3.56}$$

which is $\beta = 0$ case of (3.50). Suppose that the congruence (3.50) is true for $\beta \geq 0$. Utilizing (2.15) in (3.50), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(48 \cdot 5^{2\beta+1} n + 34 \cdot 5^{2\beta+1}\right) q^n \equiv 16 q^2 f_5^{13} \pmod{32}, \tag{3.57}$$

which implies (3.51) and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(48 \cdot 5^{2\beta+2} n + 26 \cdot 5^{2\beta+2}\right) q^n \equiv 16 f_1^{13} \pmod{32}, \tag{3.58}$$

which shows that the congruence (3.50) is true for $\beta + 1$. By induction, the congruence (3.50) holds for all integer $\beta \geq 0$. ◻

Theorem 3.5. *For all $n\geq 0$ and $\alpha\geq 0$, we have*

$$\sum_{n=0}^{\infty}\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+3}n+2^{2\alpha+4}\right)q^n\equiv 2\frac{f_1^6f_6^3}{f_4^2}+16f_4^4 \pmod{32}, \tag{3.59}$$

$$\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+6}n+23\cdot 2^{2\alpha+3}\right)\equiv 0 \pmod{32}. \tag{3.60}$$

Proof. Using (2.3) and (2.9) in (3.5) and invoking (3.8) and (3.9), we arrive at

$$\sum_{n=0}^{\infty}\overline{p}_{3,6}\left(6n+1\right)q^n\equiv 2\frac{f_2^2f_3^2}{f_6} \pmod{8} \tag{3.61}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{3,6}\left(6n+4\right)q^n\equiv 2\frac{f_1^2f_2^6f_6^3}{f_4^4}+8q\frac{f_3^2f_4^4}{f_6} \pmod{32}. \tag{3.62}$$

Substituting (2.2) in (3.62), we get

$$\sum_{n=0}^{\infty}\overline{p}_{3,6}\left(12n+4\right)q^n\equiv 2\frac{f_1^7f_3^3f_4^5}{f_2^6f_8^2}+8\frac{f_4^2f_{12}}{f_6^2} \pmod{32} \tag{3.63}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{3,6}\left(12n+10\right)q^n\equiv 28\frac{f_3^3f_8^2}{f_1f_4}+16qf_4^2f_{12}^3 \pmod{32}. \tag{3.64}$$

Employing (2.4) and (2.9) in (3.63), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{3,6}\left(24n+4\right)q^n\equiv 2\frac{f_2^2f_3^2}{f_6} \pmod{8} \tag{3.65}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{3,6}\left(24n+16\right)q^n\equiv 2\frac{f_1^6f_6^3}{f_4^2}+16f_4^4 \pmod{32}, \tag{3.66}$$

which is $\alpha=0$ case of (3.59). Let us consider the congruence (3.59) is true for $\alpha\geq 0$. Utilizing (2.2) and (2.4) in (3.59), we arrive at

$$\sum_{n=0}^{\infty}\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+4}n+2^{2\alpha+4}\right)q^n\equiv 2\frac{f_2^6f_3^3f_4}{f_1f_8^2}+16f_2^4+16q\frac{f_3^3f_4^2f_8^2}{f_1} \pmod{32} \tag{3.67}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+4}n+5\cdot 2^{2\alpha+3}\right)q^n\equiv 20\frac{f_3^3f_8^2}{f_1f_4} \pmod{32}. \tag{3.68}$$

Using (2.9) in (3.67), we see that

$$\sum_{n=0}^{\infty}\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}n+2^{2\alpha+6}\right)q^n\equiv 2\frac{f_1^6f_6^3}{f_4^2}+16f_4^4 \pmod{32}, \tag{3.69}$$

which proves that the congruence (3.59) is true for $\alpha+1$. By induction, the congruence (3.59) holds for all integer $\alpha\geq 0$.

Employing (2.9) in (3.68), we get

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}n+5\cdot 2^{2\alpha+3}\right)q^n \equiv 20\frac{f_2^2f_3^2f_4^2}{f_1^2f_6} \pmod{32} \tag{3.70}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}n+11\cdot 2^{2\alpha+3}\right)q^n \equiv 20\frac{f_4^2f_6^3}{f_2^2} \pmod{32}. \tag{3.71}$$

The equation (3.71) implies (3.60). □

Theorem 3.6. *For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have*

$$\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+6}n+17\cdot 2^{2\alpha+3}\right) \equiv 0 \pmod{8}, \tag{3.72}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+6}\cdot 7^{2\gamma}n+5\cdot 2^{2\alpha+3}\cdot 7^{2\gamma}\right)q^n \equiv 4f_1^5 \pmod{8}, \tag{3.73}$$

$$\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+6}\cdot 7^{2\gamma+1}(7n+i)+11\cdot 2^{2\alpha+3}\cdot 7^{2\gamma+1}\right) \equiv 0 \pmod{8}, \tag{3.74}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+6}\cdot 5^{2\beta}n+11\cdot 2^{2\alpha+3}\cdot 5^{2\beta}\right)q^n \equiv 4f_2f_3^3 \pmod{8}, \tag{3.75}$$

$$\overline{p}_{3,6}\left(3\cdot 2^{2\alpha+6}\cdot 5^{2\beta+1}(5n+j)+7\cdot 2^{2\alpha+3}\cdot 5^{2\beta+1}\right) \equiv 0 \pmod{8}, \tag{3.76}$$

$$\overline{p}_{3,6}(48n+2) \equiv \begin{cases} 4 \pmod{8} & \textit{if } n \textit{ is a pentagonal number,} \\ 0 \pmod{8} & \textit{otherwise,} \end{cases} \tag{3.77}$$

$$\overline{p}_{3,6}(24n+4) \equiv \overline{p}_{3,6}(6n+1) \pmod{8}, \tag{3.78}$$

where $i = 0, 2, 3, 4, 5, 6$ and $j = 0, 1, 3, 4$.

Proof. From the equations (3.70) and (3.71), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+5}n+5\cdot 2^{2\alpha+3}\right)q^n \equiv 4f_2^5 \pmod{8} \tag{3.79}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+6}n+11\cdot 2^{2\alpha+3}\right)q^n \equiv 4f_2f_3^3 \pmod{8}. \tag{3.80}$$

The equation (3.79) implies (3.72) and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3\cdot 2^{2\alpha+6}n+5\cdot 2^{2\alpha+3}\right)q^n \equiv 4f_1^5 \pmod{8}. \tag{3.81}$$

The rest of the proofs of the identities (3.73)-(3.76) are similar to the proofs of the identities (3.22)-(3.25). So, we omit the details.

From the equation (3.55), we obtain (3.77).

In view of the congruences (3.61) and (3.65), we obtain (3.78). □

Theorem 3.7. *For any $n \geq 0$ and $\alpha, \beta \geq 0$, we have*

$$\overline{p}_{3,6}\left(48n+46\right) \equiv 0 \pmod{32}, \tag{3.82}$$

$$\overline{p}_{3,6}\left(48n+34\right) \equiv 0 \pmod{8}, \tag{3.83}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(48 \cdot 7^{2\alpha}n + 10 \cdot 7^{2\alpha}\right) q^n \equiv 4f_1^5 \pmod{8}, \tag{3.84}$$

$$\overline{p}_{3,6}\left(48 \cdot 7^{2\alpha+1}(7n+i) + 22 \cdot 7^{2\alpha+1}\right) \equiv 0 \pmod{8}, \tag{3.85}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(48 \cdot 5^{2\beta}n + 22 \cdot 5^{2\beta}\right) q^n \equiv 4f_2f_3^3 \pmod{8}, \tag{3.86}$$

$$\overline{p}_{3,6}\left(48 \cdot 5^{2\beta+1}(5n+j) + 14 \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{8}, \tag{3.87}$$

where $i = 0, 2, 3, 4, 5, 6$ and $j = 0, 1, 3, 4$.

Proof. Using (2.9) in (3.64), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(24n+22\right) q^n \equiv 12\frac{f_4^2 f_6^3}{f_2^2} \pmod{32} \tag{3.88}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(24n+10\right) q^n \equiv 4f_2^5 \pmod{8}. \tag{3.89}$$

Collecting the coefficients of q^{2n+1} from the equations (3.88) and (3.89), we obtain (3.82) and (3.83) respectively.

The equations (3.88) and (3.89) implies

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(48n+22\right) q^n \equiv 4f_2f_3^3 \pmod{8} \tag{3.90}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(48n+10\right) q^n \equiv 4f_1^5 \pmod{8}. \tag{3.91}$$

The rest of the proof is similar to the proofs of the identities (3.22)-(3.25). So, we omit the details. □

Theorem 3.8. *For all $n \geq 0$ and $\alpha, \beta \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(2 \cdot 3^{\alpha+1} \cdot 5^{2\beta}n + 3^{\alpha+1} \cdot 5^{2\beta}\right) q^n \equiv 2f_1^3 f_3^3 \pmod{4}, \tag{3.92}$$

$$\overline{p}_{3,6}\left(2 \cdot 3^{\alpha+2}n + 5 \cdot 3^{\alpha+1}\right) \equiv 0 \pmod{4}, \tag{3.93}$$

$$\overline{p}_{3,6}\left(2 \cdot 3^{\alpha+1} \cdot 5^{2\beta+1}(5n+i) + 3^{\alpha+1} \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{4}, \tag{3.94}$$

$$\overline{p}_{3,6}\left(2 \cdot 3^{\alpha+2}(4n+j) + 3^{\alpha+1}\right) \equiv 0 \pmod{4}, \tag{3.95}$$

$$\overline{p}_{3,6}\left(8 \cdot 3^{\alpha+2}n + 3^{\alpha+1}\right) \equiv \begin{cases} 2 \pmod{4} & \textit{if } n \textit{ is a pentagonal number,} \\ 0 \pmod{4} & \textit{otherwise,} \end{cases} \tag{3.96}$$

where $i = 0, 1, 3, 4$ and $j = 1, 2, 3$.

Proof. Invoking (3.8) in (3.4), we see that

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(3n\right) q^{n} \equiv \frac{f_4 f_3^2}{f_1^2 f_2 f_6} \pmod{4}. \tag{3.97}$$

Utilizing (2.8) in (3.97) and then collecting the coefficients of q^{2n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(6n+3\right) q^{n} \equiv 2 f_1^3 f_3^3 \pmod{4}, \tag{3.98}$$

which is $\alpha = \beta = 0$ case of (3.92). Suppose that the congruence (3.92) is true for $\alpha \geq 0$ and $\beta = 0$. Using (2.13) in (3.92) with $\beta = 0$, we obtain (3.93) and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(2 \cdot 3^{\alpha+2} n + 3^{\alpha+1}\right) q^{n} \equiv 2 f_4 \pmod{4}, \tag{3.99}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(2 \cdot 3^{\alpha+2} n + 3^{\alpha+2}\right) q^{n} \equiv 2 f_1^3 f_3^3 \pmod{4}, \tag{3.100}$$

which proves that the congruence (3.92) is true for $\alpha+1$ with $\beta = 0$. By induction, the congruence (3.92) holds for all integer $\alpha \geq 0$. Suppose that the congruence (3.92) is true for $\alpha, \beta \geq 0$. Using (2.15) in (3.92), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(2 \cdot 3^{\alpha+1} \cdot 5^{2\beta+1} n + 3^{\alpha+1} \cdot 5^{2\beta+1}\right) q^{n} \equiv 2 q^2 f_5^3 f_{15}^3 \pmod{4}, \tag{3.101}$$

which implies (3.94) and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(2 \cdot 3^{\alpha+1} \cdot 5^{2\beta+2} n + 3^{\alpha+1} \cdot 5^{2\beta+2}\right) q^{n} \equiv 2 f_1^3 f_3^3 \pmod{4}, \tag{3.102}$$

which implies that the congruence (3.92) is true for $\beta + 1$. By induction, the congruence (3.92) holds for all integers $\alpha, \beta \geq 0$.

The equation (3.99) implies (3.95) and

$$\sum_{n=0}^{\infty} \overline{p}_{3,6}\left(8 \cdot 3^{\alpha+2} n + 3^{\alpha+1}\right) q^{n} \equiv 2 f_1 \pmod{4}. \tag{3.103}$$

From the equation (3.103), we obtain (3.96). □

4. Congruences for $\overline{p}_{5,10}(n)$

Theorem 4.1. *For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}\right) q^{n} \equiv 4 f_1^9 \pmod{8}, \tag{4.1}$$

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 11 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}\right) q^{n} \equiv 4 f_2 f_3^3 \pmod{8}, \tag{4.2}$$

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}\left(8\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+19\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)q^n\equiv 4f_1f_6^3 \pmod 8, \tag{4.3}$$

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}\left(8\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma+1}\right)q^n\equiv 4q^2f_{10}f_{15}^3 \pmod 8, \tag{4.4}$$

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}\left(8\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+23\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}\right)q^n\equiv 4q^3f_5f_{30}^3 \pmod 8, \tag{4.5}$$

$$\overline{p}_{5,10}\left(8\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}(5n+i)+3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma+1}\right)\equiv 0 \pmod 8, \tag{4.6}$$

$$\overline{p}_{5,10}\left(8\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}(5n+j)+23\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}\right)\equiv 0 \pmod 8, \tag{4.7}$$

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}\left(8\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma+1}\right)q^n\equiv 4qf_5^9 \pmod 8, \tag{4.8}$$

$$\overline{p}_{5,10}\left(8\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}(5n+k)+3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma+1}\right)\equiv 0 \pmod 8, \tag{4.9}$$

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}\left(8\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma+1}\right)q^n\equiv 4q^2f_7^9 \pmod 8, \tag{4.10}$$

$$\overline{p}_{5,10}\left(8\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}(7n+m)+3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma+1}\right)\equiv 0 \pmod 8, \tag{4.11}$$

where $i=0,1,3,4,\ j=0,1,2,4,\ k=0,2,3,4$ *and* $m=0,1,3,4,5,6$.

Proof. Setting $j=5$ and $k=10$ in (1.2), we see that

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}(n)\,q^n=\frac{f_2f_5^2f_{20}}{f_1^2f_{10}^3}. \tag{4.12}$$

Employing (2.5) in (4.12) and then collecting the coefficients of q^{2n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}(2n+1)\,q^n=2\frac{f_2^3f_{10}^2}{f_1^4f_5^2}. \tag{4.13}$$

Invoking (3.8) in (4.13), we see that

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}(2n+1)\,q^n\equiv 2f_2f_5^2 \pmod 8. \tag{4.14}$$

Using (2.2) in (4.14), we get

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}(4n+1)\,q^n\equiv 2\frac{f_1f_5f_{20}}{f_{10}^2} \pmod 8 \tag{4.15}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}(4n+3)\,q^n\equiv 4q^2\frac{f_2f_5f_{20}^3}{f_1} \pmod 8. \tag{4.16}$$

Utilizing (2.5) in (4.16), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8n+3\right)q^n \equiv 4qf_1^3f_{10}^3 \pmod{8} \tag{4.17}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8n+7\right)q^n \equiv 4qf_5^9 \pmod{8}. \tag{4.18}$$

The equation (4.18) implies

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(40n+15\right)q^n \equiv 4f_1^9 \pmod{8}, \tag{4.19}$$

which is $\alpha = \beta = \gamma = 0$ case of (4.1). Let us consider the congruence (4.1) is true for $\alpha \geq 0$ and $\beta = \gamma = 0$. Employing (2.13) in (4.1) with $\beta = \gamma = 0$, we get

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(40\cdot 3^{4\alpha+1}n + 5\cdot 3^{4\alpha+1}\right)q^n \equiv 4f_1^3 + 3qf_3^9 \pmod{8}. \tag{4.20}$$

Substituting (2.13) in (4.20), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(40\cdot 3^{4\alpha+2}n + 5\cdot 3^{4\alpha+3}\right)q^n \equiv 4f_1^9 + 3f_3^3 \pmod{8}. \tag{4.21}$$

Again, using (2.13) in (4.21), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(40\cdot 3^{4\alpha+3}n + 5\cdot 3^{4\alpha+3}\right)q^n \equiv 4qf_3^9 \pmod{8}, \tag{4.22}$$

which implies

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(40\cdot 3^{4\alpha+4}n + 5\cdot 3^{4\alpha+5}\right)q^n \equiv 4f_1^9 \pmod{8}, \tag{4.23}$$

which implies that the congruence (4.1) is true for $\alpha + 1$. By induction, the congruence (4.1) holds for all $\alpha \geq 0$ with $\beta = \gamma = 0$. Suppose that the congruence (4.1) is true for $\alpha, \beta \geq 0$ and $\gamma = 0$. Substituting (2.15) in (4.1) with $\gamma = 0$, we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8\cdot 3^{4\alpha}\cdot 5^{2\beta+2}n + 7\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\right)q^n \equiv 4qf_5^9 \pmod{8}, \tag{4.24}$$

which implies

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8\cdot 3^{4\alpha}\cdot 5^{2\beta+3}n + 3^{4\alpha+1}\cdot 5^{2\beta+3}\right)q^n \equiv 4f_1^9 \pmod{8}, \tag{4.25}$$

which implies that the congruence (4.1) is true for $\beta + 1$. By induction, the congruence (4.1) holds for all $\alpha, \beta \geq 0$ with $\gamma = 0$. Suppose that the congruence

(4.1) is true for all $\alpha, \beta, \gamma \geq 0$. Utilizing (2.16) in (4.1), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} n + 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma+1}\right) q^n \equiv 4q^2 f_7^9 \pmod{8}, \tag{4.26}$$

which implies

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} n + 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2}\right) q^n \equiv 4 f_1^9 \pmod{8}, \tag{4.27}$$

which implies that the congruence (4.1) is true for $\gamma + 1$. By induction, the congruence (4.1) holds for all integers $\alpha, \beta, \gamma \geq 0$.

Employing (2.13) in (4.1) and then collecting the coefficients of q^{3n+1} and q^{3n+2}, we get (4.2) and (4.3) respectively.

From the equations (4.2) and (4.3) along with (2.15), we arrive at (4.4) and (4.5) respectively.

From the congruences (4.4) and (4.5), we obtain (4.6) and (4.7) respectively.

Using (2.15) in (4.1) and then comparing the coefficients of q^{5n+4} on both sides, we get (4.8).

The congruence (4.8) implies (4.9).

Using (2.16) in (4.1) and then extracting the coefficients of q^{7n+4} from both sides, we get (4.10).

From the equation (4.10), we obtain (4.11). □

Theorem 4.2. *For all $n \geq 0$ and $\beta \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 5^{2\beta} n + 3 \cdot 5^{2\beta}\right) q^n \equiv 4q f_1^3 f_{10}^3 \pmod{8}, \tag{4.28}$$

$$\overline{p}_{5,10}\left(8 \cdot 5^{2\beta}(5n+i) + 3 \cdot 5^{2\beta}\right) \equiv 0 \pmod{8}, \tag{4.29}$$

$$\overline{p}_{5,10}\left(8 \cdot 5^{2\beta+1}(5n+j) + 7 \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{8}, \tag{4.30}$$

where $i = 0, 3$ and $i = 3, 4$.

Proof. The equation (4.17) is $\beta = 0$ case of (4.28). Suppose that the congruence (4.28) is true for $\beta \geq 0$. Employing (2.15) in (4.28), we obtain (4.29) and

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 5^{2\beta+1} n + 7 \cdot 5^{2\beta+1}\right) q^n \equiv 4 f_2^3 f_5^3 \pmod{8}. \tag{4.31}$$

Using (2.15) in (4.31), we get (4.30) and

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 5^{2\beta+2} n + 3 \cdot 5^{2\beta+2}\right) q^n \equiv 4q f_1^3 f_{10}^3 \pmod{8}, \tag{4.32}$$

which shows that the congruence (4.28) is true for $\beta + 1$. By induction, the congruence (4.28) holds for all integer $\beta \geq 0$. □

Theorem 4.3. *For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) q^n \equiv 2f_1^3 \pmod 4, \tag{4.33}$$

$$\begin{aligned} &\overline{p}_{5,10}\left(8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) \\ &\equiv \begin{cases} 2 \pmod 4 & \text{if } n = k(3k+1)/2, \\ 0 \pmod 4 & \text{otherwise}, \end{cases} \end{aligned} \tag{4.34}$$

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) q^n \equiv 2f_3^3 \pmod 4, \tag{4.35}$$

$$\overline{p}_{5,10}\left(8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 17 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) \equiv 0 \pmod 4, \tag{4.36}$$

$$\overline{p}_{5,10}\left(8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma}(3n+i) + 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) \equiv 0 \pmod 4, \tag{4.37}$$

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}\right) q^n \equiv 2f_5^3 \pmod 4, \tag{4.38}$$

$$\overline{p}_{5,10}\left(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}(5n+j) + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}\right) \equiv 0 \pmod 4, \tag{4.39}$$

$$\overline{p}_{5,10}\left(8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}(5n+k) + 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}\right) \equiv 0 \pmod 4, \tag{4.40}$$

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}\right) q^n \equiv 2f_7^3 \pmod 4, \tag{4.41}$$

$$\overline{p}_{5,10}\left(8 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1}(7n+m) + 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}\right) \equiv 0 \pmod 4, \tag{4.42}$$

where $i = 1, 2$, $j = 2, 4$, $k = 1, 2, 3, 4$ and $m = 1, 2, 3, 4, 5, 6$.

Proof. The equation (4.15) becomes

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}(4n+1) q^n \equiv 2\frac{f_2 f_5}{f_1} \pmod 4. \tag{4.43}$$

Substituting (2.5) in (4.43), we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}(8n+1) q^n \equiv 2f_1^3 \pmod 4 \tag{4.44}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}(8n+5) q^n \equiv 2f_5^3 \pmod 4. \tag{4.45}$$

The equation (4.44) is $\alpha = \beta = \gamma = 0$ case of (4.33). Let us consider the congruence (4.33) is true for $\alpha \geq 0$ and $\beta, \gamma \geq 0$. Using (2.13) in (4.33) with $\beta, \gamma \geq 0$, we arrive at

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8 \cdot 3^{2\alpha+1} n + 3^{2\alpha+2}\right) q^n \equiv 2f_3^3 \pmod 4, \tag{4.46}$$

which implies

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8\cdot 3^{2\alpha+2}n+3^{2\alpha+2}\right)q^n \equiv 2f_1^3 \pmod 4, \tag{4.47}$$

which proves that the congruence (4.33) is true for $\alpha+1$ with $\beta=\gamma=0$. By induction, the congruence (4.33) holds for all $\alpha\geq 0$. Suppose that the congruence (4.33) is true for $\alpha,\beta\geq 0$ and $\gamma=0$. Utilizing (2.15) in (4.33) with $\gamma=0$, we get

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+3^{2\alpha}\cdot 5^{2\beta+2}\right)q^n \equiv 2f_5^3 \pmod 4, \tag{4.48}$$

which implies

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8\cdot 3^{2\alpha}\cdot 5^{2\beta+2}n+3^{2\alpha}\cdot 5^{2\beta+2}\right)q^n \equiv 2f_1^3 \pmod 4, \tag{4.49}$$

which shows that the congruence (4.33) is true for $\beta+1$ with $\gamma=0$. By induction, the congruence (4.33) holds for all $\alpha,\beta\geq 0$. Suppose that the congruence (4.33) is true for $\alpha,\beta,\gamma\geq 0$. Using (2.16) in (4.33), we have

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv 2f_7^3 \pmod 4, \tag{4.50}$$

which implies

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv 2f_1^3 \pmod 4, \tag{4.51}$$

which implies that the congruence (4.33) is true for $\gamma+1$. By induction, the congruence (4.33) holds for all integers $\alpha,\beta,\gamma\geq 0$.

Using (2.13) in (4.33) and then collecting the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we obtain (4.34), (4.35) and (4.36) respectively.

From the equation (4.35), we get (4.37).

From the equation (4.33) along with (2.15), we arrive at (4.38) and (4.39).

The equation (4.38) implies (4.40).

Employing (2.16) in (4.33) and then collecting the coefficients of q^{7n+6}, we obtain (4.41).

From the equation (4.41), we arrive at (4.42). □

Theorem 4.4. *For all $n\geq 0$ and $\alpha,\beta,\gamma\geq 0$, we have*

$$\overline{p}_{5,10}\left(8(5n+k)+5\right)\equiv 0 \pmod 4, \tag{4.52}$$

$$\sum_{n=0}^{\infty} \overline{p}_{5,10}\left(8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)q^n \equiv 2f_1^3 \pmod 4, \tag{4.53}$$

$$\overline{p}_{5,10}\left(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right) \equiv \begin{cases} 2 \pmod 4 & \text{if } n=k(3k+1)/2, \\ 0 \pmod 4 & \text{otherwise,} \end{cases} \tag{4.54}$$

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}\left(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)q^n\equiv 2f_3^3 \pmod 4, \tag{4.55}$$

$$\overline{p}_{5,10}\left(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+17\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)\equiv 0 \pmod 4, \tag{4.56}$$

$$\overline{p}_{5,10}\left(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}(3n+i)+3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)\equiv 0 \pmod 4, \tag{4.57}$$

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}\left(8\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}\right)q^n\equiv 2f_5^3 \pmod 4, \tag{4.58}$$

$$\overline{p}_{5,10}\left(8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}(5n+j)+3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)\equiv 0 \pmod 4, \tag{4.59}$$

$$\overline{p}_{5,10}\left(8\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}(5n+k)+3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}\right)\equiv 0 \pmod 4, \tag{4.60}$$

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}\left(8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}\right)q^n\equiv 2f_7^3 \pmod 4, \tag{4.61}$$

$$\overline{p}_{5,10}\left(8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}(7n+m)+3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}\right)\equiv 0 \pmod 4, \tag{4.62}$$

where $i=1,2,\ j=2,4,\ k=1,2,3,4$ *and* $m=1,2,3,4,5,6$.

Proof. The equation (4.45) implies (4.52) and

$$\sum_{n=0}^{\infty}\overline{p}_{5,10}(40n+5)q^n\equiv 2f_1^3 \pmod 4. \tag{4.63}$$

The rest of the proof is similar to the proofs of the identities (4.33)-(4.42). So, we omit the details. □

5. Congruences for $\overline{p}_{9,18}(n)$

Theorem 5.1. *For all* $n\geq 0$ *and* $\alpha\geq 0$, *we have*

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha}(6n+3)\right)q^n\equiv 8\frac{f_1f_2f_3^7}{f_6^2}+32qf_3^6f_6^3 \pmod{128}, \tag{5.1}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+4}(6n+5)\right)\equiv\overline{p}_{9,18}\left(3^{4\alpha+2}(6n+5)\right) \pmod{128}, \tag{5.2}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+4}(24n+23)\right)\equiv\overline{p}_{9,18}\left(3^{4\alpha+2}(24n+23)\right)\equiv 0 \pmod{128}. \tag{5.3}$$

Proof. Setting $j=9$ and $k=18$ in (1.2), we see that

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}(n)q^n=\frac{f_2f_9^2f_{36}}{f_1^2f_{18}^3}. \tag{5.4}$$

Employing (2.6) in (5.4) and then collecting the coefficients of q^{2n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}(2n+1)q^n=2\frac{f_2^2f_6^2f_{18}}{f_1^4f_9^2}. \tag{5.5}$$

Using (2.11) in (5.5) and invoking (3.10), we get

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}(6n+1)q^n=2\frac{f_2^{10}f_3^{10}}{f_1^{16}f_6^5}+32q\frac{f_2^7f_3f_6^4}{f_1^{13}}, \tag{5.6}$$

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(6n+3\right)q^n\equiv 8\frac{f_1f_2f_3^7}{f_6^2}+32qf_3^6f_6^3 \pmod{128} \tag{5.7}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(6n+5\right)q^n=24\frac{f_2^8f_3^4f_6}{f_1^{14}}. \tag{5.8}$$

The congruence (5.7) is $\alpha=0$ case of (5.1). Suppose that the congruence (5.1) is true for $\alpha\geq 0$. Using (2.14) in (5.1), we get

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}(6n+1)\right)q^n\equiv 8\frac{f_1^6f_3^4}{f_2f_6^2} \pmod{128}, \tag{5.9}$$

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+2}(2n+1)\right)q^n\equiv 120\frac{f_1^7f_3f_6}{f_2^2}+32f_1^6f_2^3 \pmod{128} \tag{5.10}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}(6n+5)\right)q^n\equiv 112\frac{f_2f_6^4}{f_3^2} \pmod{128}. \tag{5.11}$$

Employing (2.12) and (2.13) in (5.10), we get

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+2}(6n+1)\right)q^n\equiv 120\frac{f_2^2f_3^{10}}{f_6^5}+32\frac{f_2f_4f_6^2}{f_1^2f_{12}} \pmod{128}, \tag{5.12}$$

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+3}(2n+1)\right)q^n\equiv 56\frac{f_1f_2f_3^7}{f_6^2}+32qf_3^6f_6^3+64f_1^3f_3^3 \pmod{128} \tag{5.13}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+2}(6n+5)\right)q^n\equiv 32\frac{f_2^2f_6^3}{f_1^2}+32\frac{f_3^2f_4f_{12}}{f_2} \pmod{128}. \tag{5.14}$$

Utilizing (2.13) and (2.14) in (5.13), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+3}(6n+1)\right)q^n\equiv 56\frac{f_1^6f_3^4}{f_2f_6^2}+64f_4 \pmod{128}, \tag{5.15}$$

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+4}(2n+1)\right)q^n\equiv 72\frac{f_1^7f_3f_6}{f_2^2}+32f_1^6f_2^3+64f_1^3f_3^3 \pmod{128} \tag{5.16}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+3}(6n+5)\right)q^n\equiv 16\frac{f_2f_6^4}{f_3^2} \pmod{128}. \tag{5.17}$$

Employing (2.12) and (2.13) in (5.16), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+4}(6n+1)\right)q^n\equiv 72\frac{f_2^2f_3^{10}}{f_6^5}+32\frac{f_2f_4f_6^2}{f_1^2f_{12}}+64f_4 \pmod{128}, \tag{5.18}$$

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+4}(6n+3)\right)q^n\equiv 8\frac{f_1f_2f_3^7}{f_6^2}+32qf_3^6f_6^3 \pmod{128} \tag{5.19}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+4}(6n+5)\right)q^n\equiv 32\frac{f_2^2f_6^3}{f_1^2}+32\frac{f_3^2f_4f_{12}}{f_2}\pmod{128}. \tag{5.20}$$

The congruence (5.19) shows that the congruence (5.1) is true for $\alpha+1$. By induction, the congruence (5.1) is true for all $\alpha\geq 0$.

From (5.14) and (5.20), we obtain (5.2).

Employing (2.1) and (2.2) in (5.20), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+4}(12n+5)\right)q^n\equiv 32\frac{f_3^3f_4}{f_1^3}+32\frac{f_2f_3f_{12}}{f_1f_6}\pmod{128} \tag{5.21}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+4}(12n+11)\right)q^n\equiv 64\frac{f_2^7f_3^3}{f_1}+64\frac{f_2f_3^3f_{12}^3}{f_1}\pmod{128}. \tag{5.22}$$

Employing (2.9) in (5.22), we obtain (5.3) and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+4}(24n+11)\right)q^n\equiv 64f_1^{11}+64q\frac{f_6^6}{f_1}\pmod{128}. \tag{5.23}$$

□

Theorem 5.2. *For all $n\geq 0$ and $\alpha,\beta\geq 0$, we have*

$$\overline{p}_{9,18}\left(3^{4\alpha+4}(12n+5)\right)\equiv\overline{p}_{9,18}\left(3^{4\alpha+2}(12n+5)\right)\equiv 0\pmod{64}, \tag{5.24}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+4}(24n+11)\right)\equiv\overline{p}_{9,18}\left(3^{4\alpha+2}(24n+11)\right)\equiv 0\pmod{64}, \tag{5.25}$$

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}\cdot 5^{2\beta}(24n+19)\right)q^n\equiv 32f_1^{19}+32f_1f_6^3\pmod{64}, \tag{5.26}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+1}\cdot 5^{2\beta+1}(24(5n+i)+23)\right)\equiv 0\pmod{64}, \tag{5.27}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+3}(24n+19)\right)\equiv\overline{p}_{9,18}\left(3^{4\alpha+1}(24n+19)\right)\pmod{64}, \tag{5.28}$$

where $i=0,1,2,4$.

Proof. From (5.21) and (5.23), we obtain (5.24) and (5.25) respectively.

The congruence (5.9) can be written as

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}(6n+1)\right)q^n\equiv 8\frac{f_2^3f_3^4}{f_1^2f_6^2}\pmod{64}. \tag{5.29}$$

Using (2.1) and (2.4) in (5.29), we get

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}(12n+1)\right)q^n\equiv 8\frac{f_4^5f_6^2}{f_1^2f_3^4f_8^2}\pmod{64} \tag{5.30}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}(12n+7)\right)q^n\equiv 16\frac{f_2^2f_4^3}{f_1^2}+32qf_2f_{12}^3\pmod{64}. \tag{5.31}$$

Using (2.1) in (5.31), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}(24n+7)\right)q^n \equiv 16\frac{f_2^3f_4}{f_1^3} \pmod{64} \tag{5.32}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}(24n+19)\right)q^n \equiv 32f_1^{19}+32f_1f_6^3 \pmod{64}, \tag{5.33}$$

which is $\beta=0$ case of (5.26). Suppose that the congruence (5.26) is true for $\beta\geq 0$. Employing (2.15) in (5.26) and then collecting the coefficients of q^{5n+4} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}\cdot 5^{2\beta+1}(24n+23)\right)q^n \equiv 32q^3f_5^{19}+32q^3f_5f_{30}^3 \pmod{64}, \tag{5.34}$$

which implies (5.27) and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}\cdot 5^{2\beta+2}(24n+19)\right)q^n \equiv 32f_1^{19}+32f_1f_6^3 \pmod{64}, \tag{5.35}$$

which implies that the congruence (5.26) is true for $\beta+1$. By induction, the congruence (5.26) holds for all $\beta\geq 0$.

The equation (5.15) becomes

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+3}(6n+1)\right)q^n \equiv 56\frac{f_2^3f_3^4}{f_1^2f_6^2} \pmod{64}. \tag{5.36}$$

Substituting (2.1) and (2.4) in (5.36), we have

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+3}(12n+1)\right)q^n \equiv 56\frac{f_4^5f_6^2}{f_1^2f_3^4f_8^2} \pmod{64} \tag{5.37}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+3}(12n+7)\right)q^n \equiv 48\frac{f_2^2f_4^3}{f_1^2}+32qf_2f_{12}^3 \pmod{64}. \tag{5.38}$$

Employing (2.1) in (5.38), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+3}(24n+7)\right)q^n \equiv 48\frac{f_2^3f_4}{f_1^3} \pmod{64} \tag{5.39}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+3}(24n+19)\right)q^n \equiv 32f_1^{19}+32f_1f_6^3 \pmod{64}. \tag{5.40}$$

From the equations (5.33) and (5.40), we obtain (5.28). □

Theorem 5.3. *For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have*

$$\overline{p}_{9,18}\left(3^{4\alpha+3}(6n+5)\right) \equiv \overline{p}_{9,18}\left(3^{4\alpha+1}(6n+5)\right) \pmod{32}, \tag{5.41}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+1}(12n+11)\right) \equiv 0 \pmod{32}, \tag{5.42}$$

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(3^{4\alpha+1} \cdot 7^{2\gamma}(24n+5)\right) q^n \equiv 16 f_1^5 \pmod{32}, \tag{5.43}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+1} \cdot 7^{2\gamma+1}(24(7n+i)+11)\right) \equiv 0 \pmod{32}, \tag{5.44}$$

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(3^{4\alpha+1} \cdot 5^{2\beta}(24n+7)\right) q^n \equiv 16 f_1^7 \pmod{32}, \tag{5.45}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+1} \cdot 5^{2\beta+1}(24(5n+j)+11)\right) \equiv 0 \pmod{32}, \tag{5.46}$$

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(3^{4\alpha+1} \cdot 5^{2\beta}(24n+13)\right) q^n \equiv 16 f_1^{13} \pmod{32}, \tag{5.47}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+1} \cdot 5^{2\beta+1}(25(5n+k)+17)\right) \equiv 0 \pmod{32}, \tag{5.48}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+3}(24n+7)\right) \equiv \overline{p}_{9,18}\left(3^{4\alpha+1}(24n+7)\right) \pmod{32}, \tag{5.49}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+3}(24n+13)\right) \equiv \overline{p}_{9,18}\left(3^{4\alpha+1}(24n+13)\right) \pmod{32}, \tag{5.50}$$

where $i = 0, 2, 3, 4, 5, 6$, $j = 0, 2, 3, 4$ *and* $k = 0, 1, 3, 4$.

Proof. From the equations (5.11) and (5.17), we obtain (5.41).

From (5.11), we have

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(3^{4\alpha+1}(6n+5)\right) q^n \equiv 16 f_2 f_6^3 \pmod{32}, \tag{5.51}$$

which implies (5.42) and

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(3^{4\alpha+1}(12n+5)\right) q^n \equiv 16 \frac{f_2 f_3^3}{f_1} \pmod{32}. \tag{5.52}$$

Using (2.9) in (5.52), we get

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(3^{4\alpha+1}(24n+5)\right) q^n \equiv 16 f_1^5 \pmod{32}, \tag{5.53}$$

which is $\gamma = 0$ case of (5.43). The rest of the proofs of the identities (5.43) and (5.44) are similar to the proofs of the identities (3.22) and (3.23). So, we omit the details.

The congruence (5.32) reduces to

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(3^{4\alpha+1}(24n+7)\right) q^n \equiv 16 f_1^7 \pmod{32}, \tag{5.54}$$

which is $\beta = 0$ case of (5.45). Suppose that the congruence (5.45) is true for $\beta \geq 0$. Employing (2.15) in (5.45), we get

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(3^{4\alpha+1} \cdot 5^{2\beta+1}(24n+11)\right) q^n \equiv 16 q f_5^7 \pmod{32}, \tag{5.55}$$

which implies (5.46) and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}\cdot 5^{2\beta+2}(24n+7)\right)q^n \equiv 16f_1^7 \pmod{32}, \tag{5.56}$$

which implies that the congruence (5.45) is true for $\beta+1$. By induction, the congruence (5.45) holds for all $\beta \geq 0$.

From the congruence (5.30), we have

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}(12n+1)\right)q^n \equiv 8\frac{f_4}{f_1^2} \pmod{32}. \tag{5.57}$$

Using (2.1) in (5.57), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}(24n+1)\right)q^n \equiv 8\frac{f_4}{f_1f_2} \pmod{32} \tag{5.58}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+1}(24n+13)\right)q^n \equiv 16f_1^{13} \pmod{32}, \tag{5.59}$$

which is $\beta=0$ case of (5.47). The rest of the proofs of the identities (5.47) and (5.48) are similar to the proofs of the identities (5.45) and (5.46). So, we omit the details.

From the equations (5.32) and (5.39), we obtain (5.49).

The equation (5.37) becomes

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+3}(12n+1)\right)q^n \equiv 24\frac{f_4}{f_1^2} \pmod{32}. \tag{5.60}$$

Employing (2.1) in (5.60), we have

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+3}(24n+1)\right)q^n \equiv 24\frac{f_4}{f_1f_2} \pmod{32} \tag{5.61}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_{9,18}\left(3^{4\alpha+3}(24n+13)\right)q^n \equiv 16f_1^{13} \pmod{32}. \tag{5.62}$$

From the equations (5.59) and (5.62), we obtain (5.50). □

Theorem 5.4. *For all $n \geq 0$ and $\alpha, \gamma \geq 0$, we have*

$$\overline{p}_{9,18}\left(3^{4\alpha+4}(6n+1)\right) \equiv \overline{p}_{9,18}\left(3^{4\alpha+2}(6n+1)\right) \pmod{16}, \tag{5.63}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+1}(24n+1)\right) \equiv \begin{cases} 8 \pmod{16} & \text{if } n \text{ is a pentagonal number,} \\ 0 \pmod{16} & \text{otherwise,} \end{cases} \tag{5.64}$$

$$\overline{p}_{9,18}\left(3^{4\alpha+3}(24n+1)\right) \equiv \overline{p}_{9,18}\left(3^{4\alpha+1}(24n+1)\right) \pmod{16}, \tag{5.65}$$

$$\overline{p}_{9,18}(12n+11) \equiv 0 \pmod{16}, \tag{5.66}$$

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(7^{2\gamma}(24n+5)\right) q^n \equiv 8f_1^5 \pmod{16}, \tag{5.67}$$

$$\overline{p}_{9,18}\left(7^{2\gamma+1}(24(7n+i)+11)\right) \equiv 0 \pmod{16}, \tag{5.68}$$

where $i = 0, 2, 3, 4, 5, 6$.

Proof. From the equations (5.12) and (5.18), we get (5.63).

The equation (5.58) becomes

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(3^{4\alpha+1}(24n+1)\right) q^n \equiv 8f_1 \pmod{16}, \tag{5.69}$$

which proves (5.64).

From the equations (5.58) and (5.61), we obtain (5.65).

From (5.8), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}(6n+5) q^n \equiv 8f_2f_6^3 \pmod{16}, \tag{5.70}$$

which implies (5.66) and

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}(12n+5) q^n \equiv 8\frac{f_2f_3^3}{f_1} \pmod{16}. \tag{5.71}$$

Employing (2.9) in (5.71), we get

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}(24n+5) q^n \equiv 8f_1^5 \pmod{16}, \tag{5.72}$$

which is $\gamma = 0$ case of (5.67). The rest of the proofs of the identities (5.67) and (5.68) are similar to the proofs of the identities (3.22) and (3.23). So, we omit the details. □

Theorem 5.5. *For all* $n \geq 0$ *and* $\beta \geq 0$, *we have*

$$\overline{p}_{9,18}(24n+7) \equiv 0 \pmod{8}, \tag{5.73}$$

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(5^{2\beta}(24n+19)\right) q^n \equiv 4f_1f_6^3 \pmod{8}, \tag{5.74}$$

$$\overline{p}_{9,18}\left(5^{2\beta+1}(24(5n+i)+23)\right) \equiv 0 \pmod{8}, \tag{5.75}$$

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\left(5^{2\beta}(24n+13)\right) q^n \equiv 4f_1^{13} \pmod{8}, \tag{5.76}$$

$$\overline{p}_{9,18}\left(5^{2\beta+1}(24(5n+j)+17)\right) \equiv 0 \pmod{8}, \tag{5.77}$$

$$\overline{p}_{9,18}(24n+1) \equiv \begin{cases} 2 \pmod{4} & \text{if } n \text{ is a pentagonal number,} \\ 0 \pmod{4} & \text{otherwise,} \end{cases} \tag{5.78}$$

where $i = 0, 1, 2, 4$ *and* $j = 0, 1, 3, 4$.

Proof. The equation (5.6) becomes

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\,(6n+1)\,q^n \equiv 2\frac{f_2^2 f_3^2}{f_6} \pmod 8. \tag{5.79}$$

Using (2.2) in (5.79), we get

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\,(12n+1)\,q^n \equiv 2\frac{f_1^2 f_{12}}{f_6^2} \pmod 8 \tag{5.80}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\,(12n+7)\,q^n \equiv 4q f_2 f_{12}^3 \pmod 8. \tag{5.81}$$

The equation (5.81) implies (5.73) and

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\,(24n+19)\,q^n \equiv 4 f_1 f_6^3 \pmod 8, \tag{5.82}$$

which is $\beta = 0$ case of (5.74). The rest of the proofs of the identities (5.74)-(5.75) are similar to the proofs of the identities (3.24)-(3.25). So, we omit the details.

Employing (2.2) in (5.80), we have

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\,(24n+1)\,q^n \equiv 2\frac{f_1 f_4 f_6}{f_2^2 f_3^2} \pmod 8 \tag{5.83}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{9,18}\,(24n+13)\,q^n \equiv 4 f_1^{13} \pmod 8. \tag{5.84}$$

The equation (5.84) is $\beta = 0$ case of (5.76). The rest of the proofs of the identities (5.76)-(5.77) are similar to the proofs of the identities (5.45)-(5.46). So, we omit the details.

From (5.83), we obtain (5.78). □

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