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EXPLORING A NONLOCAL IMPLICIT GENERALIZED KATUGAMPOLA FRACTIONAL COUPLED SYSTEM

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Abstract. This paper discusses a coupled system of implicit type with nonlocal initial conditions involving fractional differential equations to establish the desired existence and uniqueness results. The generalized Katugampola fractional derivative is employed to formulate the fractional differential equations, and the Katugampola fractional integral is utilized to describe the initial conditions. The existence result is carried out by means of a fixed point theorem of Krasnosel'skii-Dhage type, and the second result for the uniqueness of the solution is obtained by using the Banach contraction principle. A suitable example is provided at the end to illustrate our main findings.

Keywords. Generalized Katugampola derivative, implicit fractional differential equation, nonlocal initial value problem
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1. Introduction

Fractional calculus and its widespread relevance in various fields of engineering, applied sciences and economics have become a center of attraction in many researchers' studies. The development in fractional calculus helped many mathematicians model various complex phenomena in Aerodynamics, Bio-engineering, Control theory, Material viscoelasticity theory, Physics, etc. See [13, 18, 20, 19] and references therein.

Certainly, due to its applicability, fractional differential equations remain a popular subject among the authors. The most favored approach among researchers to derive results on the existence and uniqueness of solutions of fractional differential equations is the use of fixed point theorems. See [2, 5, 7, 22, 14, 17, 23, 24, 25] and references therein. Banach contraction principle, Schauder, Leray–Schauder nonlinear alternative, Schaefer's, Krasnosel'skii and Krasnosel'skii-Dhage type fixed point theorems are some of the most commonly used fixed point theorems by researchers to establish the existence results.

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The theory of implicit fractional differential equations has also garnered the remarkable attention of numerous academicians. In [8], Benchohra et al. obtained the existence result for the implicit Caputo-type fractional differential equation with periodic boundary conditions by means of the coincidence degree theory. In [4], Bagwan and Pachpatte investigated the implicit fractional differential equation with nonlocal initial conditions to obtain the results of existence and uniqueness.

A coupled system is composed of a single independent variable and a pair of differential equations with dependent variables. As a result of their extensive applications in real-world problems, including ecological models, disease models, anomalous diffusion, and chaotic systems, coupled systems of fractional differential equations are gaining increasing attention in contemporary research nowadays.

In [1], S. Abbas et al. derived the existence results for two types of coupled systems of random fractional differential equations using Itoh's random fixed point theorem. In [26], Zhai and Jiang investigated the coupled system of fractional differential equations and derived their main results on existence and uniqueness. In [6], using Krasnosel'skii fixed point theorem, Bagwan et al. investigated the coupled fractional differential equations with the generalized Katugampola derivative.

In [27], W. Zhang et al. studied the following coupled system of nonlinear implicit fractional differential equations to establish the existence results:

$$\begin{aligned}\mathcal{D}^{\mu}_{0^+}v(s) &= f_1\big(s, s^{1-\nu}w(s), \mathcal{D}^{\nu}_{0^+}w(s)\big),\\ \mathcal{D}^{\nu}_{0^+}w(s) &= f_2\big(s, s^{1-\mu}v(s), \mathcal{D}^{\mu}_{0^+}v(s)\big),\end{aligned}$$

$$\begin{aligned}\lim_{s\to 0^+} s^{1-\mu}v(s) &= v(1),\\ \lim_{s\to 0^+} s^{1-\nu}w(s) &= w(1),\end{aligned}$$

where $s \in [0,1]$, $0 < \mu, \nu \leq 1$, $f_1, f_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$. $\mathcal{D}^{(\cdot)}_{0^+}$ is the standard R–L derivative of fractional order.

Motivated by all above work, in this paper we considered a coupled system of implicit fractional differential equations involving a generalized Katugampola fractional derivative of the form:

$$ {}^{\varrho}D^{\omega_1,\vartheta_1}_{a_1^+}\mathfrak{u}(\mathfrak{t}) = f_1\left(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}), {}^{\varrho}D^{\omega_1,\vartheta_1}_{a_1^+}\mathfrak{u}(\mathfrak{t})\right) \tag{1.1}$$

$$ {}^{\varrho}D^{\omega_2,\vartheta_2}_{a_1^+}\mathfrak{v}(\mathfrak{t}) = f_2\left(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}), {}^{\varrho}D^{\omega_2,\vartheta_2}_{a_1^+}\mathfrak{v}(\mathfrak{t})\right) \tag{1.2}$$

with the following non-local initial conditions

$$ {}^{\varrho}I^{1-\chi_1}_{a_1^+}\mathfrak{u}(a_1) = \sum_{i=1}^{m}\lambda_{1i}\mathfrak{u}(\varpi_i) \tag{1.3}$$

$$ {}^{\varrho}I^{1-\chi_2}_{a_1^+}\mathfrak{v}(a_1) = \sum_{i=1}^{m}\lambda_{2i}\mathfrak{v}(\varpi_i) \tag{1.4}$$

where $f_j : (a_1, a_2] \times \mathfrak{E} \times \mathfrak{E} \times \mathfrak{E} \to \mathfrak{E},\ j = 1, 2$, $(\mathfrak{E}, \|\cdot\|_{\mathfrak{E}})$ is a real Banach space. The operators ${}^{\varrho}D^{\omega_j,\vartheta_j}_{a_1^+},\ j = 1, 2$ are the generalized Katugampola fractional derivatives of order $\omega_j \in (0, 1)$ and type $\vartheta_j \in [0, 1],\ \varrho > 0$ and ${}^{\varrho}I^{1-\chi_j}_{a_1^+}$ are the Katugampola fractional integrals of order $1 - \chi_j,\ j = 1, 2$ with $a_1 > 0$ and $\omega_j \leq \chi_j = \omega_j + \vartheta_j - \omega_j\vartheta_j \leq 1,\ j = 1, 2$ with $\chi_1 \leq \chi_2$. $\varpi_i \in (a_1, a_2]$ are prescribed points such that $a_1 < \varpi_1 \leq \varpi_2 \leq \varpi_3 \leq ... \leq \varpi_m \leq a_2, \lambda_{ji} \geq 0$ for all $i = 1, 2, 3..., m$ and $j = 1, 2$.

The main contribution of the study is to establish the existence and uniqueness results for a coupled implicit system with nonlocal conditions given by Katugampola integrals, posed with the generalized Katugampola derivative (which covers the Riemann–Liouville, Hilfer, Hadamard and Katugampola cases), with existence under a nonlinear D-contraction condition that is weaker than Lipschitz.

This paper is structured such that Section 2 outlines fundamental definitions, significant results, and preliminary notions concerning generalized Katugampola derivatives, which are needed throughout the study. In addition, the equivalent Volterra integral equations of mixed type for the coupled system (1.1)-(1.4) are obtained that are useful to investigate our prime theorems. In Section 3, the existence result is obtained by employing Krasnosel'skii-Dhage type fixed point theorem and the uniqueness result is derived by means of Banach contraction principle. An example verifying the relevance of the established results is discussed in Section 4.

2. Preliminaries

Let $0 < s_1 < s_2 < \infty$ and let $\mathcal{C}[s_1, s_2]$ be the Banach space of all continuous mappings $\Lambda : [s_1, s_2] \to \mathbb{R}$ with associated norm

$$\|\Lambda\|_{\mathcal{C}} = \max\{|\Lambda(z)| : z \in [s_1, s_2]\}.$$

For $0 \leq \chi_j \leq 1,\ \varrho > 0$, let

$$\mathcal{C}_{\chi_j,\varrho}[a_1, a_2] = \left\{\mathfrak{h} : (a_1, a_2] \to \mathbb{R} : \left(\frac{\mathfrak{t}^{\varrho} - a_1^{\varrho}}{\varrho}\right)^{\chi_j} \mathfrak{h}(\mathfrak{t}) \in \mathcal{C}[a_1, a_2]\right\},\ \ j = 1, 2$$

be the weighted space on $(a_1, a_2]$ with the norm

$$\|\mathfrak{h}\|_{\mathcal{C}_{\chi_j,\varrho}} = \left\|\left(\frac{\mathfrak{t}^{\varrho} - a_1^{\varrho}}{\varrho}\right)^{\chi_j} \mathfrak{h}(\mathfrak{t})\right\|_{\mathcal{C}} = \max_{t\in[a_1,a_2]}\left|\left(\frac{\mathfrak{t}^{\varrho} - a_1^{\varrho}}{\varrho}\right)^{\chi_j} \mathfrak{h}(\mathfrak{t})\right|. \tag{2.1}$$

where $\mathfrak{h}$ is a continuous function. Observe that $\mathcal{C}_{0,\varrho}[a_1, a_2] = \mathcal{C}[a_1, a_2]$, for any $\varrho > 0$.

Let $\varrho > 0$ and consider the parameters $\omega_j, \vartheta_j, \alpha_j, \chi_j$ such that $\chi_j = \omega_j + \vartheta_j - \omega_j\vartheta_j$ for $\omega_j, \vartheta_j, \chi_j \in (0, 1)$ and $\alpha_j \in [0, 1),\ \ j = 1, 2$. We define $\mathcal{C}_{1-\chi_j}[a_1, a_2]$, the weighted space of the continuous functions ϕ as

$$\mathcal{C}_{1-\chi_j}[a_1, a_2] = \left\{\phi(\mathfrak{t}) : [a_1, a_2] \to \mathbb{R}|\ (\mathfrak{t} - a_1)^{1-\chi_j}\phi(\mathfrak{t}) \in \mathcal{C}[a_1, a_2]\right\},$$

where $[a_1, a_2]$ is a finite interval.

Obviously, $\mathcal{C}_{1-\chi_j}[a_1, a_2]$ is a weighted Banach space with the norm

$$\|\phi\|_{\mathcal{C}_{1-\chi_j}} = \left\|(\mathfrak{t} - a_1)^{1-\chi_j}\phi(\mathfrak{t})\right\|_{\mathcal{C}}.$$

Moreover, using the norm given in (2.1) we define the following weighted spaces

$$\mathcal{C}^{\omega_j,\vartheta_j}_{1-\chi_j,\alpha_j}[a_1,a_2]=\left\{g\in\mathcal{C}_{1-\chi_j,\varrho}[a_1,a_2],\ {}^{\varrho}D^{\omega_j,\vartheta_j}_{a_1^+}g\in\mathcal{C}_{\alpha_j,\varrho}[a_1,a_2]\right\}$$

and

$$\mathcal{C}^{\chi_j}_{1-\chi_j,\varrho}[a_1,a_2]=\left\{g\in\mathcal{C}_{1-\chi_j,\varrho}[a_1,a_2],\ {}^{\varrho}D^{\chi_j}_{a_1^+}g\in\mathcal{C}_{1-\chi_j,\varrho}[a_1,a_2]\right\}.$$

Further, consider $\mathcal{C}^{\omega_1,\vartheta_1}_{1-\chi_1}\times\mathcal{C}^{\omega_2,\vartheta_2}_{1-\chi_2}$ as a product weighted space with the norm

$$\|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{C}^{\omega_1,\vartheta_1}_{1-\chi_1}\times\mathcal{C}^{\omega_2,\vartheta_2}_{1-\chi_2}}=\|\mathfrak{u}\|_{\mathcal{C}^{\omega_1,\vartheta_1}_{1-\chi_1}}+\|\mathfrak{v}\|_{\mathcal{C}^{\omega_2,\vartheta_2}_{1-\chi_2}}.$$

As in [18], for $\varsigma\in\mathbb{R}$ and $m\in[1,\infty]$ consider the space $Z^m_\varsigma(a_1,a_2)$ of those complex valued Lebesgue measurable functions Θ on $[a_1,a_2]$ for which $\|\Theta\|_{Z^m_\varsigma}<\infty$, where

$$\|\Theta\|_{Z^m_\varsigma}=\left(\int_{a_1}^{a_2}\left|y^{\varsigma}\Theta(y)\right|^m\frac{dy}{y}\right)^{1/m}<\infty,\quad \varsigma\in\mathbb{R},1\le m<\infty$$

and for $m=\infty$

$$\|\Theta\|_{Z^\infty_\varsigma}=ess\sup_{y\in[a_1,a_2]}\left[y^{\varsigma}\left|\Theta(y)\right|\right],\qquad(\varsigma\in\mathbb{R}).$$

The basic definitions of Katugampola fractional integral, Katugampola fractional derivative, generalized Katugampola fractional derivative operators and their related lemmas can be found in [15, 16, 21] and references therein.

Lemma 2.1. [4] *Let $\Lambda:(a_1,a_2]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, be such that $\Lambda\left(\cdot,\mathfrak{F}(\cdot),{}^{\varrho}D^{\omega,\vartheta}_{a_1^+}\mathfrak{F}(\cdot)\right)\in\mathcal{C}_{1-\chi,\varrho}[a_1,a_2]$ for any $\mathfrak{F}\in\mathcal{C}_{1-\chi,\varrho}[a_1,a_2]$ where $\chi=\omega+\vartheta(1-\omega)$ with $\omega\in(0,1)$, $\vartheta\in[0,1]$. A function $\mathfrak{F}\in\mathcal{C}^{\chi}_{1-\chi,\varrho}[a_1,a_2]$ is a solution of generalized Katugampola type implicit fractional differential equation:*

$${}^{\varrho}D^{\omega,\vartheta}_{a_1^+}\mathfrak{F}(\mathfrak{t})=\Lambda\left(\mathfrak{t},\mathfrak{F}(\mathfrak{t}),{}^{\varrho}D^{\omega,\vartheta}_{a_1^+}\mathfrak{F}(\mathfrak{t})\right),$$

with

$${}^{\varrho}I^{1-\chi}_{a_1^+}\mathfrak{F}\left(a_1^+\right)=\mathfrak{F}_0,$$

if and only if $\mathfrak{F}$ satisfies the following Volterra type integral equation:

$$\mathfrak{F}(\mathfrak{t})=\frac{\mathfrak{F}_0}{\Gamma(\chi)}\left(\frac{\mathfrak{t}^{\varrho}-{a_1}^{\varrho}}{\varrho}\right)^{\chi-1}+\frac{1}{\Gamma(\omega)}\int_{a_1}^{\mathfrak{t}}\left(\frac{\mathfrak{t}^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega-1}x^{\varrho-1}$$
$$\cdot\Lambda\left(x,\mathfrak{F}(x),{}^{\varrho}D^{\omega,\vartheta}_{a_1^+}\mathfrak{F}(x)\right)dx.$$

Now, here are a few concepts that we will use in the subsequent results.

Let the category of continuous functions $\varphi:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ and the class of functions $\tau:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$, such that

(1) $z\mapsto\tau(z,y)$ is measurable $\forall\ y\in\mathbb{R}$,
(2) $y\mapsto\tau(z,y)$ is continuous for a.e. $z\in\mathbb{R}$

is represented by $\mathcal{C}(\mathcal{J}\times\mathbb{R},\mathbb{R})$. Further, the mappings of Carathéodory type on $\mathcal{J}\times\mathbb{R}$, which are Lebesgue integrable functions when bounded by a Lebesgue integrable function on $\mathcal{J}$ is referred to as $\mathcal{C}(\mathcal{J}\times\mathbb{R},\mathbb{R})$.

Definition 2.2. ([9, 10, 11]) A function $\Upsilon:\mathbb{R}_+\to\mathbb{R}_+$ which is upper semi-continuous and non-decreasing is said to be a $\mathcal{D}$-function if $\Upsilon(0)=0$. Furthermore, the category of each $\mathcal{D}$-functions on $\mathbb{R}_+$ is denoted by $\mathfrak{D}$.

Definition 2.3. ([9, 10, 11]) For an operator $\Delta:\mathcal{C}_{1-\chi}(\mathcal{J},\mathbb{R})\to\mathcal{C}_{1-\chi}(\mathcal{J},\mathbb{R})$, if there is a $\mathcal{D}$-function $\Upsilon_\Delta\in\mathfrak{D}$, where

$$\|\Delta\phi_1-\Delta\phi_2\|\leq\Upsilon_\Delta(\|\phi_1-\phi_2\|),$$

for all $\phi_1,\phi_2\in\mathcal{C}_{1-\chi}(\mathcal{J},\mathbb{R})$, where $0<\Upsilon_\Delta(\alpha)<\alpha$ for all $\alpha>0$, then the operator Δ is referred to as a nonlinear $\mathcal{D}$-contraction on $\mathcal{C}_{1-\chi}(\mathcal{J},\mathbb{R})$.

Theorem 2.4. ([3, 12]) *Let $\mathfrak{C}\subset\mathcal{C}_{1-\chi}(\mathcal{J},\mathbb{R})$ be closed, convex and bounded. Let the operators $\Phi:\mathcal{C}_{1-\chi}(\mathcal{J},\mathbb{R})\to\mathcal{C}_{1-\chi}(\mathcal{J},\mathbb{R})$ and $\varpi:\mathfrak{C}\to\mathcal{C}_{1-\chi}(\mathcal{J},\mathbb{R})$ that satisfy:*

(1) *Φ is a nonlinear $\mathcal{D}$-contraction,*
(2) *ϖ is completely continuous and*
(3) *$\mathfrak{d}=\Phi\mathfrak{d}+\varpi\mathfrak{w}\Rightarrow\mathfrak{d}\in\mathfrak{C}$ for all $\mathfrak{w}\in\mathfrak{C}$.*

Then, the operator $\Phi+\varpi$ has a fixed point in $\mathfrak{C}$.

To ensure conciseness, let $A_{\mathfrak{u}}:(a_1,a_2]\to\mathbb{R}$ that satisfy

$${}^{\varrho}D^{\omega_1,\vartheta_1}_{a_1^+}\mathfrak{u}(\mathfrak{t})=A_{\mathfrak{u}}(\mathfrak{t})=f_1(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{v}(\mathfrak{t}),A_{\mathfrak{u}}(\mathfrak{t}))$$

and $A_{\mathfrak{v}}:(a_1,a_2]\to\mathbb{R}$ that satisfy

$${}^{\varrho}D^{\omega_2,\vartheta_2}_{a_1^+}\mathfrak{v}(\mathfrak{t})=A_{\mathfrak{v}}(\mathfrak{t})=f_2(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{v}(\mathfrak{t}),A_{\mathfrak{v}}(\mathfrak{t})).$$

Clearly, $A_{\mathfrak{u}}\in C_{1-\chi_1,\varrho}[a_1,a_2]$ and $A_{\mathfrak{v}}\in C_{1-\chi_2,\varrho}[a_1,a_2]$.

Theorem 2.5. *Let $f_1:(a_1,a_2]\times\mathfrak{E}\times\mathfrak{E}\times\mathfrak{E}\to\mathfrak{E}$, that satisfy $f_1(\cdot,\mathfrak{u}(\cdot),\mathfrak{v}(\cdot),{}^{\varrho}D^{\omega_1,\vartheta_1}_{a_1^+}\mathfrak{u}(\cdot))\in\mathcal{C}_{1-\chi_1,\varrho}[a_1,a_2]$ for any $\mathfrak{u},\mathfrak{v}\in\mathcal{C}_{1-\chi_1,\varrho}[a_1,a_2]$ where $\chi_1=\omega_1+\vartheta_1(1-\omega_1)$ with $\omega_1\in(0,1)$, $\vartheta_1\in[0,1]$. A function $\mathfrak{u}\in\mathcal{C}^{\chi_1}_{1-\chi_1,\varrho}[a_1,a_2]$ is a solution of generalized Katugampola type implicit fractional differential equation (1.1) with the initial condition (1.3) precisely when it satisfies the following mixed-type Volterra integral equation*

$$\mathfrak{u}(t)=\frac{\mathfrak{M}_1}{\Gamma(\omega_1)}\left(\frac{t^{\varrho}-{a_1}^{\varrho}}{\varrho}\right)^{\chi_1-1}\sum_{i=1}^{m}\lambda_{1i}\int_{a_1}^{\varpi_i}\left(\frac{{\varpi_i}^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega_1-1}x^{\varrho-1}A_{\mathfrak{u}}(x)\,dx$$
$$+\frac{1}{\Gamma(\omega_1)}\int_{a_1}^{t}\left(\frac{t^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega_1-1}x^{\varrho-1}A_{\mathfrak{u}}(x)\,dx \tag{2.2}$$

where $\mathfrak{M}_1=\left\{\Gamma(\chi_1)-\sum_{i=1}^{m}\lambda_{1i}\left(\frac{{\varpi_i}^{\varrho}-{a_1}^{\varrho}}{\varrho}\right)^{\chi_1-1}\right\}^{-1}$, provided $\Gamma(\chi_1)\neq\sum_{i=1}^{m}\lambda_{1i}\left(\frac{{\varpi_i}^{\varrho}-{a_1}^{\varrho}}{\varrho}\right)^{\chi_1-1}$.

Proof. Proof of this theorem can be obtained by repeating the process of Theorem 3 of article [4] with some obvious modifications as per implicit fractional differential equation (1.1) and initial condition (1.3). Hence, it is omitted. □

Corollary 2.6. *Let* $f_2 : (a_1, a_2] \times \mathfrak{E} \times \mathfrak{E} \times \mathfrak{E} \to \mathfrak{E}$, *that satisfy* $f_2(\cdot, \mathfrak{u}(\cdot), \mathfrak{v}(\cdot), {}^{\varrho}D^{\omega_2,\vartheta_2}_{a_1^+}\mathfrak{v}(\cdot)) \in \mathcal{C}_{1-\chi_2,\varrho}[a_1, a_2]$ *for any* $\mathfrak{u}, \mathfrak{v} \in \mathcal{C}_{1-\chi_2,\varrho}[a_1, a_2]$ *where* $\chi_2 = \omega_2 + \vartheta_2(1-\omega_2)$ *with* $\omega_2 \in (0,1)$, $\vartheta_2 \in [0,1]$. *A function* $\mathfrak{v} \in \mathcal{C}^{\chi_2}_{1-\chi_2,\varrho}[a_1, a_2]$ *is a solution of generalized Katugampola type implicit fractional differential equation* (1.2) *with the initial condition* (1.4) *precisely when it satisfies the following mixed-type Volterra integral equation*

$$\mathfrak{v}(t) = \frac{\mathfrak{M}_2}{\Gamma(\omega_2)}\left(\frac{t^{\varrho} - {a_1}^{\varrho}}{\varrho}\right)^{\chi_2-1}\sum_{i=1}^{m}\lambda_{2i}\int_{a_1}^{\varpi_i}\left(\frac{{\varpi_i}^{\varrho} - x^{\varrho}}{\varrho}\right)^{\omega_2-1}x^{\varrho-1}A_{\mathfrak{v}}(x)\,dx + \frac{1}{\Gamma(\omega_2)}\int_{a_1}^{t}\left(\frac{t^{\varrho} - x^{\varrho}}{\varrho}\right)^{\omega_2-1}x^{\varrho-1}A_{\mathfrak{v}}(x)\,dx \tag{2.3}$$

where $\mathfrak{M}_2 = \left\{\Gamma(\chi_2) - \sum_{i=1}^{m}\lambda_{2i}\left(\frac{{\varpi_i}^{\varrho}-{a_1}^{\varrho}}{\varrho}\right)^{\chi_2-1}\right\}^{-1}$, *provided* $\Gamma(\chi_2) \neq \sum_{i=1}^{m}\lambda_{2i}\left(\frac{{\varpi_i}^{\varrho}-{a_1}^{\varrho}}{\varrho}\right)^{\chi_2-1}$.

3. Main results

In this section, the following hypotheses are assumed:
Q1: Suppose for $j = 1, 2$, a continuous function $f_j : (a_1, a_2] \times \mathfrak{E} \times \mathfrak{E} \times \mathfrak{E} \to \mathfrak{E}$, be such that $f_j(\cdot, \mathfrak{u}(\cdot), \mathfrak{v}(\cdot), \mathfrak{w}(\cdot)) \in \mathcal{C}^{\vartheta_j(1-\omega_j)}_{1-\chi_j,\varrho}[a_1, a_2]$ for any $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in \mathcal{C}_{1-\chi_j,\varrho}[a_1, a_2]$.
Q2: The following weak contraction condition holds for the functions $f_j(\cdot, \mathfrak{u}(\cdot), \mathfrak{v}(\cdot), \mathfrak{w}(\cdot))$

$$\begin{aligned}|f_j(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}), \mathfrak{w}(\mathfrak{t})) - f_j(\mathfrak{t}, \overline{\mathfrak{u}}(\mathfrak{t}), \overline{\mathfrak{v}}(\mathfrak{t}), \overline{\mathfrak{w}}(\mathfrak{t}))| \\ \leq \psi_{A,j}(|\mathfrak{u} - \overline{\mathfrak{u}}|) + \psi_{B,j}(|\mathfrak{v} - \overline{\mathfrak{v}}|) + L_j|\mathfrak{w} - \overline{\mathfrak{w}}|,\end{aligned}$$

for $\mathfrak{t} \in (a_1, a_2]$ and $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \overline{\mathfrak{u}}, \overline{\mathfrak{v}}, \overline{\mathfrak{w}} \in \mathfrak{E}$, $\psi_{A,j}, \psi_{B,j}$ are $\mathcal{D}$-functions and $L_j \in (0,1)$, $j = 1, 2$.
Q3: The constant

$$\xi_j := \frac{\mathbf{B}(\omega_j, \chi_j)}{(1-L_j)\Gamma(\omega_j)}\left\{|\mathfrak{M}_j|\sum_{i=1}^{m}\lambda_{ji}\left(\frac{{\varpi_i}^{\varrho} - a_1^{\varrho}}{\varrho}\right)^{\omega_j+\chi_j-1} + \left(\frac{a_2^{\varrho} - a_1^{\varrho}}{\varrho}\right)^{\omega_j}\right\} \tag{3.1}$$

where $\mathfrak{M}_1, \mathfrak{M}_2$ are as described in Theorem 2.5 and Corollary 2.6 respectively. $\mathbf{B}(\cdot,\cdot)$ is the usual Beta function. Moreover, we observe that $\xi_j < 1$.

Theorem 3.1. *Assume that the hypotheses* $Q1, Q2$ *and* $Q3$ *hold. Then, the coupled system (1.1)-(1.4) admits at least one solution in* $\mathcal{C}^{\omega_1,\vartheta_1}_{1-\chi_1} \times \mathcal{C}^{\omega_2,\vartheta_2}_{1-\chi_2}$.

Proof. To obtain the desired result, the existence results for mixed type Volterra integral equations (2.2) and (2.3) are sufficient according to Theorem 2.5 and Corollary 2.6.

Fixed $\mathfrak{v}$, we define the operator $\Delta_1 : \mathcal{C}_{1-\chi_1} \to \mathcal{C}_{1-\chi_1}$ by

$$(\Delta_1 \mathfrak{u})(\mathfrak{t}) = \frac{\mathfrak{M}_1}{\Gamma(\omega_1)} \left(\frac{\mathfrak{t}^\varrho - a_1^\varrho}{\varrho} \right)^{\chi_1 - 1} \sum_{i=1}^{m} \lambda_{1i} \int_{a_1}^{\varpi_i} \left\{ \left(\frac{\varpi_i{}^\varrho - x^\varrho}{\varrho} \right)^{\omega_1 - 1} x^{\varrho - 1} A_{\mathfrak{u}}(x) \right\} dx$$

$$+ \frac{1}{\Gamma(\omega_1)} \int_{a_1}^{\mathfrak{t}} \left\{ \left(\frac{\mathfrak{t}^\varrho - x^\varrho}{\varrho} \right)^{\omega_1 - 1} x^{\varrho - 1} A_{\mathfrak{u}}(x) \right\} dx. \tag{3.2}$$

Fixed $\mathfrak{u}$, we define the operator $\Delta_2 : \mathcal{C}_{1-\chi_2} \to \mathcal{C}_{1-\chi_2}$ by

$$(\Delta_2 \mathfrak{v})(\mathfrak{t}) = \frac{\mathfrak{M}_2}{\Gamma(\omega_2)} \left(\frac{\mathfrak{t}^\varrho - a_1^\varrho}{\varrho} \right)^{\chi_2 - 1} \sum_{i=1}^{m} \lambda_{2i} \int_{a_1}^{\varpi_i} \left\{ \left(\frac{\varpi_i{}^\varrho - x^\varrho}{\varrho} \right)^{\omega_2 - 1} x^{\varrho - 1} A_{\mathfrak{v}}(x) \right\} dx$$

$$+ \frac{1}{\Gamma(\omega_2)} \int_{a_1}^{\mathfrak{t}} \left\{ \left(\frac{\mathfrak{t}^\varrho - x^\varrho}{\varrho} \right)^{\omega_2 - 1} x^{\varrho - 1} A_{\mathfrak{v}}(x) \right\} dx. \tag{3.3}$$

Consider the continuous operator $\Delta : \mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1} \times \mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2} \to \mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1} \times \mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}$ defined by:

$$(\Delta(\mathfrak{u}, \mathfrak{v}))(\mathfrak{t}) = ((\Delta_1 \mathfrak{u})(\mathfrak{t}), (\Delta_2 \mathfrak{v})(\mathfrak{t})). \tag{3.4}$$

Clearly, $\Delta_j,\ j = 1, 2$ are well defined and map $\mathcal{C}_{1-\chi_j}$ into $\mathcal{C}_{1-\chi_j}$. Hence, Δ is also well defined and maps $\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1} \times \mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}$ into itself. Obviously, the fixed points of Δ are solutions to the coupled system (1.1)-(1.4).
For j = 1,2, let $\hat{f}_j = \max |f_j(x, 0, 0, 0)|$ and

$$\eta_j := \frac{2\mathbf{B}(\omega_j, \chi_j)}{\Gamma(\omega_j)} \left\{ |\mathfrak{M}_j| \sum_{i=1}^{m} \lambda_{ji} \left(\frac{\varpi_i{}^\varrho - a_1^\varrho}{\varrho} \right)^{\omega_j + \chi_j - 1} + \left(\frac{a_2^\varrho - a_1^\varrho}{\varrho} \right)^{\omega_j} \right\} \left\| \hat{f}_j \right\|_{\mathcal{C}_{1-\chi_j}}. \tag{3.5}$$

Consider a ball $\mathfrak{B}_s := \left\{ (\mathfrak{u}, \mathfrak{v}) \in \mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1} \times \mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2} : \|(\mathfrak{u}, \mathfrak{v})\|_{\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1} \times \mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}} \leq s \right\} = B(0, s)$ with $\frac{\eta_j}{1-\xi_j} \leq s,\ (\xi_j < 1),\ j = 1, 2$.
Now, let us subdivide Δ_1 as follows,

$$(\mathfrak{F}_1 \mathfrak{u})(\mathfrak{t}) = \frac{\mathfrak{M}_1}{\Gamma(\omega_1)} \left(\frac{\mathfrak{t}^\varrho - a_1^\varrho}{\varrho} \right)^{\chi_1 - 1} \sum_{i=1}^{m} \lambda_{1i} \int_{a_1}^{\varpi_i} \left\{ \left(\frac{\varpi_i{}^\varrho - x^\varrho}{\varrho} \right)^{\omega_1 - 1} x^{\varrho - 1} A_{\mathfrak{u}}(x) \right\} dx, \tag{3.6}$$

and

$$(\mathfrak{G}_1 \mathfrak{u})(\mathfrak{t}) = \frac{1}{\Gamma(\omega_1)} \int_{a_1}^{\mathfrak{t}} \left\{ \left(\frac{\mathfrak{t}^\varrho - x^\varrho}{\varrho} \right)^{\omega_1 - 1} x^{\varrho - 1} A_{\mathfrak{u}}(x) \right\} dx.$$

Similarly, subdivide Δ_2 as follows,

$$(\mathfrak{F}_2 \mathfrak{v})(\mathfrak{t}) = \frac{\mathfrak{M}_2}{\Gamma(\omega_2)} \left(\frac{\mathfrak{t}^\varrho - a_1^\varrho}{\varrho} \right)^{\chi_2 - 1} \sum_{i=1}^{m} \lambda_{2i} \int_{a_1}^{\varpi_i} \left\{ \left(\frac{\varpi_i{}^\varrho - x^\varrho}{\varrho} \right)^{\omega_2 - 1} x^{\varrho - 1} A_{\mathfrak{v}}(x) \right\} dx$$

and

$$\left(\mathfrak{G}_2\mathfrak{v}\right)(\mathfrak{t}) = \frac{1}{\Gamma(\omega_2)}\int_{a_1}^{\mathfrak{t}}\left\{\left(\frac{\mathfrak{t}^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega_2-1}x^{\varrho-1}A_{\mathfrak{v}}(x)\right\}dx.$$

Consequently, from equation (3.4) we have

$$\begin{aligned}\left(\Delta(\mathfrak{u},\mathfrak{v})\right)(\mathfrak{t}) &= \left(\left(\Delta_1\mathfrak{u}\right)(\mathfrak{t}),\left(\Delta_2\mathfrak{v}\right)(\mathfrak{t})\right)\\ &= \left(\left(\mathfrak{F}_1+\mathfrak{G}_1\right)(\mathfrak{u}),\left(\mathfrak{F}_2+\mathfrak{G}_2\right)(\mathfrak{v})\right)\\ &= \left(\left(\mathfrak{F}_1\right)\mathfrak{u},\left(\mathfrak{F}_2\right)\mathfrak{v}\right)+\left(\left(\mathfrak{G}_1\right)\mathfrak{u},\left(\mathfrak{G}_2\right)\mathfrak{v}\right) = \overline{\mathfrak{F}}(\mathfrak{u},\mathfrak{v})+\overline{\mathfrak{G}}(\mathfrak{u},\mathfrak{v}),\end{aligned}$$

where $\overline{\mathfrak{F}}(\mathfrak{u},\mathfrak{v}) = \left(\left(\mathfrak{F}_1\right)\mathfrak{u},\left(\mathfrak{F}_2\right)\mathfrak{v}\right)$ and $\overline{\mathfrak{G}}(\mathfrak{u},\mathfrak{v}) = \left(\left(\mathfrak{G}_1\right)\mathfrak{u},\left(\mathfrak{G}_2\right)\mathfrak{v}\right)$.
The proof is subdivided as follows.
Step I: $\overline{\mathfrak{F}}$ is non-linear $\mathcal{D}-$contraction.
From (3.6), for any $(\mathfrak{u},\mathfrak{v}),(\overline{\mathfrak{u}},\overline{\mathfrak{v}})\in\mathfrak{B}_s$ we have

$$\left|\left\{\left(\mathfrak{F}_1\mathfrak{u}\right)(\mathfrak{t})-\left(\mathfrak{F}_1\overline{\mathfrak{u}}\right)(\mathfrak{t})\right\}\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\right| \le \frac{|\mathfrak{M}_1|}{\Gamma(\omega_1)}\sum_{i=1}^{m}\lambda_{1i}\int_{a_1}^{\varpi_i}\left(\frac{\varpi_i{}^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega_1-1}x^{\varrho-1}\cdot\left(\left|A_{\mathfrak{u}}(x)-A_{\overline{\mathfrak{u}}}(x)\right|\right)dx. \tag{3.7}$$

From hypothesis $[Q1]$, we have

$$\left|A_{\mathfrak{u}}(\mathfrak{t})-A_{\overline{\mathfrak{u}}}(\mathfrak{t})\right| \le \frac{\psi_{A,1}\left(\left|\mathfrak{u}(\mathfrak{t})-\overline{\mathfrak{u}}(\mathfrak{t})\right|\right)}{(1-L_1)}. \tag{3.8}$$

Using (3.8) in (3.7) and the fact

$$\int_{a_1}^{\mathfrak{t}}\left(\frac{\mathfrak{t}^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega_1-1}x^{\varrho-1}\left|\mathfrak{u}(x)-\overline{\mathfrak{u}}(x)\right|dx \le \left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_1+\chi_1-1}\mathbf{B}(\omega_1,\chi_1)\cdot\left\|\mathfrak{u}-\overline{\mathfrak{u}}\right\|_{\mathcal{C}_{1-\chi_1,\varrho}}, \tag{3.9}$$

we get

$$\left|\left\{\left(\mathfrak{F}_1\mathfrak{u}\right)(\mathfrak{t})-\left(\mathfrak{F}_1\overline{\mathfrak{u}}\right)(\mathfrak{t})\right\}\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\right| \le \frac{|\mathfrak{M}_1|\mathbf{B}(\omega_1,\chi_1)}{(1-L_1)\Gamma(\omega_1)}\sum_{i=1}^{m}\lambda_{1i}\left(\frac{\varpi_i{}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_1+\chi_1-1}\psi_{A,1}\left(\left\|\mathfrak{u}-\overline{\mathfrak{u}}\right\|_{\mathcal{C}_{1-\chi_1}}\right). \tag{3.10}$$

Similarly,

$$\left|\left\{\left(\mathfrak{F}_2\mathfrak{v}\right)(\mathfrak{t})-\left(\mathfrak{F}_2\overline{\mathfrak{v}}\right)(\mathfrak{t})\right\}\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_2}\right| \le \frac{|\mathfrak{M}_2|\mathbf{B}(\omega_2,\chi_2)}{(1-L_2)\Gamma(\omega_2)}\sum_{i=1}^{m}\lambda_{2i}\left(\frac{\varpi_i{}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_2+\chi_2-1}\psi_{B,2}\left(\left\|\mathfrak{v}-\overline{\mathfrak{v}}\right\|_{\mathcal{C}_{1-\chi_2}}\right). \tag{3.11}$$

Consequently, the hypothesis $Q2$ implies that

$$\begin{aligned}\left\|\overline{\mathfrak{F}}\left(\mathfrak{u},\mathfrak{v}\right)-\overline{\mathfrak{F}}\left(\overline{\mathfrak{u}},\overline{\mathfrak{v}}\right)\right\|_{\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1}\times\mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}}&=\left\|\mathfrak{F}_1\mathfrak{u}-\mathfrak{F}_1\overline{\mathfrak{u}}\right\|_{\mathcal{C}_{1-\chi_1}}+\left\|\mathfrak{F}_2\mathfrak{v}-\mathfrak{F}_2\overline{\mathfrak{v}}\right\|_{\mathcal{C}_{1-\chi_2}}\\&\leq\xi_1\psi_{A,1}\left(\left\|\mathfrak{u}-\overline{\mathfrak{u}}\right\|_{\mathcal{C}_{1-\chi_1}}\right)+\xi_2\psi_{B,2}\left(\left\|\mathfrak{v}-\overline{\mathfrak{v}}\right\|_{\mathcal{C}_{1-\chi_2}}\right)\\&\leq\xi\left\{\psi_{A,1}\left(\left\|\mathfrak{u}-\overline{\mathfrak{u}}\right\|_{\mathcal{C}_{1-\chi_1}}\right)+\psi_{B,2}\left(\left\|\mathfrak{v}-\overline{\mathfrak{v}}\right\|_{\mathcal{C}_{1-\chi_2}}\right)\right\}\\&\leq\xi\psi_D\left(\left\|\left(\mathfrak{u},\mathfrak{v}\right)-\left(\overline{\mathfrak{u}},\overline{\mathfrak{v}}\right)\right\|_{\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1}\times\mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}}\right),\end{aligned}$$

where $\xi=\max\left\{\xi_1,\xi_2\right\}$ and

$$\psi_D\left(\left\|\left(\mathfrak{u},\mathfrak{v}\right)-\left(\overline{\mathfrak{u}},\overline{\mathfrak{v}}\right)\right\|_{\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1}\times\mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}}\right)=\psi_{A,1}\left(\left\|\mathfrak{u}-\overline{\mathfrak{u}}\right\|_{\mathcal{C}_{1-\chi_1}}\right)+\psi_{B,2}\left(\left\|\mathfrak{v}-\overline{\mathfrak{v}}\right\|_{\mathcal{C}_{1-\chi_2}}\right).$$

Hence, the operator $\overline{\mathfrak{F}}$ is non-linear $\mathcal{D}-$contraction.

Step II: $\overline{\mathfrak{G}}$ is completely continuous.

Consider a sequence $\{(\mathfrak{u}_n,\mathfrak{v}_n)\}_{n=1}^{\infty}\in\mathfrak{B}_s$ satisfying $(\mathfrak{u}_n,\mathfrak{v}_n)\to(\mathfrak{u},\mathfrak{v})$. Consequently, the definitions of the spaces $\mathcal{C}_{1-\chi_j,\varrho}$ yield the following:

$$\begin{aligned}\left\|\overline{\mathfrak{G}}\left(\mathfrak{u}_n,\mathfrak{v}_n\right)-\overline{\mathfrak{G}}\left(\mathfrak{u},\mathfrak{v}\right)\right\|_{\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1}\times\mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}}&=\left\|\mathfrak{G}_1\mathfrak{u}_n-\mathfrak{G}_1\mathfrak{u}\right\|_{\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1}}+\left\|\mathfrak{G}_2\mathfrak{v}_n-\mathfrak{G}_2\mathfrak{v}\right\|_{\mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}}\\&=\max_{\mathfrak{t}\in[a_1,a_2]}\left|\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\left(\left(\mathfrak{G}_1\mathfrak{u}_n\right)\left(\mathfrak{t}\right)-\left(\mathfrak{G}_1\mathfrak{u}\right)\left(\mathfrak{t}\right)\right)\right|\\&\quad+\max_{\mathfrak{t}\in[a_1,a_2]}\left|\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_2}\left(\left(\mathfrak{G}_2\mathfrak{v}_n\right)\left(\mathfrak{t}\right)-\left(\mathfrak{G}_2\mathfrak{v}\right)\left(\mathfrak{t}\right)\right)\right|.\end{aligned}\tag{3.12}$$

Next,

$$\begin{aligned}\left|\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\left(\left(\mathfrak{G}_1\mathfrak{u}_n\right)\left(\mathfrak{t}\right)-\left(\mathfrak{G}_1\mathfrak{u}\right)\left(\mathfrak{t}\right)\right)\right|&\leq\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\frac{1}{\Gamma\left(\omega_1\right)}\\&\quad\cdot\int_{a_1}^{\mathfrak{t}}\left(\frac{\mathfrak{t}^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega_1-1}x^{\varrho-1}\left(\left|A_{\mathfrak{u}_n}\left(x\right)-A_{\mathfrak{u}}\left(x\right)\right|\right)dx.\end{aligned}\tag{3.13}$$

From hypothesis $[Q1]$, we have

$$\left|A_{\mathfrak{u}_n}\left(x\right)-A_{\mathfrak{u}}\left(x\right)\right|\leq\frac{\psi_{A,1}\left(\left|\mathfrak{u}_n\left(x\right)-\mathfrak{u}\left(x\right)\right|\right)}{\left(1-L_1\right)}.\tag{3.14}$$

Using (3.14) in (3.13), we get

$$\begin{aligned}&\left|\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\left(\left(\mathfrak{G}_1\mathfrak{u}_n\right)\left(\mathfrak{t}\right)-\left(\mathfrak{G}_1\mathfrak{u}\right)\left(\mathfrak{t}\right)\right)\right|\\&\qquad\leq\frac{\mathbf{B}\left(\omega_1,\chi_1\right)}{\Gamma\left(\omega_1\right)}\left(\frac{a_2^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_1}\frac{\psi_{A,1}\left(\left\|\mathfrak{u}_n-\mathfrak{u}\right\|_{\mathcal{C}_{1-\chi_1}}\right)}{\left(1-L_1\right)}\\&\qquad\to0\text{ as }n\to\infty.\end{aligned}$$

Thus, $\max_{\mathfrak{t}\in[a_1,a_2]}\left|\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\left(\left(\mathfrak{G}_1\mathfrak{u}_n\right)(\mathfrak{t})-\left(\mathfrak{G}_1\mathfrak{u}\right)(\mathfrak{t})\right)\right|\to 0$ as $n\to\infty$. In a similar manner, it can be demonstrated that $\max_{\mathfrak{t}\in[a_1,a_2]}\left|\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_2}\left(\left(\mathfrak{G}_2\mathfrak{v}_n\right)(\mathfrak{t})-\left(\mathfrak{G}_2\mathfrak{v}\right)(\mathfrak{t})\right)\right|$ $\to 0$ as $n\to\infty$. Consequently, (3.12) implies that $\left\|\overline{\mathfrak{G}}\left(\mathfrak{u}_n,\mathfrak{v}_n\right)-\overline{\mathfrak{G}}\left(\mathfrak{u},\mathfrak{v}\right)\right\|_{\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1}\times\mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}}\to 0$ as $n\to\infty$. This proves the continuity of $\overline{\mathfrak{G}}$.

Next, we prove the compactness of $\overline{\mathfrak{G}}$. For any $a_1<\mathfrak{t}_1<\mathfrak{t}_2\leq a_2$, we have

$$\begin{aligned}
&\left|\left(\mathfrak{G}_1\mathfrak{u}\right)(\mathfrak{t}_1)-\left(\mathfrak{G}_1\mathfrak{u}\right)(\mathfrak{t}_2)\right|\\
&=\left|\frac{1}{\Gamma(\omega_1)}\int_{a_1}^{\mathfrak{t}_1}\left(\frac{\mathfrak{t}_1^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega_1-1}x^{\varrho-1}A_{\mathfrak{u}}(x)\,dx\frac{1}{\Gamma(\omega_1)}\int_{a_1}^{\mathfrak{t}_2}\left(\frac{\mathfrak{t}_2^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega_1-1}x^{\varrho-1}A_{\mathfrak{u}}(x)\,dx\right|\\
&\leq\frac{\|A_{\mathfrak{u}}\|_{\mathcal{C}_{1-\chi_1}}}{\Gamma(\omega_1)}\mathbf{B}(\omega_1,\chi_1)\left|\left(\frac{\mathfrak{t}_1^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_1+\chi_1-1}-\left(\frac{\mathfrak{t}_2^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_1+\chi_1-1}\right|,
\end{aligned}\tag{3.15}$$

that tends to 0 as $\mathfrak{t}_2\to\mathfrak{t}_1$, whether $\omega_1+\chi_1-1\geq 0$ or $\omega_1+\chi_1-1<0$. Thus, $\mathfrak{G}_1$ is equicontinuous. Also, we have that

$$\begin{aligned}
&\left|\left(\mathfrak{G}_2\mathfrak{v}\right)(\mathfrak{t}_1)-\left(\mathfrak{G}_2\mathfrak{v}\right)(\mathfrak{t}_2)\right|\\
&\leq\frac{\|A_{\mathfrak{v}}\|_{\mathcal{C}_{1-\chi_2}}}{\Gamma(\omega_2)}\mathbf{B}(\omega_2,\chi_2)\left|\left(\frac{\mathfrak{t}_1^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_2+\chi_2-1}-\left(\frac{\mathfrak{t}_2^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_2+\chi_2-1}\right|,
\end{aligned}\tag{3.16}$$

that tends to 0 as $\mathfrak{t}_2\to\mathfrak{t}_1$, whether $\omega_2+\chi_2-1\geq 0$ or $\omega_2+\chi_2-1<0$. Therefore, $\mathfrak{G}_2$ is equicontinuous.

Now, by employing (3.15) and (3.16) and the definitions of the spaces $\mathcal{C}_{1-\chi_j,\varrho}$, we can deduce that

$$\begin{aligned}
\left\|\overline{\mathfrak{G}}\left(\mathfrak{u},\mathfrak{v}\right)(\mathfrak{t}_1)-\overline{\mathfrak{G}}\left(\mathfrak{u},\mathfrak{v}\right)(\mathfrak{t}_2)\right\|_{\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1}\times\mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}}&=\left\|\left(\mathfrak{G}_1\mathfrak{u}\right)(\mathfrak{t}_1)-\left(\mathfrak{G}_1\mathfrak{u}\right)(\mathfrak{t}_2)\right\|_{\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1}}\\
&\quad+\left\|\left(\mathfrak{G}_2\mathfrak{v}\right)(\mathfrak{t}_1)-\left(\mathfrak{G}_2\mathfrak{v}\right)(\mathfrak{t}_2)\right\|_{\mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}}\to 0
\end{aligned}$$

as $\mathfrak{t}_2\to\mathfrak{t}_1$. Hence, $\overline{\mathfrak{G}}$ is equicontinuous. For $\mathfrak{t}\in(a_1,a_2]$, we have

$$\left|\left(\mathfrak{G}_1\mathfrak{u}\right)(\mathfrak{t})\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\right|\leq\frac{1}{\Gamma(\omega_1)}\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\int_{a_1}^{\mathfrak{t}}\left(\frac{\mathfrak{t}^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega_1-1}x^{\varrho-1}\left|A_{\mathfrak{u}}(x)\right|dx.$$

From the hypothesis $[Q1]$, we have

$$\begin{aligned}
\left|A_{\mathfrak{u}}(\mathfrak{t})\right|&=\left|f_1\left(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{v}(\mathfrak{t}),A_{\mathfrak{u}}(\mathfrak{t})\right)-f_1(\mathfrak{t},0,0,0)+f_1(\mathfrak{t},0,0,0)\right|\\
&\leq\frac{\psi_{A,1}\left(\left|\mathfrak{u}(\mathfrak{t})\right|\right)+\psi_{B,1}\left(\left|\mathfrak{v}(\mathfrak{t})\right|\right)+\hat{f}_1}{(1-L_1)}.
\end{aligned}\tag{3.17}$$

Now, using (3.17) and then the fact (3.9), we have

$$\left|\left(\mathfrak{G}_1\mathfrak{u}\right)(\mathfrak{t})\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\right| \leq \frac{\mathbf{B}\left(\omega_1,\chi_1\right)}{(1-L_1)\Gamma\left(\omega_1\right)}\left(\frac{a_2^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_1} \cdot\left(\psi_{A,1}\left(\|\mathfrak{u}\|_{\mathcal{C}_{1-\chi_1}}\right)+\psi_{B,1}\left(\|\mathfrak{v}\|_{\mathcal{C}_{1-\chi_2}}\right)+\left\|\hat{f}_1\right\|_{\mathcal{C}_{1-\chi_1}}\right),$$

which gives

$$\left\|\left(\mathfrak{G}_1\mathfrak{u}\right)\right\|_{\mathcal{C}_{1-\chi_1}} \leq \frac{\mathbf{B}\left(\omega_1,\chi_1\right)}{(1-L_1)\Gamma\left(\omega_1\right)}\left(\frac{a_2^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_1}\left(\psi_{A,1}\left(\|\mathfrak{u}\|_{\mathcal{C}_{1-\chi_1}}\right)+\psi_{B,1}\left(\|\mathfrak{v}\|_{\mathcal{C}_{1-\chi_2}}\right)+\left\|\hat{f}_1\right\|_{\mathcal{C}_{1-\chi_1}}\right). \tag{3.18}$$

Similarly, we get

$$\left\|\left(\mathfrak{G}_2\mathfrak{v}\right)\right\|_{\mathcal{C}_{1-\chi_2}} \leq \frac{\mathbf{B}\left(\omega_2,\chi_2\right)}{(1-L_2)\Gamma\left(\omega_2\right)}\left(\frac{a_2^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_2}\left(\psi_{A,2}\left(\|\mathfrak{u}\|_{\mathcal{C}_{1-\chi_1}}\right)+\psi_{B,2}\left(\|\mathfrak{v}\|_{\mathcal{C}_{1-\chi_2}}\right)+\left\|\hat{f}_2\right\|_{\mathcal{C}_{1-\chi_2}}\right). \tag{3.19}$$

From equations (3.18) and (3.19) we can infer that $\mathfrak{G}_j,\ j=1,2$ are uniformly bounded. Consequently, $\overline{\mathfrak{G}}$ is uniformly bounded. Hence, the operator $\overline{\mathfrak{G}}$ is compact on $\mathfrak{B}_s$ by the Arzela-Ascoli theorem. Consequently, $\overline{\mathfrak{G}}$ is completely continuous.

Step III: If $(\overline{\mathfrak{u}},\overline{\mathfrak{v}})\in\mathfrak{B}_s$ and $(\mathfrak{u},\mathfrak{v})\in\mathfrak{E}$, such that

$$(\mathfrak{u},\mathfrak{v})=\Delta\left(\mathfrak{u},\mathfrak{v}\right)=\overline{\mathfrak{F}}\left(\mathfrak{u},\mathfrak{v}\right)+\overline{\mathfrak{G}}\left(\overline{\mathfrak{u}},\overline{\mathfrak{v}}\right),$$

then $(\mathfrak{u},\mathfrak{v})\in\mathfrak{B}_s$. To prove this, let

$$\begin{aligned}\|(\mathfrak{u},\mathfrak{v})\|=\|\Delta\left(\mathfrak{u},\mathfrak{v}\right)\| &= \left\|\overline{\mathfrak{F}}\left(\mathfrak{u},\mathfrak{v}\right)+\overline{\mathfrak{G}}\left(\overline{\mathfrak{u}},\overline{\mathfrak{v}}\right)\right\| \\ &\leq \left\|\overline{\mathfrak{F}}\left(\mathfrak{u},\mathfrak{v}\right)\right\|+\left\|\overline{\mathfrak{G}}\left(\overline{\mathfrak{u}},\overline{\mathfrak{v}}\right)\right\| \\ &\leq \left\|\left(\mathfrak{F}_1\right)\mathfrak{u}\right\|+\left\|\left(\mathfrak{F}_2\right)\mathfrak{v}\right\|+\left\|\left(\mathfrak{G}_1\right)\overline{\mathfrak{u}}\right\|+\left\|\left(\mathfrak{G}_2\right)\overline{\mathfrak{v}}\right\|. \end{aligned}\tag{3.20}$$

Next, for each $\mathfrak{t}\in(a_1,a_2]$, we have

$$\left|\left(\mathfrak{F}_1\mathfrak{u}\right)(\mathfrak{t})\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\right| \leq \frac{|\mathfrak{M}_1|}{\Gamma\left(\omega_1\right)}\sum_{i=1}^{m}\lambda_{1i}\int_{a_1}^{\varpi_i}\left\{\left(\frac{\varpi_i{}^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega_1-1}x^{\varrho-1}\left|A_{\mathfrak{u}}\left(x\right)\right|\right\}dx. \tag{3.21}$$

Using (3.17) and (3.9) in (3.21), we get

$$\left|\left(\mathfrak{F}_1\mathfrak{u}\right)(\mathfrak{t})\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\right| \leq \frac{|\mathfrak{M}_1|}{\Gamma\left(\omega_1\right)}\sum_{i=1}^{m}\lambda_{1i}\left\{\left(\frac{\varpi_i^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_1+\chi_1-1}\mathbf{B}\left(\omega_1,\chi_1\right)\right\} \cdot\left(\frac{\psi_{A,1}\left(\|\mathfrak{u}\|_{\mathcal{C}_{1-\chi_1}}\right)+\psi_{B,1}\left(\|\mathfrak{v}\|_{\mathcal{C}_{1-\chi_2}}\right)+\left\|\hat{f}_1\right\|_{\mathcal{C}_{1-\chi_1}}}{(1-L_1)}\right),$$

which gives

$$\|(\mathfrak{F}_1\mathfrak{u})\|_{\mathcal{C}_{1-\chi_1}} \leq \frac{|\mathfrak{M}_1|\,\mathbf{B}\,(\omega_1,\chi_1)}{(1-L_1)\Gamma\,(\omega_1)}\sum_{i=1}^{m}\lambda_{1i}\left\{\left(\frac{\varpi_i^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_1+\chi_1-1}\right.$$
$$\left.\cdot\left(\psi_{A,1}\left(\|\mathfrak{u}\|_{\mathcal{C}_{1-\chi_1}}\right)+\psi_{B,1}\left(\|\mathfrak{v}\|_{\mathcal{C}_{1-\chi_2}}\right)+\left\|\hat{f}_1\right\|_{\mathcal{C}_{1-\chi_1}}\right)\right\}. \quad (3.22)$$

Similarly, it can be shown that

$$\|(\mathfrak{F}_2\mathfrak{v})\|_{\mathcal{C}_{1-\chi_2}} \leq \frac{|\mathfrak{M}_2|\,\mathbf{B}\,(\omega_2,\chi_2)}{(1-L_2)\Gamma\,(\omega_2)}\sum_{i=1}^{m}\lambda_{2i}\left\{\left(\frac{\varpi_i^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_2+\chi_2-1}\right.$$
$$\left.\cdot\left(\psi_{A,2}\left(\|\mathfrak{u}\|_{\mathcal{C}_{1-\chi_1}}\right)+\psi_{B,2}\left(\|\mathfrak{v}\|_{\mathcal{C}_{1-\chi_2}}\right)+\left\|\hat{f}_2\right\|_{\mathcal{C}_{1-\chi_2}}\right)\right\}. \quad (3.23)$$

Now, let

$$S := \|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{C}^{1-\chi_1}_{\omega_1,\vartheta_1}\times\mathcal{C}^{1-\chi_2}_{\omega_2,\vartheta_2}} = \|\mathfrak{u}\|_{\mathcal{C}^{1-\chi_1}_{\omega_1,\vartheta_1}} + \|\mathfrak{v}\|_{\mathcal{C}^{1-\chi_2}_{\omega_2,\vartheta_2}}.$$

Suppose that $(\mathfrak{u},\mathfrak{v})$ satisfies the fixed point equation

$$(\mathfrak{u},\mathfrak{v}) = \Delta(\mathfrak{u},\mathfrak{v}) = \overline{\mathfrak{F}}(\mathfrak{u},\mathfrak{v}) + \overline{\mathfrak{G}}(\mathfrak{u},\mathfrak{v}).$$

Then, by the triangle inequality,

$$S = \|(\mathfrak{u},\mathfrak{v})\| \;\leq\; \|\overline{\mathfrak{F}}(\mathfrak{u},\mathfrak{v})\| + \|\overline{\mathfrak{G}}(\mathfrak{u},\mathfrak{v})\|.$$

Using equations (3.18), (3.19), (3.22), (3.23) and the definitions of ξ_j in (3.1) and η_j in (3.5), there exist $\xi = \max\{\xi_1,\xi_2\} < 1$ and $\eta = \max\{\eta_1,\eta_2\}$ such that

$$\|\overline{\mathfrak{F}}(\mathfrak{u},\mathfrak{v})\| + \|\overline{\mathfrak{G}}(\mathfrak{u},\mathfrak{v})\| \;\leq\; \xi\big(\psi_A(\|\mathfrak{u}\|) + \psi_B(\|\mathfrak{v}\|)\big) + \eta. \quad (3.24)$$

Since by hypothesis $[Q2]$ ψ_A and ψ_B are D-functions, we have $\psi_A(\alpha) < \alpha$ and $\psi_B(\alpha) < \alpha$ for all $\alpha > 0$. Thus,

$$\psi_A(\|\mathfrak{u}\|) + \psi_B(\|\mathfrak{v}\|) \;<\; \|\mathfrak{u}\| + \|\mathfrak{v}\| \;=\; S.$$

Hence, equation (3.24) implies

$$S \leq \xi S + \eta \leq \frac{\eta}{1-\xi}.$$

By construction of the ball $\mathfrak{B}_s$, we have

$$\frac{\eta}{1-\xi} \leq \max_{j=1,2}\frac{\eta_j}{1-\xi_j} \leq s.$$

Hence $S \leq s$, which implies $(\mathfrak{u},\mathfrak{v}) \in \mathfrak{B}_s$. Therefore, all three conditions of Theorem 2.4 hold. It follows that $\Delta = \overline{\mathfrak{F}} + \overline{\mathfrak{G}}$ has a fixed point in $\mathfrak{B}_s$, i.e., the system (1.1) - (1.4) has a solution in $\mathcal{C}^{\omega_1,\vartheta_1}_{1-\chi_1}\times\mathcal{C}^{\omega_2,\vartheta_2}_{1-\chi_2}$. □

Theorem 3.2. *Assume that $Q1$ and the following hypotheses hold:*
Q4: *The nonlinearities $f_j(\mathfrak{t},\mathfrak{u},\mathfrak{v},w)$ for $j=1,2$ satisfy the global Lipschitz condition:*

$$\big|f_j(\mathfrak{t},\mathfrak{u}_1,\mathfrak{v}_1,\mathfrak{w}_1) - f_j(\mathfrak{t},\mathfrak{u}_2,\mathfrak{v}_2,\mathfrak{w}_2)\big| \leq b_j|\mathfrak{u}_1-\mathfrak{u}_2| + c_j|\mathfrak{v}_1-\mathfrak{v}_2| + d_j|\mathfrak{w}_1-\mathfrak{w}_2|, \quad (3.25)$$

where $\mathfrak{t} \in (a_1, a_2]$ *and* $b_j, c_j, d_j \geq 0$ *are constants.*

Q5: *The constant*

$$\zeta_j := \frac{\mathbf{B}(\omega_j, \chi_j) b_j}{(1-d_j)\,\Gamma(\omega_j)} \left\{ |\mathfrak{M}_j| \sum_{i=1}^{m} \lambda_{ji} \left(\frac{\varpi_i{}^{\varrho} - a_1^{\varrho}}{\varrho} \right)^{\omega_j+\chi_j-1} + \left(\frac{a_2^{\varrho} - a_1^{\varrho}}{\varrho} \right)^{\omega_j} \right\} < 1 \tag{3.26}$$

If there exists $\zeta = max\,(\zeta_1, \zeta_2) < 1$ *with*

$$\|\Delta(\mathfrak{u}_1, \mathfrak{v}_1) - \Delta(\mathfrak{u}_2, \mathfrak{v}_2)\| \leq \zeta \,\|(\mathfrak{u}_1, \mathfrak{v}_1) - (\mathfrak{u}_2, \mathfrak{v}_2)\| \quad \textit{for all } (\mathfrak{u}_1, \mathfrak{v}_1), (\mathfrak{u}_2, \mathfrak{v}_2) \in \mathfrak{B}_s,$$

then by Banach contraction principle, the coupled system (1.1)-(1.4) *has a unique solution in* $\mathfrak{B}_s$.

Proof. Let $(\mathfrak{u}_1, \mathfrak{v}_1), (\mathfrak{u}_2, \mathfrak{v}_2) \in \mathfrak{B}_s$. Using (3.4) and the triangle inequality, we have

$$\|\Delta(\mathfrak{u}_1, \mathfrak{v}_1) - \Delta(\mathfrak{u}_2, \mathfrak{v}_2)\| \leq \|\overline{\mathfrak{F}}(\mathfrak{u}_1, \mathfrak{v}_1) - \overline{\mathfrak{F}}(\mathfrak{u}_2, \mathfrak{v}_2)\| + \|\overline{\mathfrak{G}}(\mathfrak{u}_1, \mathfrak{v}_1) - \overline{\mathfrak{G}}(\mathfrak{u}_2, \mathfrak{v}_2)\|. \tag{3.27}$$

For any $(\mathfrak{u}_1, \mathfrak{v}_1), (\mathfrak{u}_2, \mathfrak{v}_2) \in \mathfrak{B}_s$, and the operator $\mathfrak{F}_1$ we have

$$\left| \{(\mathfrak{F}_1 \mathfrak{u}_1)(\mathfrak{t}) - (\mathfrak{F}_1 \mathfrak{u}_2)(\mathfrak{t})\} \left(\frac{\mathfrak{t}^{\varrho} - a_1^{\varrho}}{\varrho} \right)^{1-\chi_1} \right| \leq \frac{|\mathfrak{M}_1|}{\Gamma(\omega_1)} \sum_{i=1}^{m} \lambda_{1i} \int_{a_1}^{\varpi_i} \left(\frac{\varpi_i{}^{\varrho} - x^{\varrho}}{\varrho} \right)^{\omega_1-1} \cdot x^{\varrho-1} \left(|A_{\mathfrak{u}_1}(x) - A_{\mathfrak{u}_2}(x)| \right) dx. \tag{3.28}$$

From (3.25), we have

$$\begin{aligned} |A_{\mathfrak{u}_1}(\mathfrak{t}) - A_{\mathfrak{u}_2}(\mathfrak{t})| &= |f_1(\mathfrak{t}, \mathfrak{u}_1(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}), A_{\mathfrak{u}_1}(\mathfrak{t})) - f_1(\mathfrak{t}, \mathfrak{u}_2(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}), A_{\mathfrak{u}_2}(\mathfrak{t}))| \\ &\leq \frac{b_1 \left(|\mathfrak{u}_1(\mathfrak{t}) - \mathfrak{u}_2(\mathfrak{t})| \right)}{(1-d_1)}. \end{aligned} \tag{3.29}$$

Using (3.29) in (3.28) and using the fact (3.9), we get

$$\begin{aligned} &\left| \{(\mathfrak{F}_1 \mathfrak{u}_1)(\mathfrak{t}) - (\mathfrak{F}_1 \mathfrak{u}_2)(\mathfrak{t})\} \left(\frac{\mathfrak{t}^{\varrho} - a_1^{\varrho}}{\varrho} \right)^{1-\chi_1} \right| \\ &\qquad \leq \frac{|\mathfrak{M}_1| \mathbf{B}(\omega_1, \chi_1)}{(1-d_1)\Gamma(\omega_1)} \sum_{i=1}^{m} \lambda_{1i} \left(\frac{\varpi_i{}^{\varrho} - a_1^{\varrho}}{\varrho} \right)^{\omega_1+\chi_1-1} b_1 \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}}, \end{aligned}$$

which gives,

$$\|\mathfrak{F}_1 \mathfrak{u}_1 - \mathfrak{F}_1 \mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}} \leq \Xi_1 \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}}, \tag{3.30}$$

where $\Xi_1 = \frac{|\mathfrak{M}_1| \mathbf{B}(\omega_1, \chi_1) b_1}{(1-d_1)\Gamma(\omega_1)} \sum_{i=1}^{m} \lambda_{1i} \left(\frac{\varpi_i{}^{\varrho} - a_1^{\varrho}}{\varrho} \right)^{\omega_1+\chi_1-1}$.

Similarly, it can be shown that

$$\|\mathfrak{F}_2 \mathfrak{v}_1 - \mathfrak{F}_2 \mathfrak{v}_2\|_{\mathcal{C}_{1-\chi_2}} \leq \Xi_2 \|\mathfrak{v}_1 - \mathfrak{v}_2\|_{\mathcal{C}_{1-\chi_2}}, \tag{3.31}$$

where $\Xi_2 = \frac{|\mathfrak{M}_2| \mathbf{B}(\omega_2, \chi_2) b_2}{(1-d_2)\Gamma(\omega_2)} \sum_{i=1}^{m} \lambda_{2i} \left(\frac{\varpi_i{}^{\varrho} - a_1^{\varrho}}{\varrho} \right)^{\omega_2+\chi_2-1}$.

Thus, from equations (3.30) and (3.31) we have

$$\left\| \overline{\mathfrak{F}}(\mathfrak{u}_1, \mathfrak{v}_1) - \overline{\mathfrak{F}}(\mathfrak{u}_2, \mathfrak{v}_2) \right\|_{\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1} \times \mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}} \leq \Xi_1 \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}} + \Xi_2 \|\mathfrak{v}_1 - \mathfrak{v}_2\|_{\mathcal{C}_{1-\chi_2}} \tag{3.32}$$

Now, for any $(\mathfrak{u}_1,\mathfrak{v}_1),(\mathfrak{u}_2,\mathfrak{v}_2)\in\mathfrak{B}_s$, and the operator $\mathfrak{G}_1$ we have

$$\left|\left\{\left(\mathfrak{G}_1\mathfrak{u}_1\right)(\mathfrak{t})-\left(\mathfrak{G}_1\mathfrak{u}_2\right)(\mathfrak{t})\right\}\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\right|\leq\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\frac{1}{\Gamma(\omega_1)}$$
$$\cdot\int_{a_1}^{\mathfrak{t}}\left(\frac{\mathfrak{t}^{\varrho}-x^{\varrho}}{\varrho}\right)^{\omega_1-1}x^{\varrho-1}\left(\left|A_{\mathfrak{u}_1}(x)-A_{\mathfrak{u}_2}(x)\right|\right)dx. \tag{3.33}$$

Using (3.29) in (3.33), we get

$$\left|\left(\frac{\mathfrak{t}^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{1-\chi_1}\left(\left(\mathfrak{G}_1\mathfrak{u}_n\right)(\mathfrak{t})-\left(\mathfrak{G}_1\mathfrak{u}\right)(\mathfrak{t})\right)\right|\leq\frac{\mathbf{B}(\omega_1,\chi_1)}{\Gamma(\omega_1)}\left(\frac{a_2^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_1}$$
$$\cdot\frac{b_1\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}}}{(1-d_1)},$$

which gives

$$\|\mathfrak{G}_1\mathfrak{u}_1-\mathfrak{G}_1\mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}}\leq\Omega_1\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}}, \tag{3.34}$$

where $\Omega_1=\frac{\mathbf{B}(\omega_1,\chi_1)b_1}{\Gamma(\omega_1)(1-d_1)}\left(\frac{a_2^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_1}$.
Similarly, it can be shown that

$$\|\mathfrak{G}_2\mathfrak{v}_1-\mathfrak{G}_2\mathfrak{v}_2\|_{\mathcal{C}_{1-\chi_2}}\leq\Omega_2\|\mathfrak{v}_1-\mathfrak{v}_2\|_{\mathcal{C}_{1-\chi_2}}, \tag{3.35}$$

where $\Omega_2=\frac{\mathbf{B}(\omega_2,\chi_2)b_2}{\Gamma(\omega_2)(1-d_2)}\left(\frac{a_2^{\varrho}-a_1^{\varrho}}{\varrho}\right)^{\omega_2}$.
Thus, from equations (3.34) and (3.35), we have

$$\left\|\overline{\mathfrak{G}}(\mathfrak{u}_1,\mathfrak{v}_1)-\overline{\mathfrak{G}}(\mathfrak{u}_2,\mathfrak{v}_2)\right\|_{\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1}\times\mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}}\leq\Omega_1\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}}+\Omega_2\|\mathfrak{v}_1-\mathfrak{v}_2\|_{\mathcal{C}_{1-\chi_2}}. \tag{3.36}$$

Using (3.32) and (3.36) in (3.27), we get

$$\begin{aligned}\|\Delta(\mathfrak{u}_1,\mathfrak{v}_1)-\Delta(\mathfrak{u}_2,\mathfrak{v}_2)\|&\leq\left\{\Xi_1\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}}+\Xi_2\|\mathfrak{v}_1-\mathfrak{v}_2\|_{\mathcal{C}_{1-\chi_2}}\right\}\\&\quad+\left\{\Omega_1\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}}+\Omega_2\|\mathfrak{v}_1-\mathfrak{v}_2\|_{\mathcal{C}_{1-\chi_2}}\right\}\\&\leq\zeta_1\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}}+\zeta_2\|\mathfrak{v}_1-\mathfrak{v}_2\|_{\mathcal{C}_{1-\chi_2}}\\&\leq\zeta\cdot\left(\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\mathcal{C}_{1-\chi_1}}+\|\mathfrak{v}_1-\mathfrak{v}_2\|_{\mathcal{C}_{1-\chi_2}}\right)\\&=\zeta\left\|(\mathfrak{u}_1,\mathfrak{v}_1)-(\mathfrak{u}_2,\mathfrak{v}_2)\right\|,\end{aligned} \tag{3.37}$$

where $\zeta=max(\zeta_1,\zeta_2)<1$. Hence Δ is a contraction on $\mathfrak{B}_s$. By Banach contraction principle Δ has a unique fixed point in $\mathfrak{B}_s$, and hence the coupled system (1.1)-(1.4) admits at most one solution in $\mathfrak{B}_s$. □

4. Applications

Example 4.1. Consider the following coupled system of implicit fractional differential equations

$${}^{1}D_{a_1^+}^{0.7,0.3}\mathfrak{u}(\mathfrak{t})=0.25\tanh(\mathfrak{u})+0.18\tan^{-1}(\mathfrak{v})+0.33\,{}^{1}D_{a_1^+}^{0.7,0.3}\mathfrak{u}(\mathfrak{t})+0.05\sin(\mathfrak{t}), \tag{4.1}$$

$$ {}^{1}D_{a_1^+}^{0.6,0.4}\mathfrak{v}\left(\mathfrak{t}\right)=0.22\ \frac{\mathfrak{u}}{1+|\mathfrak{u}|}+0.2\ \tanh\left(\mathfrak{v}\right)+0.3\ {}^{1}D_{a_1^+}^{0.6,0.4}\mathfrak{v}\left(\mathfrak{t}\right)+0.03\ \cos(\mathfrak{t}),\quad (4.2) $$

with the following non-local initial conditions

$$ {}^{1}I_{a_1^+}^{1-0.79}\mathfrak{u}\left(a_1\right)=0.002\ \mathfrak{u}\left(1.02\right)+0.0015\ \mathfrak{u}\left(1.08\right), \quad (4.3) $$

$$ {}^{1}I_{a_1^+}^{1-0.76}\mathfrak{v}\left(a_1\right)=0.001\ \mathfrak{v}\left(1.02\right)+0.0025\ \mathfrak{v}\left(1.08\right), \quad (4.4) $$

where $\mathfrak{t}\in(a_1,a_2]=(1,1.1]$. Set

$$ f_1\left(\mathfrak{t},\mathfrak{u}\left(\mathfrak{t}\right),\mathfrak{v}\left(\mathfrak{t}\right),\mathfrak{w}\left(\mathfrak{t}\right)\right)=0.25\ \tanh\left(\mathfrak{u}\right)+0.18\ \tan^{-1}\left(\mathfrak{v}\right)+0.33\ \mathfrak{w}+0.05\ \sin(\mathfrak{t}), $$

and

$$ f_2\left(\mathfrak{t},\mathfrak{u}\left(\mathfrak{t}\right),\mathfrak{v}\left(\mathfrak{t}\right),\mathfrak{w}\left(\mathfrak{t}\right)\right)=0.22\ \frac{\mathfrak{u}}{1+|\mathfrak{u}|}+0.2\ \tanh\left(\mathfrak{v}\right)+0.3\ \mathfrak{w}+0.03\ \cos(\mathfrak{t}). $$

Clearly, both f_1 and f_2 are continuous. Also, they are measurable in $\mathfrak{t}$ and locally bounded on $[a_1,a_2]$. So, the condition $Q1$ holds.

Further, for any $\mathfrak{u},\mathfrak{v},\mathfrak{w},\overline{\mathfrak{u}},\overline{\mathfrak{v}},\overline{\mathfrak{w}}\in\mathfrak{E}$, $\mathfrak{t}\in(1,1.1]$ and for f_1 let us choose the $\mathcal{D}$-functions $\psi_{A,1}\left(s\right)=0.25s$, $\psi_{B,1}\left(s\right)=0.18s$, and $L_1=0.33$. Similarly, for f_2 let us choose the $\mathcal{D}$-functions $\psi_{A,2}\left(s\right)=0.22s$, $\psi_{B,2}\left(s\right)=0.2s$, and $L_2=0.3$. Then, the condition $Q2$ holds for both f_1 and f_2.
Moreover, with some elementary computation we obtained $\mathfrak{M}_1=0.8557557776$ and $\mathfrak{M}_2=0.8297414296$. Consequently, we obtain $\xi_1=0.3964190345<1$ and $\xi_2=0.4907231305<1$. That is $\xi_j<1$, for $j=1,2$ thus condition $Q3$ holds.

Hence, it follows that all the hypotheses $Q1,Q2$ and $Q3$ of Theorem 3.1 hold for the above coupled system (4.1) - (4.4). Therefore, Theorem 3.1 guarantees at least one solution in the product space $\mathcal{C}_{1-\chi_1}^{\omega_1,\vartheta_1}\times\mathcal{C}_{1-\chi_2}^{\omega_2,\vartheta_2}$.

Next, for any $\mathfrak{u}_1,\mathfrak{v}_1,\mathfrak{w}_1,\mathfrak{u}_2,\mathfrak{v}_2,\mathfrak{w}_2\in\mathfrak{E}$, $\mathfrak{t}\in(1,1.1]$ and for f_1 let us choose the constants $b_1=0.25,c_1=0.18$, and $d_1=0.33$ and use the bounds $|\tanh'\left(\mathfrak{u}\right)|\leq 1$ and $\left|(\tan^{-1}\left(\mathfrak{u}\right))'\right|\leq 1$. Similarly, for f_2 let us choose the constants $b_2=0.22,c_2=0.2$, and $d_2=0.3$ and using the bounds $\left|\frac{d}{du}\left(\frac{\mathfrak{u}}{1+|\mathfrak{u}|}\right)\right|\leq 1$ and $|\tanh'\left(\mathfrak{u}\right)|\leq 1$.
Then, the condition $Q4$ holds for both f_1 and f_2.
Moreover, with some elementary computation we obtained $\zeta_1=0.099105<1$ and $\zeta_2=0.107959<1$. Thus, $\zeta<1$, which implies that the condition $Q5$ holds. Thus, the Theorem 3.2 implies that there exists a unique solution in $\mathfrak{B}_s$ for the coupled system (4.1) - (4.4).

5. Conclusion

A coupled system of implicit fractional differential equations with non-local initial conditions involving a generalized Katugampola derivative operator was investigated to establish the result over the existence of solution and guarantees the uniqueness. Equivalent Volterra integral equations are obtained for the system (1.1)-(1.4). The main existence result was established in the weighted Banach space by applying the Krasnoselskii-Dhage type fixed point theorem. Further, the uniqueness result is obtained by means of the Banach contraction principle. Ultimately, an application was presented to verify the usefulness of the findings. The coupled system of implicit fractional differential equations involving a generalized

Katugampola derivative operator (1.1)-(1.4) discussed in this paper generalizes the existing results available in the literature. However, it would be interesting to see the impact of including the control term in the system (1.1)-(1.4). This concept may serve as an open question for future research.

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