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A CHARACTERIZATION OF GENERALIZED HIGHER LIE DERIVATIONS ON ALGEBRAS

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. In this paper, under certain conditions, a characterization of generalized higher Lie derivations from $\mathcal{A}$ into $\mathcal{B}$ as a sequence of generalized Lie derivations will be presented.

1. Introduction and preliminaries

Throughout this paper, $\mathcal{A}$ and $\mathcal{B}$ will represent two algebras (associative or non-associative). Let $\sigma : \mathcal{A} \to \mathcal{B}$ be a linear mapping and $\mathcal{X}$ be a $\mathcal{B}$-bimodule. A linear mapping $d : \mathcal{A} \to \mathcal{X}$ is called a σ-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ holds for all $a, b \in \mathcal{A}$. Hosseini et al. [4] extended this concept and defined generalized σ-derivation as follows:
A linear mapping $\delta : \mathcal{A} \to \mathcal{X}$ is called a generalized σ-derivation if $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)d(b)$, where d is a σ-derivation (for more details see [3], [4] and [5]).

Like most authors, we denote the commutator $ab - ba$ by $[a, b]$ for all pair $a, b \in \mathcal{A}$. Let $\theta : \mathcal{A} \to \mathcal{B}$ be a linear mapping. A linear mapping $\delta : \mathcal{A} \to \mathcal{B}$ is called a θ- Lie derivation if it satisfies $\delta[a, b] = [\delta(a), \theta(b)] + [\theta(a), \delta(b)]$ for all $a, b \in \mathcal{A}$. If $\mathcal{A} = \mathcal{B}$ and $\theta = id$, the identity mapping on $\mathcal{A}$, then δ is called a Lie derivation. Furthermore, a linear mapping $\Delta : \mathcal{A} \to \mathcal{B}$ is called a generalized Lie derivation if there exists a Lie derivation $\delta : \mathcal{A} \to \mathcal{B}$ such that $\Delta([a, b]) = [\Delta(a), b] + [a, \delta(b)]$ holds for all $a, b \in \mathcal{A}$. For many years, there has been increasing interest on the study of Lie-type mappings of associative rings and operator algebras, such as Lie isomorphisms and Lie derivations. Recall that a linear mapping $T : \mathcal{A} \to \mathcal{B}$ is called a Lie homomorphism if $T([a, b]) = [T(a), T(b)]$ holds for all $a, b \in \mathcal{A}$. There is an important decomposition of Lie derivations on some Banach algebras. For example, Johnson showed in [6] that every continuous Lie derivation from a symmetrically amenable Banach algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ decomposes into a sum of an ordinary derivation from $\mathcal{A}$ into $\mathcal{X}$ and a linear mapping from $\mathcal{A}$ into the center of $\mathcal{X}$. Alaminos, et al [1] jointly proved that every Lie derivation on a symmetrically amenable semisimple Banach algebra also has the same decomposition. Moreover, Mathieu and Villena [8] obtained

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that every (not necessarily bounded) Lie derivation D on a C^*-algebra $\mathcal{A}$ can be uniquely decomposed into the sum of a derivation d of $\mathcal{A}$ and a linear mapping ψ from $\mathcal{A}$ into its center $Z(\mathcal{A})$ vanishing on all commutators.

First, in this note, we define generalized higher Lie derivations and then, by getting idea from [7], a useful characterization of them is presented as a sequence of generalized Lie derivations. This is a very effective method for characterizing generalized higher Lie derivations. A sequence $\{d_n\}$ of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a higher Lie derivation if $d_n([a,b]) = \sum_{k=0}^{n}[d_k(a), d_{n-k}(b)]$ for each $a, b \in \mathcal{A}$ and each non-negative integer n. A sequence $\{D_n\}$ of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ is said to be a *generalized higher Lie derivation* if there exists a higher Lie derivation $\{d_n\}$ from $\mathcal{A}$ into $\mathcal{B}$ such that $D_n([a,b]) = \sum_{k=0}^{n}[D_k(a), d_{n-k}(b)]$ for each $a, b \in \mathcal{A}$ and each non-negative integer n. Let d_n be a higher Lie derivation. Then d_0 is a Lie homomorphism and d_1 is a d_0-Lie derivation that is $d_1([a,b]) = [d_0(a), d_1(b)] + [d_1(a), d_0(b)]$ for all $a, b \in \mathcal{A}$. So, if d_0 is onto, then $\tilde{d_0} : \mathcal{A}/ker(d_0) \to \mathcal{B}$ defined by $\tilde{d_0}(a + ker(d_0)) = d_0(a)$ is a Lie isomorphism. It is evident that, for each $n \in \mathbb{N}$, $\tilde{d_n} : \mathcal{A}/ker(d_0) \to \mathcal{B}$ defined by $\tilde{d_n}(a + ker(d_0)) = d_n(a)$ is a well-defined linear map if and only if $ker(d_0) \subseteq ker(d_n)$.

2. Results and proofs

Definition 2.1. i) A sequence $\{d_n\}$ of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a *higher Lie derivation* if $d_n([a,b]) = \sum_{k=0}^{n}[d_k(a), d_{n-k}(b)]$ holds for each pair $a, b \in \mathcal{A}$ and each non-negative integer n.
ii) A sequence $\{D_n\}$ of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a *generalized higher Lie derivation* if there exists a higher Lie derivation $\{d_n\}$ from $\mathcal{A}$ into $\mathcal{B}$ such that$D_n([a,b]) = \sum_{k=0}^{n}[D_k(a), d_{n-k}(b)]$ holds for each pair $a, b \in \mathcal{A}$ and each non-negative integer n.

We begin our results with the following proposition which will be used extensively to prove our theorems.

Proposition 2.2. *Suppose that $\{d_n\}$ is a higher Lie derivation from $\mathcal{A}$ into $\mathcal{B}$ with $d_0(\mathcal{A}) = \mathcal{B}$ and $ker(d_0) \subseteq ker(d_n)$ $(n \in \mathbb{N})$. Then, there is a sequence $\{\delta_n\}$ of Lie derivations on $\mathcal{B}$ such that*

$$(n+1)\tilde{d}_{n+1} = \sum_{k=0}^{n} \delta_{k+1}\tilde{d}_{n-k}, \tag{2.1}$$

for each non-negative integer n.

Proof. Let us take an inductive approach for the index n. For $n = 0$, we define $\delta_1 : \mathcal{B} \to \mathcal{B}$ by $\delta_1 = \tilde{d_1}\tilde{d_0}^{-1}$. Let b_1 and b_2 be two arbitrary fixed elements of $\mathcal{B}$. Since $\tilde{d_0}$ is a Lie isomorphism, there exist $a_1, a_2 \in \mathcal{A}$ such that $d_0(a_1) = b_1$ and

$d_0(a_2) = b_2$. We, therefore, have

$$\begin{aligned}\delta_1([b_1, b_2]) &= \tilde{d}_1\tilde{d}_0^{-1}([d_0(a_1), d_0(a_2)])\\ &= d_1([a_1, a_2])\\ &= [d_0(a_1), d_1(a_2)] + [d_1(a_1), d_0(a_2)]\\ &= [d_0(a_1), \tilde{d}_1\tilde{d}_0^{-1}(d_0(a_2)] + [\tilde{d}_1\tilde{d}_0^{-1}(d_0(a_1), d_0(a_2)]\\ &= [b_1, \delta_1(b_2)] + [\delta_1(b_1), b_2].\end{aligned}$$

Thus, δ_1 is a Lie derivation. Note that $\tilde{d}_1 = \delta_1\tilde{d}_0$. Now assume that δ_k is defined and is a Lie derivation for $k \leq n$ satisfying (2.1). Putting $\delta_{n+1} = \Big((n+1)\tilde{d}_{n+1} - \sum_{n=0}^{n-1}\delta_{k+1}\tilde{d}_{n-k}\Big)\tilde{d}_0^{-1}$. Thus, the well-defined mapping δ_{n+1} is a Lie derivation. For $b_1, b_2 \in \mathcal{B}$, there are $a_1, a_2 \in \mathcal{A}$ such that $\delta_1, \delta_2, ...\delta_n$ are Lie derivation and also $d_0(a_1) = b_1$ and $d_0(a_2) = b_2$. Consequently

$$\begin{aligned}\delta_{n+1}\big([b_1, b_2]\big) &= \Big((n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1}\delta_{k+1}\tilde{d}_{n-k}\Big)\tilde{d}_0^{-1}\big([d_0(a_1), d_0(a_2)]\big)\\ &= \Big((n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1}\delta_{k+1}\tilde{d}_{n-k}\Big)\big([a_1 + ker(d_0), a_2 + ker(d_0)]\big)\\ &= (n+1)d_{n+1}\big([a_1, a_2]\big) - \sum_{k=0}^{n-1}\delta_{k+1}d_{n-k}\big([a_1, a_2]\big)\\ &= (n+1)\sum_{k=0}^{n+1}\Big[d_k(a_1), d_{n+1-k}(a_2)\Big] - \sum_{k=0}^{n-1}\delta_{k+1}\Big(\sum_{l=0}^{n-k}\Big[d_l(a_1), d_{n-k-l}(a_2)\Big]\Big)\\ &= \sum_{k=0}^{n+1}(k+n+1-k)\Big[d_k(a), d_{n+1-k}(a_1)\Big] - \sum_{k=0}^{n-1}\delta_{k+1}\Big(\sum_{l=0}^{n-k}\Big[d_l(a), d_{n-k-l}(a_2)\Big]\Big)\\ &= \sum_{k=0}^{n+1}k\Big[d_k(a_1), d_{n+1-k}(a_2)\Big] + \sum_{k=0}^{n+1}(n+1-k)\Big[d_k(a), d_{n+1-k}(b)\Big]\\ &\quad - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\Big(\Big[\delta_{k+1}d_l(a_1), d_{n-k-l}(a_2)\Big] + \Big[d_l(a_1), \delta_{k+1}d_{n-k-l}(a_2)\Big]\Big).\end{aligned}$$

Writing

$$K = \sum_{k=0}^{n+1}k\Big[d_k(a_1), d_{n+1-k}(a_2)\Big] - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\Big[\delta_{k+1}d_l(a_1), d_{n-k-i}(a_2)\Big],$$

and

$$L = \sum_{k=0}^{n+1}(n+1-k)\Big[d_k(a_1), d_{n+1-k}(a_2)\Big] - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\Big[d_l(a_1), \delta_{k+1}d_{n-k-l}(a_2)\Big],$$

we have $\delta_{n+1}\big([b_1, b_2]\big) = K + L$. Our next task is to compute K and L. In the

summation $\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}$, we have $0 \leq k+l \leq n$ and $k \neq n$. So, it can be written as the form $\sum_{r=0}^{n}\sum_{k+l=r,k\neq n}$ where $r = k+l$. Putting $l = r-k$, it is obtained that

$$\begin{aligned}
K &= \sum_{k=0}^{n+1} k\Big[d_k(a_1), d_{n+1-k}(a_2)\Big] - \sum_{r=0}^{n}\sum_{k+l=r,k\neq n}\Big[\delta_{k+1}d_{r-k}(a), d_{n-r}(a_2)\Big] \\
&= \sum_{k=0}^{n+1} k\Big[d_k(a), d_{n+1-k}(b)\Big] - \sum_{r=0}^{n-1}\sum_{k=0}^{r}\Big[\delta_{k+1}d_{r-k}(a_1), d_{n-r}(a_2)\Big] \\
&- \sum_{k=0}^{n-1}\Big[\delta_{k+1}d_{n-k}(a_1), d_0(a_2).\Big].
\end{aligned}$$

We substitute $r+1$ for k in the first summation and also, by our assumption, for $r = 0, 1, ..., n-1$ we arrive at $(r+1)\tilde{d}_{r+1}([a,b]) = \sum_{k=0}^{r}\delta_{k+1}\tilde{d}_{r-k}([a,b])$. Hence, it can be concluded that

$$\begin{aligned}
K &= \sum_{r=-1}^{n}(r+1)\Big[d_{r+1}(a_1), d_{n-r}(a_2)\Big] - \sum_{r=0}^{n-1}\sum_{k=0}^{r}\Big[\delta_{k+1}d_{r-k}(a_1), d_{n-r}(a_2)\Big] \\
&- \sum_{k=0}^{n-1}\Big[\delta_{k+1}d_{n-k}(a_1), d_0(a_2)\Big] \\
&= \sum_{r=0}^{n-1}\Big[\Big((r+1)d_{r+1} - \sum_{k=0}^{r}\delta_{k+1}d_{r-k}\Big)(a_1), d_{n-r}(a_2)\Big] \\
&+ (n+1)\Big[d_{n+1}(a_1), d_0(a_2)\Big] - \sum_{k=0}^{n-1}\Big[\delta_{k+1}d_{n-k}(a_1), d_0(a_2)\Big] \\
&= \sum_{r=0}^{n-1}\Big[\Big((r+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1}\delta_{k+1}\tilde{d}_{r-k}\Big)\tilde{d}_0^{-1}d_0(a_1), d_{n-r}(a_2)\Big] \\
&+ (n+1)\Big[d_{n+1}(a_1), d_0(a_2)\Big] - \sum_{k=0}^{n-1}\Big[\delta_{k+1}d_{n-k}(a_1), d_0(a_2)\Big] \\
&= (n+1)\Big[d_{n+1}(a_1), d_0(a_2)\Big] - \sum_{k=0}^{n-1}\Big[\delta_{k+1}d_{n-k}(a_1), d_0(a_2)\Big] \\
&= \Big[\Big((n+1)d_{n+1} - \sum_{k=0}^{n-1}\delta_{k+1}d_{n-k}\Big)(a_1), d_0(a_2)\Big] \\
&= \Big[\Big((n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1}\delta_{k+1}\tilde{d}_{n-k}\Big)\tilde{d}_0^{-1}d_0(a_1), d_0(a_2)\Big] \\
&= \Big[\delta_{n+1}\tilde{d}_0\big(\tilde{d}_0^{-1}d_0(a_1)\big), d_0(a_2)\Big] \\
&= \Big[\delta_{n+1}(b_1), b_2\Big].
\end{aligned}$$

Also in the same way

$$L = \Big[d_0(a_1), \Big((n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1}\tilde{d}_{n-k}\Big)\tilde{d}_0^{-1}d_0(a_2)\Big] = \Big[b_1, \delta_{n+1}(b_2)\Big].$$

Finally, we conclude

$$\delta_{n+1}\big([b_1, b_2]\big) = K + L = \Big[\delta_{n+1}(b_1), b_2\Big] + \Big[b_1, \delta_{n+1}(b_2)\Big],$$

for all $b_1, b_2 \in \mathcal{A}$. This implies that δ_{n+1} is a Lie derivation of $\mathcal{A}$ and it completes the proof. □

Suppose that $\{D_n\}$ is a *generalized higher Lie derivation* from $\mathcal{A}$ into $\mathcal{B}$. We define $\tilde{D}_n : \mathcal{A}/ker(D_0) \to \mathcal{B}$ with $\tilde{D}_n(a + ker(D_0)) = D_n(a)$ in which $D_0(\mathcal{A}) = \mathcal{B}$ and $ker(D_0) \subseteq ker(D_n)$.

Theorem 2.3. *Let $\{D_n\}$ be a generalized higher Lie derivation from $\mathcal{A}$ into $\mathcal{B}$ associated with a higher Lie derivation $\{d_n\}$ from $\mathcal{A}$ into $\mathcal{B}$, i.e. $D_n([a_1, a_2]) = \sum_{k=0}^{n}[D_k(a_1), d_{n-k}(a_2)]$ holds for each $a_1, a_2 \in \mathcal{A}$ and each non-negative integer n. Moreover, suppose that $d_0(\mathcal{A}) = D_0(\mathcal{A}) = \mathcal{B}$, $ker(d_0) \subseteq ker(d_n)$ and $ker(D_0) \subseteq ker(D_n)$ for each non-negative integer n. Then, there is a sequence $\{\Delta_n\}$ of generalized Lie derivations on $\mathcal{B}$ associated with a sequence $\{\delta_n\}$ of Lie derivations such that for each non-negative integer n,*

$$(n+1)\tilde{D}_{n+1} = \sum_{k=0}^{n} \Delta_{k+1}\tilde{D}_{n-k}. \tag{2.2}$$

Moreover, suppose that $\{D_n\}$ be a generalized higher Lie derivation from $\mathcal{A}$ into $\mathcal{B}$ associated with a higher Lie derivation $\{d_n\}$ from $\mathcal{A}$ into $\mathcal{B}$, i.e. $D_n([a_1, a_2]) = \sum_{k=0}^{n}[D_k(a_1), d_{n-k}(a_2)]$ holds for each $a_1, a_2 \in \mathcal{A}$ and each non-negative integer n. Furthermore,

$$D_n = \sum_{i=1}^{n}\Big(\sum_{\Sigma_{j=1}^{i} r_j = n}\big(\prod_{j=1}^{i}\frac{1}{r_j + r_{j+1} + ... + r_i}\big)\Delta_{r_1}\Delta_{r_2}...\Delta_{r_i}D_0\Big), \tag{2.3}$$

where the inner summation is taken over all positive integers $r_1, r_2, ..., r_n$ with $\sum_{j=1}^{i} r_j = n$.

Proof. We prove this theorem by using Proposition 2.2 and induction on n. For $n = 0$, let $\Delta_1 : \mathcal{B} \to \mathcal{B}$ be defined by $\Delta_1 = \tilde{D}_1\tilde{D}_0^{-1}$. Since D_0 and d_0 are surjective, there exist $a_1, a_2 \in \mathcal{A}$ such that $D_0(a_1) = b_1$ and $d_0(a_2) = b_2$, where b_1 and b_2 are

two arbitrary fixed elements of $\mathcal{B}$. Consequently,

$$\begin{aligned}\Delta_1([b_1,b_2]) &= \tilde{D}_1\tilde{D}_0^{-1}\big([D_0(a_1),d_0(a_2)]\big)\\ &= \tilde{D}_1\tilde{D}_0^{-1}\big(D_0([a_1,a_2])\big)\\ &= D_1\big([a_1,a_2]\big)\\ &= [D_0(a_1),d_1(a_2)]+[D_1(a_1),d_0(a_2)]\\ &= [D_0(a_1),\tilde{d}_1\tilde{d}_0^{-1}(d_0(a_2)]+[\tilde{D}_1\tilde{D}_0^{-1}(D_0(a_1),d_0(a_2)]\\ &= [b_1,\delta_1(b_2)]+[\Delta_1(b_1),b_2].\end{aligned}$$

Since δ_1 is a Lie derivation, Δ_1 is a generalized Lie derivation. Note that $\tilde{D}_1 = \Delta_1\tilde{D}_0$. Now, assume that Δ_k is defined and is a generalized Lie derivation for $k \leq n$ satisfying (2.2) such that $\Delta_k\big([b_1,b_2]\big) = [\Delta_k(b_1),b_2]+[b_1,\delta_k(b_2)]$ for each $b_1,b_2 \in \mathcal{B}$. Put $\Delta_{n+1} = \Big((n+1)\tilde{D}_{n+1} - \sum_{n=0}^{n-1}\Delta_{k+1}\tilde{D}_{n-k}\Big)\tilde{D}_0^{-1}$, and $a_1,a_2 \in \mathcal{A}$ such that $D_0(a_1) = b_1$ and $d_0(a_2) = b_2$. Thus,

$$\begin{aligned}\Delta_{n+1}([b_1,b_2]) &= \Big((n+1)\tilde{D}_{n+1} - \sum_{k=0}^{n-1}\Delta_{k+1}\tilde{D}_{n-k}\Big)\tilde{D}_0^{-1}\Big(D_0([a_1,a_2])\Big)\\ &= \Big((n+1)\tilde{D}_{n+1} - \sum_{k=0}^{n-1}\Delta_{k+1}\tilde{D}_{n-k}\Big)\big(\Big[a_1+kerD_0,a_2+kerD_0\Big]\big)\\ &= (n+1)D_{n+1}\big([a_1,a_2]\big) - \sum_{k=0}^{n-1}\Delta_{k+1}D_{n-k}\big([a_1,a_2]\big)\\ &= (n+1)\sum_{k=0}^{n+1}\Big[D_k(a_1),d_{n+1-k}(a_2)\Big] - \sum_{k=0}^{n-1}\Delta_{k+1}\Big(\sum_{l=0}^{n-k}\Big[D_l(a_1),d_{n-k-l}(a_2)\Big]\Big).\end{aligned}$$

Since $\Delta_k([b_1,b_2]) = [\Delta_k(b_1),b_2]+[b_1,\delta_k(b_2)]$ for $k \leq n$, we have

$$\begin{aligned}\Delta_{n+1}([b_1,b_2]) &= \sum_{k=0}^{n+1}k\Big[D_k(a_1),d_{n+1-k}(a_2)\Big] + \sum_{k=0}^{n+1}(n+1-k)\Big[D_k(a_1),d_{n+1-k}(a_2)\Big]\\ &\quad - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\Big(\Big[\Delta_{k+1}D_l(a_1),d_{n-k-l}(a_2)\Big] + \Big[D_l(a_1),\delta_{k+1}d_{n-k-l}(a_2)\Big]\Big).\end{aligned}$$

Writing

$$K = \sum_{k=0}^{n+1}k\Big[D_k(a_1),d_{n+1-k}(a_2)\Big] - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\Big[\Delta_{k+1}D_l(a_1),d_{n-k-l}(a_2)\Big],$$

$$L = \sum_{k=0}^{n+1}(n+1-k)\Big[D_k(a_1),d_{n+1-k}(a_2)\Big] - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\Big[D_l(a_1),\delta_{k+1}d_{n-k-l}(a_2)\Big],$$

we have $\Delta_{n+1}([b_1,b_2]) = K+L$. Let us compute K and L. In the summation $\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}$, we have $0 \leq k+l \leq n$ and $k \neq n$. So we can write it as the form

$\sum_{r=0}^{n}\sum_{k+l=r,k\neq n}$ where $r=k+l$. Putting $l=r-k$, we deduce that

$$\begin{aligned}K&=\sum_{k=0}^{n+1}k[D_k(a_1),d_{n+1-k}(a_2)]-\sum_{r=0}^{n}\sum_{k+l=r,k\neq n}[\Delta_{k+1}D_{r-k}(a),d_{n-r}(a_2)]\\&=\sum_{k=0}^{n+1}k[D_k(a),d_{n+1-k}(b)]-\sum_{r=0}^{n-1}\sum_{k=0}^{r}[\Delta_{k+1}D_{r-k}(a_1),d_{n-r}(a_2)]\\&-\sum_{k=0}^{n-1}[\Delta_{k+1}D_{n-k}(a_1),d_0(a_2)].\end{aligned}$$

We substitute $r+1$ for k in the first summation and also, by our assumption, for $r=0,1,...,n-1$ we get

$$(r+1)\tilde{D}_{r+1}([a,b])-\sum_{k=0}^{r}\Delta_{k+1}\tilde{D}_{r-k}([a,b])=0,$$

we can, conclude that

$$\begin{aligned}K&=\sum_{r=-1}^{n}(r+1)\Big[D_{r+1}(a_1),d_{n-r}(a_2)\Big]-\sum_{r=0}^{n-1}\sum_{k=0}^{r}\Big[\Delta_{k+1}D_{r-k}(a_1),d_{n-r}(a_2)\Big]\\&-\sum_{k=0}^{n-1}\Big[\Delta_{k+1}D_{n-k}(a_1),d_0(a_2)\Big]\\&=\sum_{r=0}^{n-1}\Big[\Big((r+1)D_{r+1}-\sum_{k=0}^{r}\Delta_{k+1}D_{r-k}\Big)(a_1),d_{n-r}(a_2)\Big]+(n+1)\Big[D_{n+1}(a_1),d_0(a_2)\Big]\\&-\sum_{k=0}^{n-1}\Big[\Delta_{k+1}D_{n-k}(a_1),d_0(a_2)\Big]\\&=\sum_{r=0}^{n-1}\Big[\Big((r+1)\tilde{D}_{n+1}-\sum_{k=0}^{r}\Delta_{k+1}\tilde{D}_{r-k}\Big)\tilde{D_0}^{-1}D_0(a_1),d_{n-r}(a_2)\Big]+(n+1)\Big[D_{n+1}(a_1),d_0(a_2)\Big]\\&-\sum_{k=0}^{n-1}\Big[\Delta_{k+1}D_{n-k}(a_1),d_0(a_2)\Big]\\&=(n+1)\Big[D_{n+1}(a_1),d_0(a_2)\Big]-\sum_{k=0}^{n-1}\Big[\Delta_{k+1}D_{n-k}(a_1),d_0(a_2)\Big]\\&=\Big[\Big((n+1)D_{n+1}-\sum_{k=0}^{n-1}\Delta_{k+1}D_{n-k}\Big)(a_1),d_0(a_2)\Big]\\&=\Big[\Big((n+1)\tilde{D}_{n+1}-\sum_{k=0}^{n-1}\Delta_{k+1}\tilde{D}_{n-k}\Big)\tilde{D_0}^{-1}D_0(a_1),d_0(a_2)\Big]\\&=\Big[\Delta_{n+1}\tilde{D}_0\big(\tilde{D_0}^{-1}D_0(a_1)\big),d_0(a_2)\Big]\end{aligned}$$

$$= \Big[\Delta_{n+1}(b_1), b_2\Big].$$

Also, in the same way and reusing Proposition 2.2, it can be achieved that

$$L = \Big[D_0(a_1), \Big((n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1}\tilde{d}_{n-k}\Big)\tilde{d}_0^{-1}d_0(a_2)\Big] = \Big[b_1, \delta_{n+1}(b_2)\Big].$$

Thus D_{n+1} is a generalized Lie derivation on $\mathcal{B}$ such that

$$\delta_{n+1}([b_1, b_2]) = K + L = \Big[\Delta_{n+1}(b_1), b_2\Big] + \Big[b_1, \delta_{n+1}(b_2)\Big].$$

Now, we show that if D_n is of the relation (2.3), then $\tilde{D}_n$ satisfies the recursive relation (2.2). Since the solution of the recursive relation is unique, this proves the theorem.

Simplifying the notation, we put $a_{r_1,r_2,\ldots,r_i} = \prod_{j=1}^{i} \dfrac{1}{r_j + r_{j+1} + \ldots + r_i}$. Note that if $r_1 + r_2 + \ldots + r_i = n+1$, then $a_{r_1,r_2,\ldots,r_i} = \dfrac{a_{r_2,\ldots,r_i}}{n+1}$ and $(n+1)a_{n+1} = 1$. Now, for each $a \in \mathcal{A}$ we get

$$\begin{aligned}
&(n+1)\tilde{D}_{n+1}\big(a + ker(D_0)\big)\\
&\quad = (n+1)D_{n+1}(a)\\
&\quad = \sum_{i=2}^{n+1}\Big(\sum_{\Sigma_{j=1}^{i} r_j = n+1} (n+1)a_{r_1,r_2,\ldots,r_i}\Delta_{r_1}\Delta_{r_2}\ldots\Delta_{r_i}D_0\Big)(a)\\
&\quad + \Delta_{n+1}D_0(a)\\
&\quad = \sum_{i=2}^{n+1}\Big(\sum_{r_1=1}^{n+2-i}\Delta_{r_1}\sum_{\Sigma_{j=2}^{i} r_j = n+1-r_1} a_{r_2,r_3,\ldots,r_i}\Delta_{r_2}\Delta_{r_3}\ldots\Delta_{r_i}D_0\Big)(a)\\
&\quad + \Delta_{n+1}D_0(a)\\
&\quad = \sum_{r_1=1}^{n}\Delta_{r_1}\sum_{i=2}^{n-(r_1-2)}\Big(\sum_{\Sigma_{j=2}^{i} r_j = n-(r_1-1)} a_{r_2,r_3,\ldots,r_i}\Delta_{r_2}\Delta_{r_3}\ldots\Delta_{r_i}D_0\Big)(a)\\
&\quad + \Delta_{n+1}D_0(a)\\
&\quad = \sum_{r_1=1}^{n}\Delta_{r_1}\sum_{k=1}^{n-(r_1-1)}\Big(\sum_{\Sigma_{j=1}^{k} r'_j = n-(r_1-1)} \Big(\prod_{j=1}^{k}\frac{1}{r'_j + r'_{j+1} + \ldots + r'_k}\Big)\Delta_{r'_1}\Delta_{r'_2}\ldots\Delta_{r'_k}D_0\Big)(a)\\
&\quad + \Delta_{n+1}D_0(a)\\
&\quad = \sum_{r_1=1}^{n}\Delta_{r_1}D_{n-(r_1-1)}(a) + \Delta_{n+1}D_0(a)\\
&\quad = \sum_{l=0}^{n}\Delta_{r_1}\tilde{D}_{n-l}(a + ker(D_0)).
\end{aligned}$$

Thus, we have $(n+1)\tilde{D}_{n+1} = \sum_{l=0}^{n} \Delta_{r_1}\tilde{D}_{n-l}$.

□

We are now ready for Theorem 2.4.

Theorem 2.4. *Let $\{d_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a higher Lie derivations from $\mathcal{A}$ into $\mathcal{B}$ with $d_0(\mathcal{A}) = \mathcal{B}$ and $ker(d_0) \subseteq ker(d_n)$ $(n \in \mathbb{N})$ and $\{\delta_n\}$ be the sequence of Lie derivations such that*

$$d_n = \sum_{i=1}^{n}\Big(\sum_{\Sigma_{j=1}^{i} r_j = n}\big(\prod_{j=1}^{i}\frac{1}{r_j + r_{j+1} + ... + r_i}\big)\delta_{r_1}\delta_{r_2}...\delta_{r_i}d_0\Big). \tag{2.4}$$

Moreover, suppose that $\alpha : \mathcal{A} \to \mathcal{B}$ be a surjective generalized identity mapping associated with d_0, i.e., $\alpha([a_1, a_2]) = [\alpha(a_1), d_0(a_2)]$, $\underline{\Delta}$ is the set of all sequences $\{\Delta_n\}_n \in \mathbb{N}$ of generalized Lie derivations on $\mathcal{B}$ such that $\Delta_n([b_1, b_2]) = [\Delta_n(b_1), b_2] + [b_1, \delta_n(b_2)]$ for each $b_1, b_2 \in \mathcal{B}$ and further, $\underline{D}$ be the set of all generalized higher Lie derivations $\{D_n\}_{n\in\mathbb{N}\cup\{0\}}$ from $\mathcal{A}$ into $\mathcal{B}$ with $D_0 = \alpha$, $ker(D_0) \subseteq ker(D_n)]$ $(n \in \mathbb{N})$ and $D_n([a_1, a_2]) = \sum_{k=0}^{n}[D_k(a_1), d_{n-k}(a_2)]$ for each $a_1, a_2 \in \mathcal{A}$. Then, there is a one to one correspondence $\varphi : \underline{\Delta} \to \underline{D}$ defined by $\varphi(\{\Delta_n\}) = \{D_n\}$, where $D_n = \sum_{i=1}^{n}\Big(\sum_{\Sigma_{j=1}^{i} r_j = n}\big(\prod_{j=1}^{i}\frac{1}{r_j + r_{j+1} + ... + r_i}\big)\Delta_{r_1}\Delta_{r_2}...\Delta_{r_i}D_0\Big)$.

Proof. Suppose that $\{\Delta_n\} \in \underline{\Delta}$ and $\{\delta_n\}$ is the sequence of Lie derivations on $\mathcal{B}$ such that $\Delta_n([b_1, b_2]) = [\Delta_n(b_1), b_2] + [b_1, \delta_n(b_2)]$ for each $b_1, b_2 \in \mathcal{B}$. Define $D_n : \mathcal{A} \to \mathcal{B}$ by $D_0 = \alpha$ and
$D_n = \sum_{i=1}^{n}\Big(\sum_{\Sigma_{j=1}^{i} r_j = n}\big(\prod_{j=1}^{i}\frac{1}{r_j + r_{j+1} + ... + r_i}\big)\Delta_{r_1}\Delta_{r_2}...\Delta_{r_i}D_0\Big)$. By Proposition 2.2 the sequences $\{\tilde{D}_n\}$ and $\{\tilde{d}_n\}$ satisfy the following relations

$$(r+1)\tilde{d}_{r+1} = \sum_{k=0}^{r}\delta_{k+1}\tilde{d}_{r-k} \quad , \quad (n+1)\tilde{D}_{n+1} = \sum_{k=0}^{n}\Delta_{k+1}\tilde{D}_{r-k}. \tag{2.5}$$

By using induction on n, we show that $\{D_n\}$ is a generalized higher Lie derivation. For $n = 0$, we get $D_0([a_1, a_2]) = \alpha([a_1, a_2]) = [D_0(a_1), d_0(a_2)]$. suppose that $D_k([a_1, a_2]) = \sum_{i=0}^{k}[D_i(a_1), d_{k-i}(a_2)]$ for $k \leq n$. Also, by using (2.5), we obtain

$$\begin{aligned}
(n+1)D_{n+1}([a_1, a_2]) &= (n+1)\tilde{D}_{n+1}([a_1 + ker(D_0), a_2 + ker(D_0)]) \\
&= \sum_{k=0}^{n}\Delta_{k+1}\tilde{D}_{n-k}([a_1 + ker(D_0), a_2 + ker(D_0)]) \\
&= \sum_{k=0}^{n}\Delta_{k+1}D_{n-k}([a_1, a_2]) \\
&= \sum_{k=0}^{n}\Delta_{k+1}\Big(\sum_{i=0}^{n-k}[D_i(a_1), d_{n-k-i}(a_2)]\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} \Big[\sum_{k=0}^{n-i} \Delta_{k+1} D_{n-k-i}(a_1), d_i(a_2) \Big] \\
&+ \sum_{i=0}^{n} \Big[D_i(a_1), \sum_{k=0}^{n-i} \delta_{k+1} d_{n-k-i}(a_2) \Big] \\
&= \sum_{i=0}^{n} \Big[\sum_{k=0}^{n-i} \Delta_{k+1} \tilde{D}_{n-k-i}(a_1 + ker(D_0)), d_i(a_2) \Big] \\
&+ \sum_{i=0}^{n} \Big[D_i(a_1), \sum_{k=0}^{n-i} \delta_{k+1} \tilde{d}_{n-k-i}(a_2 + ker(D_0)) \Big] \\
&= \sum_{i=0}^{n} \Big[(n-i+1) \tilde{D}_{n-i+1}(a_1 + ker(D_0)), d_i(a_2) \Big] \\
&+ \sum_{i=0}^{n} \Big[D_i(a_1), (n-i+1) \tilde{d}_{n-i+1}(a_2 + ker(D_0)) \Big] \\
&= \sum_{i=0}^{n} \Big[(n-i+1) D_{n-i+1}(a_1), d_i(a_2) \Big] \\
&+ \sum_{i=0}^{n} \Big[D_i(a_1), (n-i+1) d_{n-i+1}(a_2) \Big] \\
&= \sum_{i=0}^{n+1} \Big[i D_i(a_1), d_{n-i+1}(a_2) \Big] + \sum_{i=0}^{n} \Big[(n-i+1) D_i(a_1), d_{n-i+1}(a_2) \Big] \\
&= (n+1) \sum_{k=0}^{n+1} \Big[D_k(a_1), d_{n-k+1}(a_2) \Big].
\end{aligned}
$$

Hence, $\{D_n\} \in \underline{D}$. Note that for each $n \in \mathbb{N}$, $ker(D_0) \subseteq ker(D_n)$.
Conversely, suppose that $\{D_n\} \in \underline{D}$. Define $\Delta_n : \mathcal{B} \to \mathcal{B}$ by $\Delta_1 = \tilde{D}_1 \tilde{D}_0^{-1}$ and by using (2.5), we can write

$$\Delta_n = \Big(n\tilde{D}_n - \sum_{k=0}^{n-2} \Delta_{k+1} \tilde{D}_{n-1-k} \Big) \tilde{D}_0^{-1} \qquad (n \geq 2)$$

Therefore, Theorem 2.3 ensures us that $\{\Delta_n\} \in \underline{\Delta}$. Now, define $\varphi : \underline{\Delta} \to \underline{D}$ by $\varphi(\{\Delta_n\}) = \{D_n\}$, where

$$D_n = \sum_{i=1}^{n} \Big(\sum_{\Sigma_{j=1}^{i} r_j = n} \big(\prod_{j=1}^{i} \frac{1}{r_j + r_{j+1} + ... + r_i} \big) \Delta_{r_1} \Delta_{r_2} ... \Delta_{r_i} D_0 \Big).$$

Then, it is evident that φ is surjective. Our next task is to show that it is injective. Let $\{D_n\} = \varphi(\{\Delta_n\}) = \varphi(\{\Delta'_n\}) = \{D'_n\}$. We use induction on n. For $n = 1$, we get

$$\Delta_1 = \tilde{D}_1 {\tilde{D}_0}^{-1} = \tilde{D}'_1 {\tilde{D}'_0}^{-1} = \Delta'_1$$

Now, suppose that $\Delta_k = \Delta'_k$ for $k \leq n$. Similar to the proof of Theorem 2.3 the following relations is obtained:

$$(r+1)\tilde{D}'_{n+1} = \sum_{k=0}^{n} \Delta'_{k+1}\tilde{D}'_{n-k} \quad , \quad (n+1)\tilde{D}_{n+1} = \sum_{k=0}^{n} \Delta_{k+1}\tilde{D}_{n-k}.$$

for $n \geq 0$. From this we have

$$\begin{aligned}\Delta_{n+1} &= \Big((n+1)\tilde{D}_{n+1} - \sum_{k=0}^{n-1} \Delta_{k+1}\tilde{D}_{n-k}\Big)\tilde{D}_0^{-1} \\ &= \Big((n+1)\tilde{D}'_{n+1} - \sum_{k=0}^{n-1} \Delta'_{k+1}\tilde{D}'_{n-k}\Big)\tilde{D}_0^{-1} = \Delta'_{n+1}.\end{aligned}$$

So, $\{\Delta\} = \{\Delta'_{n+1}\}$ and it means that φ is one to one.

□

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