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AN ALGORITHM FOR CONSTRUCTING OPTIMAL EXPLICIT FINITE-DIFFERENCE FORMULAS

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Abstract. In this work, we develop a functional approach to constructing optimal explicit finite-difference formulas for Cauchy problems in a Hilbert space. Using the extremal function, we derive an expression for the norm of the error functional. Minimizing this norm yields a linear system for the optimal coefficients, and the existence and uniqueness of its solution are established. Discrete analogues of differential operators provide representations of the interior and endpoint coefficients of Adams-type formulas for arbitrary integers greater than or equal to three. The resulting formulas minimize the error functional norm subject to the prescribed exactness constraints.

Keywords. Hilbert space, Cauchy problem, extremal function, error functional, optimal finite-difference formula
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1. Introduction

It is known that the solutions of many practical problems lead to solutions of differential equations or their systems. Although differential equations have so many applications, only a small number of them can be solved exactly using elementary functions and their combinations. Even in the analytical analysis of differential equations, their application can be inconvenient due to the complexity of the obtained solution. If it is very difficult to obtain or impossible to find an analytic solution to a differential equation, one can find an approximate solution.

In the present paper we consider the problem of approximate solution to the first order linear ordinary differential equation

$$y' = f(x, y), \ \ x \in [0, 1] \tag{1.1}$$

with the initial condition

$$y(0) = y_0. \tag{1.2}$$

We assume that $f(x, y)$ is a suitable function and the differential equation (1.1) with the initial condition (1.2) has a unique solution on the interval $[0, 1]$.

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For the approximate solution of the problem (1.1)-(1.2), we divide the interval $[0,1]$ into N pieces of the length $h=\frac{1}{N}$ and find approximate values y_n of the function $y(x)$ for $n=0,1,...,N$ at nodes $x_n=nh$.

A classic method of approximate solution of the initial-value problem (1.1)-(1.2) is the Euler method. Using this method, the approximate solution of the differential equation is calculated as follows: to find an approximate value y_{n+1} of the function at the node x_{n+1}, the approximate value y_n at the node x_n is used, i.e.,

$$y_{n+1}=y_n+hy'_n, \tag{1.3}$$

where $y'_n=f(x_n,y_n)$, so that y_{n+1} is a linear combination of the values of the unknown function $y(x)$ and its first-order derivative at the node x_n.

The numerical solution of ordinary differential equations remains a major challenge even today. Therefore, many researchers are currently working on this problem. In [8], S. Mehrkanoon, Z. A. Majid, M. Suleiman, K. I. Othman, and Z. B. Ibrahim proposed a new approach to solving a system of first-order ordinary differential equations using three points and three steps. In [3, 20, 4], the authors presented new semi-explicit and semi-implicit predictor-corrector methods. In this study, they studied the numerical stability of these methods by constructing stability regions and clearly demonstrated that semi-explicit methods have higher numerical stability than conventional predictor-corrector algorithms. In [9], Osama Y. Ababneh presented new numerical methods for solving ordinary differential equations in both linear and nonlinear cases. In [1], Adekoya M. Odunayo and Z. O. Ogunwobi balanced the Adams-Bashforth-Multon and Milne-Simpson methods for a second-order differential equation. In [10], Sajal K. Kar used the Adams-Bashforth scheme to create a new finite-difference time predictor-corrector scheme. In [21], Emil Vitásek studied methods of arbitrarily high orders of accuracy for solving an abstract ordinary differential equation. The problem of constructing finite-difference formulas close to this work can be seen in the following works [17, 15, 14, 5, 16, 18, 7].

2. Statement of the problem

We consider explicit finite-difference formulas of the following form for the approximate solution of the problem (1.1)-(1.2) (see, for example, [11])

$$\sum_{\beta=0}^{k}C_\beta\varphi\left(h\beta\right)-h\sum_{\beta=0}^{k-1}C_\beta^{(1)}\varphi'\left(h\beta\right)\cong 0. \tag{2.1}$$

Here C_β are $C_\beta^{(1)}$ the coefficients of explicit finite-difference formulas. The class of problems under consideration is determined by the assignment of the class $W_2^{(m,m-1)}(0,1)$, i.e. we will consider functions $\varphi(x)$ from the class of real functions in the Hilbert space $W_2^{(m,m-1)}(0,1)$. The inner product in this space is determined by the formula [12]

$$\langle\varphi,\psi\rangle=\int_0^1\left(\varphi^{(m)}(x)+\varphi^{(m-1)}(x)\right)\left(\psi^{(m)}(x)+\psi^{(m-1)}(x)\right)dx. \tag{2.2}$$

Let the space $W_2^{(m,m-1)}(0,1)$ be embedded in the space $C(0,1)$ of continuous functions, then the error functional of explicit finite-difference formulas will be linear (see, [19])

$$(\ell,\varphi)=\int_{-\infty}^{\infty}\left[\sum_{\beta=0}^{k}C_\beta\delta(x-h\beta)+h\sum_{\beta=0}^{k-1}C_\beta^{(1)}\delta'(x-h\beta)\right]\varphi(x)\,dx. \tag{2.3}$$

The quantity (2.3) is called *the error* of explicit finite-difference formulas. The error of explicit finite-difference formulas on classes $W_2^{(m,m-1)}(0,1)$ is called the quantity

$$\ell\left(W_2^{(m,m-1)}\right)=\sup_{\varphi\in W_2^{(m,m-1)}}|(\ell,\varphi)|. \tag{2.4}$$

Bottom line

$$\overset{\circ}{\ell}\left(W_2^{(m,m-1)}\right)=\inf_{C_\beta,C_\beta^{(1)}}\left\|\ell\left(W_2^{(m,m-1)}\right)\right\| \tag{2.5}$$

is called the error of *the optimal explicit finite-difference formula*. If there exists an explicit finite-difference formula for which

$$\ell\left(W_2^{(m,m-1)}\right)=\overset{\circ}{\ell}\left(W_2^{(m,m-1)}\right),$$

then such an explicit finite-difference formula is called *an optimal explicit finite-difference formula* on the class under consideration. And the coefficients $\overset{\circ}{C}_\beta$ are $\overset{\circ}{C}_\beta^{(1)}$ called *optimal coefficients* of explicit finite-difference formulas.

The problem of constructing an optimal explicit finite-difference formula in a functional setting consists of minimizing the error of explicit finite-difference formulas on classes of functions $W_2^{(m,m-1)}(0,1)$. Since $W_2^{(m,m-1)}(0,1)$ is a Hilbert space, then this error on classes coincides with the norm of the error functional (2.3) of explicit finite-difference formulas. To find the error of explicit finite-difference formulas on classes we will use the extremal function [19].

Definition 2.1. If the equality

$$(\ell,\psi_\ell)=\left\|\ell\left|W_2^{(m,m-1)*}\right.\right\|\cdot\left\|\psi_\ell\left|W_2^{(m,m-1)}\right.\right\|$$

is satisfied and $\psi_\ell\in W_2^{(m,m-1)}(0,1)$, then ψ_ℓ is called the extremal function of the functional ℓ.

In a Hilbert space $W_2^{(m,m-1)}(0,1)$ the norm of a function $\varphi(x)$ is found by the equality

$$\left\|\varphi|W_2^{(m,m-1)}(0,1)\right\|=\left\{\int_0^1\left(\varphi^{(m)}(x)+\varphi^{(m-1)}(x)\right)^2dx\right\}^{1/2}.$$

Thus, as the error functional of explicit finite-difference formulas

$$\ell(x)=\sum_{\beta=0}^{k}C_\beta\delta(x-h\beta)+h\sum_{\beta=0}^{k-1}C_\beta^{(1)}\delta'(x-h\beta)$$

is defined and bounded in the Hilbert space $W_2^{(m,m-1)}(0,1)$, then

$$(\ell, x^\alpha) = 0,\ \alpha = 0, 1, ..., m-2, \tag{2.6}$$

$$\left(\ell, e^{-x}\right) = 0. \tag{2.7}$$

However, since the norm of an element $\varphi(x)$ is represented by a quadratic function and the inner product is given by equality (2.2), then by applying the Riesz theorem (see, [2]) one can find for the error functional of explicit finite-difference formulas, explicitly expressed through some new function $\psi_\ell(x)$, which itself is an element of $W_2^{(m,m-1)}(0,1)$.

The function $\psi_\ell(x)$ is a Riesz element [2]. Then by Riesz's theorem it follows that

$$(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle, \tag{2.8}$$

$$\left\| \ell | W_2^{(m,m-1)*}(0,1) \right\| = \left\| \psi_\ell | W_2^{(m,m-1)}(0,1) \right\|.$$

Theorem 2.2. *The maximizing element, i.e. the extremal function of the explicit finite-difference formulas (2.1) in the Hilbert space $W_2^{(m,m-1)}(0,1)$ is given by the equality*

$$\psi_\ell(x) = (-1)^m \ell(x) * G_m(x) + P_{m-2}(x) + de^{-x}, \tag{2.9}$$

where

$$G_m(x) = \frac{signx}{2}\left(\frac{e^x - e^{-x}}{2} - \sum_{j=1}^{m-1} \frac{x^{2j-1}}{(2j-1)!}\right)$$

is the Green's function or the fundamental solution to the following equation

$$G_m^{(2m)}(x) - G_m^{(2m-2)}(x) = \delta(x),$$

$d = const$, $sgnx = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$ $P_{m-2}(x)$- *some unknown polynomial of degree* $m-2$, $\delta(x)$ *is the Dirac delta function.*

The proof of the theorem 2.2 is given in [13].

By virtue of the Riesz theorem and the representation of the maximizing element it follows that

$$(\ell, \psi_\ell) = \left\| \ell | W_2^{(m,m-1)*} \right\| \cdot \left\| \psi_\ell | W_2^{(m,m-1)} \right\| = \left\| \ell | W_2^{(m,m-1)*} \right\|^2.$$

From this, by virtue of (2.8), it follows that

$$\ell\left(W_2^{(m,m-1)}\right) = \left\| \ell | W_2^{(m,m-1)*} \right\|^2, \overset{\circ}{\ell}\left(W_2^{(m,m-1)}\right) = \left\| \overset{\circ}{\ell} | W_2^{(m,m-1)*} \right\|^2.$$

Theorem 2.3. *The error of explicit finite-difference formulas (2.1) on classes $W_2^{(m,m-1)}(0,1)$, (i.e. the square of the norm of the error functional of explicit finite-difference formulas) is equal to*

$$\left\| \ell | W_2^{(m,m-1)*} \right\|^2 = (-1)^m \left[\sum_{\gamma=0}^{k} C_\gamma \sum_{\beta=0}^{k} C_\beta G_m(h\gamma - h\beta) \right.$$

$$-2h\sum_{\gamma=0}^{k-1}C_{\gamma}^{(1)}\sum_{\beta=0}^{k}C_{\beta}G_{m}'\left(h\gamma-h\beta\right)-h^{2}\sum_{\gamma=0}^{k-1}C_{\gamma}^{(1)}\sum_{\beta=0}^{k-1}C_{\beta}^{(1)}G_{m}''\left(h\gamma-h\beta\right)\Bigg], \quad (2.10)$$

where the functions $G_m'(x)$ *and* $G_m''(x)$ *are given by the equalities*

$$G_{m}'(x)=\frac{signx}{2}\left(\frac{e^{x}+e^{-x}}{2}-\sum_{j=1}^{m-1}\frac{x^{2j-2}}{(2j-2)!}\right),$$

$$G_{m}''(x)=\frac{signx}{2}\left(\frac{e^{x}-e^{-x}}{2}-\sum_{j=1}^{m-2}\frac{x^{2j-1}}{(2j-1)!}\right).$$

Proof. To prove this theorem we will use the maximizing element, i.e. the Riesz element and equalities (2.6)-(2.7), after some calculations we have (see, [6] for more details)

$$\left\|\ell|W_{2}^{(m,m-1)*}\right\|^{2}=(\ell,\psi_{\ell})=(-1)^{m}\Bigg[\sum_{\gamma=0}^{k}C_{\gamma}\sum_{\beta=0}^{k}C_{\beta}G_{m}\left(h\gamma-h\beta\right)$$

$$-2h\sum_{\gamma=0}^{k-1}C_{\gamma}^{(1)}\sum_{\beta=0}^{k}C_{\beta}G_{m}'\left(h\gamma-h\beta\right)-h^{2}\sum_{\gamma=0}^{k-1}C_{\gamma}^{(1)}\sum_{\beta=0}^{k-1}C_{\beta}^{(1)}G_{m}''\left(h\gamma-h\beta\right)\Bigg].$$

□

3. A system of linear algebraic equations for finding optimal coefficients

We now present the standard definitions for functions of a discrete argument (see [19]).

Definition 3.1. A function $\varphi(h\beta)$ is called *a function of a discrete argument* if it is defined on a certain set of integer values β.

Definition 3.2. *The inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is defined as the number

$$[\varphi,\psi]=\sum_{\beta\in B}\varphi(h\beta)\psi(h\beta),$$

provided that the series on the right-hand side converges absolutely.

Definition 3.3. *The convolution* $\varphi(h\beta)*\psi(h\beta)$ of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is defined as the inner product

$$\chi(h\beta)=\varphi(h\beta)*\psi(h\beta)=[\varphi(h\gamma)\psi(h\beta-h\gamma)].$$

Let us now apply the method of indefinite Lagrange multipliers to find the minimum error of explicit finite-difference formulas on classes $W_2^{(m,m-1)}(0,1)$, i.e. norms. For this purpose, we consider the auxiliary function

$$\Psi\left(C,C^{(1)},\lambda\right)=\left\|\ell|W_{2}^{(m,m-1)*}\right\|^{2}-2\left(-1\right)^{m}\sum_{\alpha=0}^{m-2}\lambda_{\alpha}\left(\ell,x^{\alpha}\right)-2\left(-1\right)^{m}\lambda_{m-1}\left(\ell,e^{-x}\right).$$

Here $C=(C_0,C_1,...,C_k)$, $C^{(1)}=\left(C_0^{(1)},C_1^{(1)},...,C_{k-1}^{(1)}\right)$, $\lambda=(\lambda_0,\lambda_1,...,\lambda_{m-1})$.

Now we calculate all partial derivatives with respect to $C_\beta^{(1)}$, $\beta=0,1,...,k-1$ and λ_α, $\alpha=0,1,...,m-1$ from the function $\Psi\left(C,C^{(1)},\lambda\right)$.

Next, equating them to zero, we have

$$\frac{\partial\Psi}{\partial C_\beta^{(1)}}=0,\ \ \beta=0,1,...,k-1,\ \ \frac{\partial\Psi}{\partial\lambda_\alpha}=0,\ \ \alpha=0,1,...,m-1.$$

These equalities give us a system of linear equations for finding $C_\beta^{(1)}$ and λ_α.

$$h\sum_{\gamma=0}^{k-1}C_\gamma^{(1)}G_m''(h\beta-h\gamma)+\alpha h\sum_{\alpha=0}^{m-2}\lambda_\alpha\left(h\beta\right)^{\alpha-1}+\lambda_{m-1}e^{-h\beta}$$

$$=-\sum_{\gamma=0}^{k}C_\gamma G_m'\left(h\beta-h\gamma\right),\ \beta=0,1,...,k, \tag{3.1}$$

$$h\alpha\sum_{\gamma=0}^{k-1}C_\gamma^{(1)}\left(h\gamma\right)^{\alpha-1}=\sum_{\gamma=0}^{k}C_\gamma\left(h\gamma\right)^{\alpha},\quad\alpha=1,2,...,m-2\,, \tag{3.2}$$

$$h\sum_{\gamma=0}^{k-1}C_\gamma^{(1)}e^{-h\gamma}=-\sum_{\gamma=0}^{k}C_\gamma e^{-h\gamma}. \tag{3.3}$$

By solving system (3.1)-(3.3), we obtain the optimal coefficients of $\overset{\circ}{C}{}_\beta^{(1)}$, $\beta=0,1,...,k-1$ explicit finite-difference formulas and the coefficients $\overset{\circ}{\lambda}_\alpha$, $\alpha=0,1,...,m-1$ of the optimal function of a discrete argument.

Turning to the consideration of formula (2.9), it is not difficult to see that from (2.9) and (3.1) it follows that $\psi_\ell'\left(h\beta\right)=0$. These are the Babuška theorems [11].

For the existence and uniqueness of solutions of system (3.1)-(3.3), the following is true:

Theorem 3.4. *A solution to system (3.1)-(3.3) exists and is unique.*

The proof of the theorem 3.4 is similar to the proof of the theorem in work [13].

4. A method for finding coefficients of optimal explicit finite-difference formulas

In this section we present one algorithm for solving a system of general form (3.1)-(3.3). To implement this algorithm, we rewrite the system (3.1)-(3.3) as

$$h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_\gamma^{(1)}G_m''(h\beta-h\gamma)+\overset{\circ}{P}_{m-3}(h\beta)+\overset{\circ}{\lambda}_{m-1}e^{-h\beta}=f_m(h\beta),\ \ \beta=0,1,...,k, \tag{4.1}$$

$$h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_\gamma^{(1)}\left(h\gamma\right)^{\alpha-1}=\frac{1}{\alpha}\sum_{\gamma=0}^{k}C_\gamma(h\gamma)^{\alpha},\ \ \alpha=1,2,...,m-2, \tag{4.2}$$

$$h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}\,e^{-h\gamma}=-\sum_{\gamma=0}^{k}C_{\gamma}e^{-h\gamma}. \tag{4.3}$$

Here $h=\frac{1}{N}$, $N=1,2...$,

$$G'_m(h\beta-h\gamma)=\frac{sign(h\beta-h\gamma)}{2}\left(\frac{e^{h\beta-h\gamma}+e^{h\gamma-h\beta}}{2}-\sum_{j=1}^{m-1}\frac{(h\beta-h\gamma)^{2j-2}}{(2j-2)!}\right), \tag{4.4}$$

$$f_m(h\beta)=-\sum_{\gamma=0}^{k}C_{\gamma}G'_m(h\beta-h\gamma),\beta=0,1,...,k \tag{4.5}$$

$$G''_m(h\beta-h\gamma)=\frac{sign(h\beta-h\gamma)}{2}\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-\sum_{j=1}^{m-2}\frac{(h\beta-h\gamma)^{2j-1}}{(2j-1)!}\right). \tag{4.6}$$

For the system (4.1)-(4.3) under consideration, the unknowns are the optimal coefficients of $\overset{\circ}{C}{}_{\beta}^{(1)}$ explicit finite-difference formulas of the form (2.1) and the coefficients $\overset{\circ}{\lambda}_{\alpha}$, $(\alpha=1,2,...,m-1)$ of the optimal function $\overset{\circ}{P}_{m-3}(h\beta)+\overset{\circ}{\lambda}_{m-1}e^{-h\beta}$ of the discrete argument.

5. Coefficients of optimal explicit finite-difference formulas of Adams type

In this section we will consider explicit finite-difference formulas of the Adams type. If the coefficients C_β in formula (2.1) are defined by the following equality

$$C_\beta=\begin{cases}1 & for\ \beta=k,\\ -1 & for\ \beta=k-1,\\ 0 & for\ \beta=0,1,\ \dots,k-2,\end{cases} \tag{5.1}$$

then the explicit finite-difference formula $\varphi(hk)-\varphi(hk-h)\cong h\sum_{\beta=0}^{k-1}C_\beta^{(1)}\varphi'(h\beta)$ is called *an explicit finite-difference formula* of Adams type [11].

Denote the set of integers $0,1,...,k-1$ by $\Omega=\{0,1,...,k-1\}$. Considering that the coefficients of explicit finite-difference formulas $\overset{\circ}{C}{}_{\beta}^{(1)}$ are equal to zero at $\beta\overline{\in}\,\Omega$, i.e. at $\beta=\begin{cases}...-3,-2,-1,\\ k,k+1,k+2,...,\end{cases}$ and using Definition 3.3, we rewrite the system (4.1)-(4.3) in the form of equations in convolutions

$$hG''_m(h\beta)*\overset{\circ}{C}{}_{\beta}^{(1)}+\overset{\circ}{P}_{m-3}(h\beta)+\overset{\circ}{\lambda}_{m-1}e^{-h\beta}=f_m(h\beta)\ for\quad \beta\in\Omega, \tag{5.2}$$

$$\overset{\circ}{C}{}_{\beta}^{(1)}=0\quad for\quad \beta\overline{\in}\,\Omega, \tag{5.3}$$

$$h\sum_{\beta=0}^{k-1}\overset{\circ}{C}{}_{\beta}^{(1)}(h\beta)^{\alpha-1}=g_\alpha,\qquad \alpha=1,2,...,m-2, \tag{5.4}$$

$$h\sum_{\beta=0}^{k-1}\overset{\circ}{C}{}_{\beta}^{(1)}\,e^{-h\gamma}=g_k. \tag{5.5}$$

Here the unknowns are the coefficients of $\overset{\circ}{C}{}_{\beta}^{(1)}$ explicit finite-difference formulas of the Adams type and the coefficients $\overset{\circ}{\lambda}_{\alpha}$, $\alpha=1,2,...,m-1$ of the optimal function $\overset{\circ}{P}_{m-3}(h\beta)+\overset{\circ}{\lambda}_{m-1}e^{-h\beta}$ of the discrete argument,

$$f_m(h\beta)=\begin{cases}\frac{1}{2}\left(\frac{e^{h\beta-hk}}{2}\left(1-e^{h}\right)+\frac{e^{hk-h\beta}}{2}\left(1-e^{-h}\right)-\right.\\ \left.-\sum_{j=2}^{m-1}\left(\frac{(h\beta-hk)^{2j-2}}{(2j-2)!}-\frac{(h\beta-hk+h)^{2j-2}}{(2j-2)!}\right)\right),\ \beta=0,1,...,k-1,\\ \frac{1}{2}\left(\frac{e^{h}+e^{-h}}{2}-\sum_{j=1}^{m-1}\frac{(h)^{2j-2}}{(2j-2)!}\right),\ \beta=k,\end{cases} \tag{5.6}$$

$$g_{\alpha}=\frac{h^{\alpha}}{\alpha}\left(k^{\alpha}-(k-1)^{\alpha}\right),\ \ \alpha=1,2,\ \ldots,m-2, \tag{5.7}$$

$$g_k=e^{-hk+h}-e^{-hk}. \tag{5.8}$$

Usually such systems of the form (5.2)-(5.5) are called discrete systems of the Wiener-Hopf type.

The following theorems are valid for representing the optimal coefficients of explicit finite-difference formulas of the Adams type.

Theorem 5.1. *For the representation of optimal coefficients* $\overset{\circ}{C}{}_{\beta}^{(1)}$, $\beta=1,2,\ \ldots,k-2$ *of explicit finite-difference formulas of the Adams type on classes* $W_2^{(m,m-1)}(0,1)$ *for* $m\geq 3$ *the following formula is valid*

$$\overset{\circ}{C}{}_{\beta}^{(1)}=\sum_{j=1}^{m-2}\left(M_j\cdot\lambda_j^{\beta}+N_j\lambda_j^{k-\beta}\right)\ \ for\ \ \beta=1,2,\ \ldots,k-2, \tag{5.9}$$

here λ_j *are the roots of the polynomial* $P_{2m-4}(\lambda)$ *of degree* $2m-4$, $|\lambda_j|<1$, M_j *and* N_j *unknown parameters.*

Proof. To find the representation of the optimal coefficients of explicit finite-difference formulas, we calculate the convolution

$$\overset{\circ}{C}{}_{\beta}^{(1)}=h^{-1}D_{m-1}(h\beta)*U(h\beta). \tag{5.10}$$

Here $D_{m-1}(h\beta)$ is a discrete analogue of the differential operator $\frac{d^{2m-2}}{dx^{2m-2}}-\frac{d^{2m-4}}{dx^{2m-4}}$ (see, [12, 13]) and the function of the discrete argument $U(h\beta)$ is defined by the formula:

$$U(h\beta)=\begin{cases}-\frac{e^{h\beta}}{4}g_k+a^{-}e^{-h\beta}+Q^{(2m-5)}(h\beta)+Q_{m-3}^{(-)}(h\beta), & \beta\leq -1,\\ f_m(h\beta), & 0\leq\beta\leq k,\\ \frac{e^{h\beta}}{4}g_k+a^{+}e^{-h\beta}-Q^{(2m-5)}(h\beta)+Q_{m-3}^{(+)}(h\beta), & \beta\geq k+1.\end{cases}.$$

Then,

$$\overset{\circ}{C}{}_{\beta}^{(1)}=h^{-1}D_{m-1}(h\beta)*U(h\beta)=h^{-1}\sum_{\gamma=-\infty}^{\infty}D_{m-1}(h\beta-h\gamma)U(h\gamma)$$

$$= h^{-1}\sum_{\gamma=-\infty}^{-1} D_{m-1}(h\beta-h\gamma)\left(-\frac{e^{h\gamma}}{4}g_k + a^- e^{-h\gamma} + Q^{(2m-5)}(h\gamma) + Q^{(-)}_{m-3}(h\gamma)\right)$$

$$+h^{-1}\sum_{\gamma=0}^{k-1} D_{m-1}(h\beta-h\gamma) f_m(h\gamma)$$

$$+ h^{-1}\sum_{\gamma=k}^{\infty} D_{m-1}(h\beta-h\gamma)\left(\frac{e^{h\beta}}{4}g_k + a^+ e^{-h\beta} - Q^{(2m-5)}(h\beta) + Q^{(+)}_{m-3}(h\beta)\right). \tag{5.11}$$

In formula (5.11) $D_{m-1}(h\beta)$ is a discrete analogue of the differential operator and the unknowns $Q^-_{m-3}(h\beta)$ and $Q^+_{m-3}(h\beta)$ polynomials of the discrete argument of degree $m-3$.

Let us denote by S_1 the sum of the form

$$S_1 = h^{-1}\sum_{\gamma=0}^{k-1} D_{m-1}(h\beta-h\gamma) f_m(h\gamma). \tag{5.12}$$

Then add and subtract rows

$h^{-1}\sum_{\gamma=-\infty}^{-1} D_{m-1}(h\beta-h\gamma) f_m(h\gamma)$ and $h^{-1}\sum_{\gamma=k}^{\infty} D_{m-1}(h\beta-h\gamma) f_m(h\gamma)$

to sum up S_1 we have:

$$S_1 = h^{-1}\sum_{\gamma=-\infty}^{\infty} D_{m-1}(h\beta-h\gamma) f_m(h\gamma) - h^{-1}\sum_{\gamma=-\infty}^{-1} D_{m-1}(h\beta-h\gamma) f_m(h\gamma)$$

$$-h^{-1}\sum_{\gamma=k}^{\infty} D_{m-1}(h\beta-h\gamma) f_m(h\gamma).$$

Since in our case $f_m(h\beta)$ the function is defined by formula (5.6), then we have:

$$h^{-1}D_{m-1}(h\beta) * f_m(h\beta) = h^{-1}\sum_{\gamma=-\infty}^{\infty} D_{m-1}(h\beta-h\gamma) f_m(h\gamma)$$

$$= h^{-1}\sum_{\gamma=-\infty}^{\infty} D_{m-1}(h\gamma) f_m(h\beta-h\gamma) = 0. \tag{5.13}$$

Due to (5.13) the expression S_1 takes the following form:

$$S_1 = -h^{-1}\sum_{\gamma=-\infty}^{-1} D_{m-1}(h\beta-h\gamma) f_m(h\gamma) - h^{-1}\sum_{\gamma=k}^{\infty} D_{m-1}(h\beta-h\gamma) f_m(h\gamma). \tag{5.14}$$

Using formulas (5.12), (5.13) and (5.14), we write expressions (5.11) in the following form:

$$\overset{\circ}{C}{}^{(1)}_{\beta} = h^{-1}\sum_{\gamma=1}^{\infty} D_{m-1}(h\beta+h\gamma)\Bigg(-\frac{e^{-h\gamma}}{4}g_k + a^- e^{h\gamma} + Q^{(2m-5)}(-h\gamma)$$

$$+Q^{(-)}_{m-3}(-h\gamma) - f_m(-h\gamma)\Bigg) + h^{-1}\sum_{\gamma=1}^{\infty} D_{m-1}(h\beta-hk-h\gamma+h)\left(\frac{e^{h\gamma+hk-h}}{4}g_k\right.$$

$$+a^{+}e^{-h\gamma-hk+h}-Q^{(2m-5)}(h\gamma+hk-h)+Q_{m-3}^{(+)}(h\gamma+hk-h)-f_m(h\gamma+hk-h)\Bigg).$$

Finally, we obtain the following representation of the coefficients $\overset{\circ}{C}{}_{\beta}^{(1)}$ of optimal explicit finite-difference formulas of the Adams type.

$$\overset{\circ}{C}{}_{\beta}^{(1)}=\sum_{j=1}^{m-2}\left(M_j\lambda_j^{\beta}+N_j\lambda_j^{k-\beta}\right)\ for\ 1\le\beta\le k-2, \tag{5.15}$$

here

$$M_j=\frac{A_j}{hp_{2m-4}^{(2m-4)}}\sum_{\gamma=1}^{\infty}\lambda_j^{\gamma-1}\left(-\frac{e^{-h\gamma}}{4}g_k+a^{-}e^{h\gamma}\right.$$

$$\left.+Q^{(2m-5)}(-h\gamma)+Q_{m-3}^{(-)}(-h\gamma)-f_m(-h\gamma)\right), \tag{5.16}$$

$$N_j=\frac{A_j}{hp_{2m-4}^{(2m-4)}}\sum_{\gamma=1}^{\infty}\lambda_j^{\gamma-2}\left(\frac{e^{h\gamma+hk-h}}{4}g_k+a^{+}e^{-h\gamma-hk+h}-Q^{(2m-5)}(h\gamma+hk-h)\right.$$

$$\left.+Q_{m-3}^{(+)}(h\gamma+hk-h)-f_m(h\gamma+hk-h)\right) \tag{5.17}$$

$$A_j=\frac{2\left(1-\lambda_j\right)^{2m-4}\left[\lambda_j\left(e^{2h}+1\right)-e^{h}\left(\lambda_j^{2}+1\right)\right]p_{2m-4}^{(2m-4)}}{\lambda_j P'_{2m-4}\left(\lambda_j\right)},$$

$$P_{2m-4}(\lambda)=\sum_{s=0}^{2m-4}p_s^{(2m-4)}\lambda^s=\left(1-e^{2h}\right)\left(1-\lambda\right)^{2m-4}-2\left(\lambda\left(e^{2h}+1\right)-e^{h}\left(\lambda^{2}+1\right)\right)$$

$$\times\left[h\left(1-\lambda\right)^{2m-6}+\frac{h^3\left(1-\lambda\right)^{2m-8}}{3!}E_2(\lambda)+...+\frac{h^{2m-5}E_{2m-6}(\lambda)}{(2m-5)!}\right],$$

$p_{2m-4}^{(2m-4)}$, $p_{2m-5}^{(2m-4)}$- coefficients of the polynomial $P_{2m-4}(\lambda)$, λ_j- roots of the polynomial $P_{2m-4}(\lambda)$, less than one in modulus, E_{2m-6} the well-known Euler polynomial of degree $2m-6$ (see, [12, 13]). □

Theorem 5.2. *The coefficients* $\overset{\circ}{C}{}_{0}^{(1)}$ *and* $\overset{\circ}{C}{}_{k-1}^{(1)}$ *of optimal explicit finite-difference formulas of Adams type on classes are expressed* $W_2^{(m,m-1)}(0,1)$ *by* $m\ge 3$ *the following representations:*

$$\overset{\circ}{C}{}_{0}^{(1)}=\frac{e^h-he^h-1}{he^h(e^{hk-h}-1)}+\frac{1}{e^{hk-h}-1}\sum_{j=1}^{m-2}\left[\frac{(M_j+N_j\lambda_j)(1-\lambda_j^{k-2})\lambda_j}{1-\lambda_j}\right.$$

$$\left.-\frac{M_j(e^{hk-2h}-\lambda_j^{k-2})(\lambda_j e^h-1)\lambda_j e^h+N_j(\lambda_j^{k-2}e^{hk-2h}-1)(e^h-\lambda_j)\lambda_j^2e^h}{(e^h-\lambda_j)(\lambda_j e^h-1)}\right], \tag{5.18}$$

$$\overset{\circ}{C}{}_{k-1}^{(1)}=\frac{he^{hk}-e^h+1}{he^h(e^{hk-h}-1)}-\frac{e^{hk-h}}{e^{hk-h}-1}\sum_{j=1}^{m-2}\left[\frac{(M_j+N_j\lambda_j)(1-\lambda_j^{k-2})\lambda_j}{1-\lambda_j}\right.$$

$$-\frac{M_j(e^{hk-2h}-\lambda_j^{k-2})(\lambda_j e^h-1)\lambda_j+N_j(\lambda_j^{k-2}e^{hk-2h}-1)(e^h-\lambda_j)\lambda_j^2}{(e^h-\lambda_j)(\lambda_j e^h-1)e^{hk-2h}}\Bigg]. \quad (5.19)$$

Proof. We will determine the representations for the optimal coefficients $\overset{\circ}{C}{}_0^{(1)}$ and $\overset{\circ}{C}{}_{k-1}^{(1)}$ explicit finite-difference formulas from the system (5.4) - (5.5). To do this, we will take one equation for $\alpha = 1$, equality (5.5) and one more equation from the system (5.4). Substituting for $\overset{\circ}{C}{}_\beta^{(1)}$ for $\beta = 1, 2, ..., k-2$ in expression (5.9), after simple simplifications we obtain a system of linear equations for finding the unknowns $\overset{\circ}{C}{}_0^{(1)}$ and $\overset{\circ}{C}{}_{k-1}^{(1)}$

$$\begin{cases} \overset{\circ}{C}{}_0^{(1)}+\overset{\circ}{C}{}_{k-1}^{(1)}=1-\sum_{\gamma=1}^{k-2}\overset{\circ}{C}{}_\gamma^{(1)}, \\ \overset{\circ}{C}{}_0^{(1)}+\overset{\circ}{C}{}_{k-1}^{(1)}e^{-hk+h}=h^{-1}\left(e^{-hk+h}-e^{-hk}\right)-\sum_{\gamma=1}^{k-2}\overset{\circ}{C}{}_\gamma^{(1)}e^{-h\gamma}. \end{cases}$$

Solving this system, we find the unknowns $\overset{\circ}{C}{}_0^{(1)}$ and $\overset{\circ}{C}{}_{k-1}^{(1)}$ □

6. Conclusion

In this work, a functional formulation of the problem of optimization of explicit finite-difference formulas in the space $W_2^{(m,m-1)}(0,1)$ is presented. Here, the extremal function of the error of explicit finite-difference formulas is determined and, using this function, the error of an explicit finite-difference formula on classes $W_2^{(m,m-1)}(0,1)$ is found. In addition, by minimizing the error on classes $W_2^{(m,m-1)}(0,1)$ by the coefficients of explicit finite-difference formulas, a system of linear algebraic equations is obtained. Further, the uniqueness of solutions to this system is proved. Finally, representations of the optimal coefficients of explicit finite-difference formulas of Adams type are found for any $m \geq 3$.

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