A CHARACTERIZATION OF GENERALIZED HIGHER LIE DERIVATIONS ON ALGEBRAS

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Abstract. Let \( A \) and \( B \) be two algebras. In this paper, under certain conditions, a characterization of generalized higher Lie derivations from \( A \) into \( B \) as a sequence of generalized Lie derivations will be presented.

1. Introduction and preliminaries

Throughout this paper, \( A \) and \( B \) will represent two algebras (associative or non-associative). Let \( \sigma : A \to B \) be a linear mapping and \( X \) be a \( B \)-bimodule. A linear mapping \( d : A \to X \) is called a \( \sigma \)-derivation if \( d(ab) = d(a)\sigma(b) + \sigma(a)d(b) \) holds for all \( a, b \in A \). Hosseini et al. [4] extended this concept and defined generalized \( \sigma \)-derivation as follows:

A linear mapping \( \delta : A \to X \) is called a generalized \( \sigma \)-derivation if \( \delta(ab) = \delta(a)\sigma(b) + \sigma(a)d(b) \), where \( d \) is a \( \sigma \)-derivation (for more details see [3], [4] and [5]).

Like most authors, we denote the commutator \( ab - ba \) by \([a, b]\) for all pair \( a, b \in A \). Let \( \theta : A \to B \) be a linear mapping. A linear mapping \( \delta : A \to B \) is called a \( \theta \)-Lie derivation if it satisfies \( \delta[a, b] = [\delta(a), \theta(b)] + [\theta(a), \delta(b)] \) for all \( a, b \in A \). If \( A = B \) and \( \theta = id \), the identity mapping on \( A \), then \( \delta \) is called a Lie derivation. Furthermore, a linear mapping \( \Delta : A \to B \) is called a generalized Lie derivation if there exists a Lie derivation \( \delta : A \to B \) such that \( \Delta([a, b]) = [\Delta(a), b] + [a, \delta(b)] \) holds for all \( a, b \in A \). For many years, there has been increasing interest on the study of Lie-type mappings of associative rings and operator algebras, such as Lie isomorphisms and Lie derivations. Recall that a linear mapping \( T : A \to B \) is called a Lie homomorphism if \( T([a, b]) = [T(a), T(b)] \) holds for all \( a, b \in A \). There is an important decomposition of Lie derivations on some Banach algebras. For example, Johnson showed in [6] that every continuous Lie derivation from a symmetrically amenable Banach algebra \( A \) into a Banach \( A \)-bimodule \( X \) decomposes into a sum of an ordinary derivation from \( A \) into \( X \) and a linear mapping from \( A \) into the center of \( X \). Alaminos, et al [1] jointly proved that every Lie derivation on a symmetrically amenable semisimple Banach algebra also has the same decomposition. Moreover, Mathieu and Villena [8] obtained...
that every (not necessarily bounded) Lie derivation $D$ on a $C^*$-algebra $A$ can be uniquely decomposed into the sum of a derivation $d$ of $A$ and a linear mapping $\psi$ from $A$ into its center $Z(A)$.

First, in this note, we define generalized higher Lie derivations and then, by getting idea from [7], a useful characterization of them is presented as a sequence of generalized Lie derivations. This is a very effective method for characterizing higher Lie derivations. A sequence $\{d_n\}$ of linear mappings from $A$ into $B$ is called a higher Lie derivation if $d_n([a, b]) = \sum_{k=0}^{n}[d_k(a), d_{n-k}(b)]$ for each $a, b \in A$ and each non-negative integer $n$. A sequence $\{D_n\}$ of linear mappings from $A$ into $B$ is said to be a generalized higher Lie derivation if there exists a higher Lie derivation $\{d_n\}$ from $A$ into $B$ such that $D_n([a, b]) = \sum_{k=0}^{n}[D_k(a), d_{n-k}(b)]$ for each $a, b \in A$ and each non-negative integer $n$. Let $d_n$ be a higher Lie derivation. Then $d_0$ is a Lie homomorphism and $d_1$ is a $d_0$-Lie derivation that is $d_1([a, b]) = [d_0(a), d_1(b)] + [d_1(a), d_0(b)]$ for all $a, b \in A$. So, if $d_0$ is onto, then $\tilde{d}_0 : A/\ker(d_0) \to B$ defined by $\tilde{d}_0(a + \ker(d_0)) = d_0(a)$ is a Lie isomorphism. It is evident that, for each $n \in \mathbb{N}$, $\tilde{d}_n : A/\ker(d_0) \to B$ defined by $\tilde{d}_n(a + \ker(d_0)) = d_n(a)$ is a well-defined linear map if and only if $\ker(d_0) \subseteq \ker(d_n)$.

2. Results and proofs

**Definition 2.1.** i) A sequence $\{d_n\}$ of linear mappings from $A$ into $B$ is called a higher Lie derivation if $d_n([a, b]) = \sum_{k=0}^{n}[d_k(a), d_{n-k}(b)]$ holds for each pair $a, b \in A$ and each non-negative integer $n$.

ii) A sequence $\{D_n\}$ of linear mappings from $A$ into $B$ is called a generalized higher Lie derivation if there exists a higher Lie derivation $\{d_n\}$ from $A$ into $B$ such that $D_n([a, b]) = \sum_{k=0}^{n}[D_k(a), d_{n-k}(b)]$ holds for each pair $a, b \in A$ and each non-negative integer $n$.

We begin our results with the following proposition which will be used extensively to prove our theorems.

**Proposition 2.2.** Suppose that $\{d_n\}$ is a higher Lie derivation from $A$ into $B$ with $d_0(A) = B$ and $\ker(d_0) \subseteq \ker(d_n)$ ($n \in \mathbb{N}$). Then, there is a sequence $\{\delta_n\}$ of Lie derivations on $B$ such that

$$(n+1)\tilde{d}_{n+1} = \sum_{k=0}^{n} \delta_{k+1} \tilde{d}_{n-k}, \quad (2.1)$$

for each non-negative integer $n$.

**Proof.** Let us take an inductive approach for the index $n$. For $n = 0$, we define $\delta_1 : B \to B$ by $\delta_1 = \tilde{d}_1^{-1}d_0$. Let $b_1$ and $b_2$ be two arbitrary fixed elements of $B$. Since $\tilde{d}_0$ is a Lie isomorphism, there exist $a_1, a_2 \in A$ such that $d_0(a_1) = b_1$ and $d_0(a_2) = b_2$. We have $\tilde{d}_0(a_1 + \ker(d_0)) = b_1$ and $\tilde{d}_0(a_2 + \ker(d_0)) = b_2$.

By applying $d_1$ to $[a_1, a_2]$, we get $d_1([a_1, a_2]) = \tilde{d}_0([a_1, a_2] + \ker(d_0)) = \tilde{d}_0(a_1a_2 - a_2a_1) = d_0(a_1a_2) - d_0(a_2a_1) = 0$.

Since $\tilde{d}_0$ is a Lie isomorphism, $a_1a_2 = a_2a_1$ in $A/\ker(d_0)$. Thus, $d_1([a_1, a_2]) = 0$.

By applying $d_2$ to $d_1([a_1, a_2])$, we get $d_2(d_1([a_1, a_2])) = \tilde{d}_0(d_1([a_1, a_2] + \ker(d_0))) = \tilde{d}_0([d_0(a_1), d_0(a_2)]) = d_0([a_1, a_2]) = 0$.

By applying $d_3$ to $d_2(d_1([a_1, a_2]))$, we get $d_3(d_2(d_1([a_1, a_2]))) = \tilde{d}_0(d_2(d_1([a_1, a_2] + \ker(d_0)))) = \tilde{d}_0([d_0([a_1, a_2]), d_0(a_2)]) = d_0([a_1, a_2]) = 0$.

Continuing this process, we get $d_n([a_1, a_2]) = 0$ for all $n \in \mathbb{N}$.

Therefore, $\delta_1 = \tilde{d}_1^{-1}d_0$ is a Lie isomorphism. It is evident that, for each $n \in \mathbb{N}$, $\delta_n : A/\ker(d_0) \to B$ defined by $\delta_n(a + \ker(d_0)) = d_n(a)$ is a well-defined linear map if and only if $\ker(d_0) \subseteq \ker(d_n)$.
\[ d_0(a_2) = b_2. \] We, therefore, have
\[
\delta_1([b_1, b_2]) = \tilde{d}_1 \tilde{d}_0^{-1}([d_0(a_1), d_0(a_2)])
\]
\[
= d_1([a_1, a_2])
\]
\[
= [d_0(a_1), \tilde{d}_1(a_2)] + [d_1(a_1), d_0(a_2)]
\]
\[
= [d_0(a_1), \tilde{d}_1 \tilde{d}_0^{-1}(d_0(a_2))] + [\tilde{d}_1 \tilde{d}_0^{-1}(d_0(a_1)), d_0(a_2)]
\]
\[
= [b_1, \delta_1(b_2)] + [\delta_1(b_1), b_2].
\]

Thus, \( \delta_1 \) is a Lie derivation. Note that \( \tilde{d}_1 = \delta_1 \tilde{d}_0 \). Now assume that \( \delta_k \) is defined and is a Lie derivation for \( k \leq n \) satisfying (2.1). Putting \( \delta_{n+1} = \left( (n + 1) \tilde{d}_{n+1} - \sum_{n=0}^{n-1} \delta_{k+1} \tilde{d}_{n-k} \right) \tilde{d}_0^{-1} \). Thus, the well-defined mapping \( \delta_{n+1} \) is a Lie derivation. For \( b_1, b_2 \in B \), there are \( a_1, a_2 \in A \) such that \( \delta_1, \delta_2, \ldots, \delta_n \) are Lie derivation and also \( d_0(a_1) = b_1 \) and \( d_0(a_2) = b_2 \). Consequently
\[
\delta_{n+1}([b_1, b_2]) = \left( (n + 1) \tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} \tilde{d}_{n-k} \right) \tilde{d}_0^{-1}([d_0(a_1), d_0(a_2)])
\]
\[
= \left( (n + 1) \tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} \tilde{d}_{n-k} \right) ([a_1 + ker(d_0), a_2 + ker(d_0)])
\]
\[
= (n + 1) d_{n+1}([a_1, a_2]) - \sum_{k=0}^{n-1} \delta_{k+1} d_{n-k}([a_1, a_2])
\]
\[
= (n + 1) \sum_{k=0}^{n+1} \left[ d_k(a_1), d_{n+1-k}(a_2) \right] - \sum_{k=0}^{n-1} \delta_{k+1} \left( \sum_{l=0}^{n-k} \left[ d_l(a_1), d_{n-k-l}(a_2) \right] \right)
\]
\[
= \sum_{k=0}^{n+1} (k + n + 1 - k) \left[ d_k(a), d_{n+1-k}(a_1) \right] + \sum_{k=0}^{n-1} \delta_{k+1} \left( \sum_{l=0}^{n-k} \left[ d_l(a), d_{n-k-l}(a_2) \right] \right)
\]
\[
= \sum_{k=0}^{n+1} k \left[ d_k(a_1), d_{n+1-k}(a_2) \right] + \sum_{k=0}^{n+1} (n + 1 - k) \left[ d_k(a), d_{n+1-k}(b) \right]
\]
\[
- \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \left( [\delta_{k+1} d_l(a_1), d_{n-k-l}(a_2)] + [d_l(a_1), \delta_{k+1} d_{n-k-l}(a_2)] \right).
\]

Writing
\[
K = \sum_{k=0}^{n+1} \sum_{l=0}^{n-k} \left[ d_k(a_1), d_{n+1-k-l}(a_2) \right] - \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \left[ \delta_{k+1} d_l(a_1), d_{n-k-l}(a_2) \right],
\]
and
\[
L = \sum_{k=0}^{n+1} (n + 1 - k) \left[ d_k(a_1), d_{n+1-k}(a_2) \right] - \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \left[ d_l(a_1), \delta_{k+1} d_{n-k-l}(a_2) \right],
\]
we have \( \delta_{n+1}([b_1, b_2]) = K + L \). Our next task is to compute \( K \) and \( L \). In the
sum\ation \sum_{k=0}^{n-1} \sum_{l=0}^{n-k}, \text{ we have } 0 \leq k + l \leq n \text{ and } k \neq n. \text{ So, it can be written as the form } \sum_{r=0}^{n} \sum_{k+l=r, k \neq n} \text{ where } r = k + l. \text{ Putting } l = r - k, \text{ it is obtained that }

\begin{align*}
K &= \sum_{k=0}^{n+1} k \left[ d_k(a_1), d_{n+1-k}(a_2) \right] - \sum_{r=0}^{n} \sum_{k+l=r, k \neq n} \left[ \delta_{k+1} d_{r-k}(a), d_{n-r}(a_2) \right] \\
&= \sum_{k=0}^{n+1} k \left[ d_k(a), d_{n+1-k}(b) \right] - \sum_{r=0}^{n-1} \sum_{k=0}^{r} \left[ \delta_{k+1} d_{r-k}(a_1), d_{n-r}(a_2) \right] \\
&\quad - \sum_{k=0}^{n-1} \left[ \delta_{k+1} d_{n-k}(a_1), d_0(a_2) \right].
\end{align*}

We substitute \( r + 1 \) for \( k \) in the first summation and also, by our assumption, for \( r = 0, 1, \ldots, n - 1 \) we arrive at \( (r+1) d_{r+1}([a, b]) = \sum_{k=0}^{r} \delta_{k+1} d_{r-k}([a, b]). \) Hence, it can be concluded that

\begin{align*}
K &= \sum_{r=-1}^{n} (r+1) \left[ d_{r+1}(a_1), d_{n-r}(a_2) \right] - \sum_{r=0}^{n-1} \sum_{k=0}^{r} \left[ \delta_{k+1} d_{r-k}(a_1), d_{n-r}(a_2) \right] \\
&\quad - \sum_{k=0}^{n-1} \left[ \delta_{k+1} d_{n-k}(a_1), d_0(a_2) \right] \\
&= \sum_{r=0}^{n-1} \left[ (r+1) \tilde{d}_{r+1} - \sum_{k=0}^{r} \delta_{k+1} \tilde{d}_{r-k} \right] (a_1), d_{n-r}(a_2) \right] \\
&\quad + (n+1) \left[ d_{n+1}(a_1), d_0(a_2) \right] - \sum_{k=0}^{n-1} \left[ \delta_{k+1} d_{n-k}(a_1), d_0(a_2) \right] \\
&= \sum_{r=0}^{n-1} \left[ (r+1) \tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} \tilde{d}_{n-k} \right] \tilde{d}_0^{-1} d_0(a_1), d_{n-r}(a_2) \right] \\
&\quad + (n+1) \left[ d_{n+1}(a_1), d_0(a_2) \right] - \sum_{k=0}^{n-1} \left[ \delta_{k+1} d_{n-k}(a_1), d_0(a_2) \right] \\
&= \left[ (n+1) \tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} \tilde{d}_{n-k} \right] (a_1), d_0(a_2) \right] \\
&= \left[ (n+1) \tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} \tilde{d}_{n-k} \right] \tilde{d}_0^{-1} d_0(a_1), d_0(a_2) \right] \\
&= \left[ \delta_{n+1} \tilde{d}_0 \left( \tilde{d}_0^{-1} d_0(a_1) \right), d_0(a_2) \right] \\
&= \left[ \delta_{n+1}(b_1), b_2 \right].
\end{align*}
Also in the same way

\[ L = \left[ d_0(a_1), \left( (n + 1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} \delta_{n-k} \right) \tilde{d}_0^{-1} d_0(a_2) \right] = \left[ b_1, \delta_{n+1}(b_2) \right]. \]

Finally, we conclude

\[ \delta_{n+1}([b_1, b_2]) = K + L = \left[ \delta_{n+1}(b_1), b_2 \right] + \left[ b_1, \delta_{n+1}(b_2) \right], \]

for all \( b_1, b_2 \in A \). This implies that \( \delta_{n+1} \) is a Lie derivation of \( A \) and it completes the proof. \( \square \)

Suppose that \( \{D_n\} \) is a generalized higher Lie derivation from \( A \) into \( B \). We define \( \tilde{D}_n : A/\ker(D_0) \to B \) with \( \tilde{D}_n(a + \ker(D_0)) = D_n(a) \) in which \( D_0(A) = B \) and \( \ker(D_0) \subseteq \ker(D_n) \).

**Theorem 2.3.** Let \( \{D_n\} \) be a generalized higher Lie derivation from \( A \) into \( B \) associated with a higher Lie derivation \( \{d_n\} \) from \( A \) into \( B \), i.e. \( D_n([a_1, a_2]) = \sum_{k=0}^{n} [D_k(a_1), d_{n-k}(a_2)] \) holds for each \( a_1, a_2 \in A \) and each non-negative integer \( n \). Moreover, suppose that \( d_0(A) = D_0(A) = B \), \( \ker(d_0) \subseteq \ker(d_n) \) and \( \ker(D_0) \subseteq \ker(D_n) \) for each non-negative integer \( n \). Then, there is a sequence \( \{\Delta_n\} \) of generalized Lie derivations on \( B \) associated with a sequence \( \{\delta_n\} \) of Lie derivations such that for each non-negative integer \( n \),

\[ (n + 1)\tilde{D}_{n+1} = \sum_{k=0}^{n} \Delta_{k+1}\tilde{D}_{n-k}. \] (2.2)

Moreover, suppose that \( \{D_n\} \) be a generalized higher Lie derivation from \( A \) into \( B \) associated with a higher Lie derivation \( \{d_n\} \) from \( A \) into \( B \), i.e. \( D_n([a_1, a_2]) = \sum_{k=0}^{n} [D_k(a_1), d_{n-k}(a_2)] \) holds for each \( a_1, a_2 \in A \) and each non-negative integer \( n \). Furthermore,

\[ D_n = \sum_{i=1}^{n} \left( \sum_{\sum_{j=1}^{i} r_j = n} \prod_{j=1}^{i} \frac{1}{r_j + r_{j+1} + \ldots + r_i} \right) \Delta_{r_1} \Delta_{r_2} \ldots \Delta_{r_i} D_0, \] (2.3)

where the inner summation is taken over all positive integers \( r_1, r_2, \ldots, r_n \) with \( \sum_{j=1}^{i} r_j = n \).

**Proof.** We prove this theorem by using Proposition 2.2 and induction on \( n \). For \( n = 0 \), let \( \Delta_1 : B \to B \) be defined by \( \Delta_1 = \tilde{D}_1 \tilde{D}_0^{-1} \). Since \( D_0 \) and \( d_0 \) are surjective, there exist \( a_1, a_2 \in A \) such that \( D_0(a_1) = b_1 \) and \( d_0(a_2) = b_2 \), where \( b_1 \) and \( b_2 \) are
two arbitrary fixed elements of $\mathcal{B}$. Consequently,
\[
\Delta_1([b_1, b_2]) = \tilde{D}_1 \tilde{D}_0^{-1}([D_0(a_1), d_0(a_2)]) \\
= \tilde{D}_1 \tilde{D}_0^{-1}(D_0([a_1, a_2])) \\
= D_1([a_1, a_2]) \\
= [D_0(a_1), d_1(a_2)] + [D_1(a_1), d_0(a_2)] \\
= [D_0(a_1), \tilde{d}_1 \tilde{D}_0^{-1}(d_0(a_2))] + [\tilde{D}_1 \tilde{D}_0^{-1}(D_0(a_1), d_0(a_2))] \\
= [b_1, \delta_1(b_2)] + [\Delta_1(b_1), b_2].
\]

Since $\delta_1$ is a Lie derivation, $\Delta_1$ is a generalized Lie derivation. Note that $\tilde{D}_1 = \Delta_1 \tilde{D}_0$. Now, assume that $\Delta_k$ is defined and is a generalized Lie derivation for $k \leq n$ satisfying (2.2) such that $\Delta_k([b_1, b_2]) = [\Delta_k(b_1), b_2] + [b_1, \delta_k(b_2)]$ for each $b_1, b_2 \in \mathcal{B}$. Put $\Delta_{n+1} = (n + 1) \tilde{D}_{n+1} - \sum_{k=0}^{n-1} \Delta_{k+1} \tilde{D}_{n-k} \tilde{D}_0^{-1}$, and $a_1, a_2 \in \mathcal{A}$ such that $D_0(a_1) = b_1$ and $d_0(a_2) = b_2$. Thus,
\[
\Delta_{n+1}([b_1, b_2]) = (n + 1) \tilde{D}_{n+1} - \sum_{k=0}^{n-1} \Delta_{k+1} \tilde{D}_{n-k} \tilde{D}_0^{-1}(D_0([a_1, a_2])) \\
= (n + 1) \tilde{D}_{n+1} - \sum_{k=0}^{n-1} \Delta_{k+1} \tilde{D}_{n-k}([a_1 + \ker D_0, a_2 + \ker D_0]) \\
= (n + 1) \tilde{D}_{n+1}([a_1, a_2]) - \sum_{k=0}^{n-1} \Delta_{k+1} D_{n-k}([a_1, a_2]) \\
= (n + 1) \sum_{k=0}^{n-1} [D_k(a_1), d_{n+1-k}(a_2)] - \sum_{k=0}^{n-1} \Delta_{k+1} \left( \sum_{l=0}^{n-k} [D_l(a_1), d_{n-k-l}(a_2)] \right).
\]

Since $\Delta_k([b_1, b_2]) = [\Delta_k(b_1), b_2] + [b_1, \delta_k(b_2)]$ for $k \leq n$, we have
\[
\Delta_{n+1}([b_1, b_2]) = \sum_{k=0}^{n+1} k [D_k(a_1), d_{n+1-k}(a_2)] + \sum_{k=0}^{n+1} (n + 1 - k) [D_k(a_1), d_{n+1-k}(a_2)] \\
- \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \left( [\Delta_{k+1} D_l(a_1), d_{n-k-l}(a_2)] + [D_l(a_1), \delta_{k+1} d_{n-k-l}(a_2)] \right).
\]

Writing
\[
K = \sum_{k=0}^{n+1} k [D_k(a_1), d_{n+1-k}(a_2)] - \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \left[ \Delta_{k+1} D_l(a_1), d_{n-k-l}(a_2) \right],
\]
\[
L = \sum_{k=0}^{n+1} (n + 1 - k) [D_k(a_1), d_{n+1-k}(a_2)] - \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \left[ D_l(a_1), \delta_{k+1} d_{n-k-l}(a_2) \right],
\]
we have $\Delta_{n+1}([b_1, b_2]) = K + L$. Let us compute $K$ and $L$. In the summation $\sum_{k=0}^{n-1} \sum_{l=0}^{n-k}$, we have $0 \leq k + l \leq n$ and $k \neq n$. So we can write it as the form
\[\sum_{r=0}^{n} \sum_{k+l=r, k \neq n} \sum_{k+l=r, k \neq n} \sum_{r=0}^{n} [\Delta_{k+1} D_{r-k}(a), d_{n-r}(a)]\]

Putting \(l = r - k\), we deduce that

\[K = \sum_{k=0}^{n+1} k[D_k(a), d_{n+1-k}(a)] - \sum_{r=0}^{n} \sum_{k=0}^{r+1} [\Delta_{k+1} D_{r-k}(a), d_{n-r}(a)]\]

\[= \sum_{k=0}^{n+1} k[D_k(a), d_{n+1-k}(b)] - \sum_{r=0}^{n-1} \sum_{k=0}^{r} [\Delta_{k+1} D_{r-k}(a), d_{n-r}(a)]\]

\[- \sum_{k=0}^{n-1} [\Delta_{k+1} D_{n-k}(a_1), d_0(a_2)].\]

We substitute \(r + 1\) for \(k\) in the first summation and also, by our assumption, for \(r = 0, 1, ..., n - 1\) we get

\[(r + 1) \tilde{D}_{r+1}([a, b]) - \sum_{k=0}^{r} \Delta_{k+1} \tilde{D}_{r-k}([a, b]) = 0,\]

we can conclude that

\[K = \sum_{r=-1}^{n} (r + 1) [D_{r+1}(a_1), d_{n-r}(a_2)] - \sum_{r=0}^{n-1} \sum_{k=0}^{r} [\Delta_{k+1} D_{r-k}(a_1), d_{n-r}(a_2)]\]

\[- \sum_{k=0}^{n-1} [\Delta_{k+1} D_{n-k}(a_1), d_0(a_2)]\]

\[= \sum_{r=0}^{n-1} \left[ (r + 1) D_{r+1} - \sum_{k=0}^{r} \Delta_{k+1} D_{r-k} \right] (a_1), d_{n-r}(a_2) + (n + 1) [D_{n+1}(a_1), d_0(a_2)]\]

\[- \sum_{k=0}^{n-1} [\Delta_{k+1} D_{n-k}(a_1), d_0(a_2)]\]

\[= \sum_{r=0}^{n-1} \left[ (r + 1) \tilde{D}_{n+1} - \sum_{k=0}^{r} \Delta_{k+1} \tilde{D}_{r-k} \right] \tilde{D}_0^{-1} D_0(a_1), d_{n-r}(a_2) + (n + 1) [D_{n+1}(a_1), d_0(a_2)]\]

\[- \sum_{k=0}^{n-1} [\Delta_{k+1} D_{n-k}(a_1), d_0(a_2)]\]

\[= (n + 1) [D_{n+1}(a_1), d_0(a_2)] - \sum_{k=0}^{n-1} [\Delta_{k+1} D_{n-k}(a_1), d_0(a_2)]\]

\[= \left[ (n + 1) D_{n+1} - \sum_{k=0}^{n-1} \Delta_{k+1} D_{n-k} \right] (a_1), d_0(a_2)\]

\[= \left[ (n + 1) \tilde{D}_{n+1} - \sum_{k=0}^{n-1} \Delta_{k+1} \tilde{D}_{n-k} \right] \tilde{D}_0^{-1} D_0(a_1), d_0(a_2)\]

\[= [\Delta_{n+1} \tilde{D}_0(\tilde{D}_0^{-1} D_0(a_1)), d_0(a_2)]\]
Thus $D_{n+1}$ is a generalized Lie derivation on $\mathcal{B}$ such that

$$
\delta_{n+1}([b_1, b_2]) = K + L = \left[ \Delta_{n+1}(b_1), b_2 \right] + \left[ b_1, \delta_{n+1}(b_2) \right].
$$

Now, we show that if $D_n$ is of the relation (2.3), then $\tilde{D}_n$ satisfies the recursive relation (2.2). Since the solution of the recursive relation is unique, this proves the theorem.

Simplifying the notation, we put $a_{r_1, r_2, \ldots, r_i} = \prod_{j=1}^{i} \frac{1}{r_j + r_j + \cdots + r_i}$. Note that if $r_1 + r_2 + \cdots + r_i = n + 1$, then $a_{r_1, r_2, \ldots, r_i} = \frac{a_{r_2, r_3, \ldots, r_i}}{n + 1}$ and $(n + 1)a_{n+1} = 1$. Now, for each $a \in \mathcal{A}$ we get

$$(n + 1)\tilde{D}_{n+1}(a + \ker(D_0)) = (n + 1)D_{n+1}(a)$$

$$= \sum_{i=2}^{n+1} \left( \sum_{\sum_{j=1}^{i} r_j = n+1} (n + 1)a_{r_1, r_2, \ldots, r_i} \Delta_{r_1} \Delta_{r_2} \cdots \Delta_{r_i} D_0 \right)(a)$$

$$+ \Delta_{n+1}D_0(a)$$

$$= \sum_{i=2}^{n+1} \left( \sum_{r_1 = 1}^{n+1-i} \Delta_{r_1} \sum_{\sum_{j=2}^{i} r_j = n+1-r_1} a_{r_2, r_3, \ldots, r_i} \Delta_{r_2} \Delta_{r_3} \cdots \Delta_{r_i} D_0 \right)(a)$$

$$+ \Delta_{n+1}D_0(a)$$

$$= \sum_{r_1 = 1}^{n} \Delta_{r_1} \sum_{i=2}^{n-(r_1-2)} \left( \sum_{\sum_{j=2}^{i} r_j = n-(r_1-1)} a_{r_2, r_3, \ldots, r_i} \Delta_{r_2} \Delta_{r_3} \cdots \Delta_{r_i} D_0 \right)(a)$$

$$+ \Delta_{n+1}D_0(a)$$

$$= \sum_{r_1 = 1}^{n} \Delta_{r_1} \sum_{k=1}^{n-(r_1-1)} \left( \prod_{j=1}^{k} \frac{1}{r_j + r_j + \cdots + r_k} \right) \Delta_{r_1} \Delta_{r_2} \cdots \Delta_{r_k} D_0(a)$$

$$+ \Delta_{n+1}D_0(a)$$

$$= \sum_{r_1 = 1}^{n} \Delta_{r_1} D_{n-(r_1-1)}(a) + \Delta_{n+1}D_0(a)$$

$$= \sum_{i=0}^{n} \Delta_{r_1} \tilde{D}_{n-i}(a + \ker(D_0)).$$
Thus, we have \((n + 1)\bar{D}_{n+1} = \sum_{l=0}^{n} \Delta_l \bar{D}_{n-l}\).

We are now ready for Theorem 2.4.

**Theorem 2.4.** Let \(\{d_n\}_{n \in \mathbb{N} \cup \{0\}}\) be a higher Lie derivations from \(A\) into \(B\) with \(d_0(A) = B\) and \(\ker(d_0) \subseteq \ker(d_n)\ (n \in \mathbb{N})\) and \(\{\delta_n\}\) be the sequence of Lie derivations such that

\[
d_n = \sum_{i=1}^{n} \left( \sum_{j=1}^{i} \frac{1}{r_j + r_{j+1} + \ldots + r_i} \right) \delta_{r_1} \delta_{r_2} \ldots \delta_{r_i} d_0.
\]

(2.4)

Moreover, suppose that \(\alpha : A \rightarrow B\) be a surjective generalized identity mapping associated with \(d_0\), i.e., \(\alpha([a_1, a_2]) = [\alpha(a_1), d_0(a_2)]\), \(\Delta\) is the set of all sequences \(\{\Delta_n\}_n \in \mathbb{N}\) of generalized Lie derivations on \(B\) such that \(\Delta_n([b_1, b_2]) = [\Delta_n(b_1), b_2] + [b_1, \delta_n(b_2)]\) for each \(b_1, b_2 \in B\) and further, \(\bar{D}\) be the set of all generalized higher Lie derivations \(\{\bar{D}_n\}_{n \in \mathbb{N} \cup \{0\}}\) from \(A\) into \(B\) with \(\bar{D}_0 = \alpha, \ker(D_0) \subseteq \ker(D_n)\ (n \in \mathbb{N})\) and \(\bar{D}_n([a_1, a_2]) = \sum_{k=0}^{n} [D_k(a_1), d_{n-k}(a_2)]\) for each \(a_1, a_2 \in A\). Then, there is a one to one correspondence \(\varphi : \Delta \rightarrow \bar{D}\) defined by \(\varphi(\{\Delta_n\}) = \{\bar{D}_n\}\), where \(\bar{D}_n = \sum_{i=1}^{n} \left( \sum_{j=1}^{i} r_j + r_{j+1} + \ldots + r_i \right) \Delta_{r_1} \Delta_{r_2} \ldots \Delta_{r_i} D_0\).

**Proof.** Suppose that \(\{\Delta_n\} \in \Delta\) and \(\{\delta_n\}\) is the sequence of Lie derivations on \(B\) such that \(\Delta_n([b_1, b_2]) = [\Delta_n(b_1), b_2] + [b_1, \delta_n(b_2)]\) for each \(b_1, b_2 \in B\). Define \(\bar{D}_n : A \rightarrow B\) by \(\bar{D}_0 = \alpha\) and

\[
\bar{D}_n = \sum_{i=1}^{n} \left( \sum_{j=1}^{i} r_j + r_{j+1} + \ldots + r_i \right) \Delta_{r_1} \Delta_{r_2} \ldots \Delta_{r_i} D_0.
\]

(2.5)

By Proposition 2.2 the sequences \(\{\bar{D}_n\}\) and \(\{\bar{d}_n\}\) satisfy the following relations

\[
(r + 1)d_{r+1} = \sum_{k=0}^{r} \delta_{r+k+1} d_{r-k}, \quad (n + 1)\bar{D}_{n+1} = \sum_{k=0}^{n} \Delta_{k+1} \bar{D}_{r-k}.
\]

(2.5)

By using induction on \(n\), we show that \(\{\bar{D}_n\}\) is a generalized higher Lie derivation. For \(n = 0\), we get \(\bar{D}_0([a_1, a_2]) = \alpha([a_1, a_2]) = [D_0(a_1), d_0(a_2)]\).suppose that \(\bar{D}_k([a_1, a_2]) = \sum_{i=0}^{k} [D_i(a_1), d_{n-i-1}(a_2)]\) for \(k \leq n\). Also, by using (2.5), we obtain

\[
(n + 1)\bar{D}_{n+1}([a_1, a_2]) = (n + 1)\bar{D}_{n+1}([a_1 + \ker(D_0), a_2 + \ker(D_0)])
\]

\[
= \sum_{k=0}^{n} \Delta_{k+1} \bar{D}_{n-k}([a_1 + \ker(D_0), a_2 + \ker(D_0)])
\]

\[
= \sum_{k=0}^{n} \Delta_{k+1} D_{n-k}([a_1, a_2])
\]

\[
= \sum_{k=0}^{n} \Delta_{k+1} \left( \sum_{i=0}^{n-k} [D_i(a_1), d_{n-i-1}(a_2)] \right)
\]
Therefore, Theorem 2.3 ensures us that \( \{ \Delta_n \} \in \mathcal{D} \). Note that for each \( n \in \mathbb{N} \), \( \ker(D_0) \subset \ker(D_n) \).

Conversely, suppose that \( \{ D_n \} \in \mathcal{D} \). Define \( \Delta_n : \mathcal{B} \rightarrow \mathcal{B} \) by \( \Delta_1 = \tilde{D}_1 \tilde{D}_0^{-1} \) and by using (2.5), we can write

\[
\Delta_n = \left( n\tilde{D}_n - \sum_{k=0}^{n-2} \Delta_{k+1} \tilde{D}_{n-1-k} \right) \tilde{D}_0^{-1} \quad (n \geq 2)
\]

Therefore, Theorem 2.3 ensures us that \( \{ \Delta_n \} \in \Delta \). Now, define \( \varphi : \Delta \rightarrow \mathcal{D} \) by \( \varphi(\{ \Delta_n \}) = \{ D_n \} \), where

\[
D_n = \sum_{i=1}^{n} \left( \sum_{\Sigma_{j=1}^{i} r_j = n} \left( \prod_{j=1}^{i} \frac{1}{r_j + r_{j+1} + \ldots + r_i} \right) \Delta_{r_1} \Delta_{r_2} \ldots \Delta_{r_i} D_0 \right).
\]

Then, it is evident that \( \varphi \) is surjective. Our next task is to show that it is injective. Let \( \{ D_n \} = \varphi(\{ \Delta_n \}) = \varphi(\{ \Delta'_n \}) = \{ D'_n \} \). We use induction on \( n \). For \( n = 1 \), we get

\[
\Delta_1 = \tilde{D}_1 \tilde{D}_0^{-1} = \tilde{D}'_1 \tilde{D}'_0^{-1} = \Delta'_1
\]
Now, suppose that $\Delta_k = \Delta'_k$ for $k \leq n$. Similar to the proof of Theorem 2.3 the following relations is obtained:

$$(r + 1)\tilde{D}'_{n+1} = \sum_{k=0}^{n} \Delta'_{k+1}\tilde{D}'_{n-k},\quad (n + 1)\tilde{D}_{n+1} = \sum_{k=0}^{n} \Delta_{k+1}\tilde{D}_{n-k}.$$

for $n \geq 0$. From this we have

$$\Delta_{n+1} = \left((n + 1)\tilde{D}_{n+1} - \sum_{k=0}^{n-1} \Delta_{k+1}\tilde{D}_{n-k}\right)\tilde{D}_0^{-1}$$

$$= \left((n + 1)\tilde{D}'_{n+1} - \sum_{k=0}^{n-1} \Delta'_{k+1}\tilde{D}'_{n-k}\right)\tilde{D}_0^{-1} = \Delta'_{n+1}.$$

So, $\{\Delta\} = \{\Delta'_{n+1}\}$ and it means that $\varphi$ is one to one.

\[\square\]

REFERENCES


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