FEKETE-SZEGO ESTIMATES AND SECOND HANKEL DETERMINANT FOR A GENERALIZED SUBFAMILY OF ANALYTIC FUNCTIONS DEFINED BY \( q \)-DIFFERENTIAL OPERATOR

AYOTUNDE O. LASODE\(^1\)* AND TIMOTHY O. OPOOLA\(^2\)

ABSTRACT. In this paper, we considered a family of analytic and univalent functions having positive real parts in the unit disk and defined by a \( q \)-difference operator. The coefficients, the Fekete-Szegö estimates and the second Hankel determinant were established for the family of functions. Our family of functions generalized some earlier known ones and by varying some parameters, our results also generalized some known ones.

1. INTRODUCTION AND PRELIMINARIES

Let the family of normalized analytic functions

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\})
\]

be denoted by \( \mathcal{A} \). Also, let \( \mathcal{S} \subset \mathcal{A} \) also denote the family of functions analytic and univalent in \( \mathbb{U} \).

The Fekete-Szegö functional \( \lambda_\sigma(f) = |a_3 - \sigma a_2^2| \) for \( f(z) \in \mathcal{S} \) is well-known for its role as a functional in determining the sharp upper bound for functions \( f(z) \in \mathcal{S} \) in geometric function theory. It was established by Fekete and Szegö [13] when they disproved the conjecture of Littlewood and Parley that the modulus of coefficients of odd functions \( f \in \mathcal{S} \) are less than or equal to 1. The functional has received great attention (see for instance [2, 5, 8, 10, 30]) particularly in many subfamilies of analytic and univalent functions. The establishment of sharp upper bound for functional \( \lambda_\sigma(f) \) for any family of functions \( \mathcal{V} \subset \mathcal{A} \) is what is known as the Fekete-Szegö problem of \( \mathcal{V} \).

In geometric function theory, the \( r \)-th Hankel determinant denoted by \( \mathcal{H}_r(k) \) \((r, k \in \mathbb{N})\), for each function \( f \in \mathcal{S} \) was considered by Pommerenke [23] and defined by

\[\sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\})\]
\[
\mathcal{H}_r(k) = \begin{vmatrix}
  a_k & a_{k+1} & \cdots & a_{k+r-1} \\
  a_{k+1} & a_{k+2} & \cdots & a_{k+r} \\
  a_{k+2} & a_{k+3} & \cdots & a_{k+r+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{k+r-1} & a_{k+r} & \cdots & a_{k+2(r-1)} \\
\end{vmatrix}
\]

where the \(a_k\)s (with \(a_1 = 1\) in \(f\)) are the corresponding coefficients of \(z^k\) \((k \in \mathbb{N})\) in \(f \in \mathcal{S}\). For some specific values of \(r\) and \(k\),

\[
|\mathcal{H}_2(1)| = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_3 - a_2^2| \quad \text{and} \quad |\mathcal{H}_2(2)| = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - a_3^2|.
\]

We note that the determinant \(|\mathcal{H}_2(1)|\) coincides with the Fekete-Szegö functional \(\lambda_1(f)\), hence, \(\lambda_\sigma(f)\) is a generalisation of \(|\mathcal{H}_2(1)|\) in (1.2). Some recent investigations in this direction includes the works in [8, 10, 19, 25]. It is worth mentioning that Pommerenke [23] gave some areas of applications of Hankel determinants in the study of singularities and the power series with integral coefficients of analytic functions. Another area of its application is in the solution of problems of orthogonal polynomials (see Junod [17]).

Jackson [15, 16] (see also [6, 7, 18]), introduced the \(q\)-derivative as follows. For functions \(f \in \mathcal{A}\) and \(0 < q < 1\), the \(q\)-differentiation of \(f\) is defined by

\[
\mathcal{D}_q f(z) = \begin{cases} 
  f'(z) & \text{for } z = 0 \\
  \frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0 
\end{cases}
\]

\[
\mathcal{D}_q^2 f(z) = \mathcal{D}_q(\mathcal{D}_q f(z))
\]

From (1.1) and (1.3), we can establish that

\[
\mathcal{D}_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}
\]

\[
\mathcal{D}_q^2 f(z) = \sum_{k=2}^{\infty} [k-1]_q [k]_q a_k z^{k-2}
\]

where \([k]_q = \frac{1-q^k}{1-q}\), \([k-1]_q = \frac{1-q^{k-1}}{1-q}\), \(\lim_{q \uparrow 1} [k]_q = k\) and \(\lim_{q \uparrow 1} [k-1]_q = k-1\). Notably, some researchers (such as [1, 2, 3, 24, 29]) have used the \(q\)-difference operator in some ways to define some families of analytic functions.

Here, the \(q\)-difference operator is employed to define a new family of analytic functions and generalize some results. The investigated family of functions is defined as follows.

**Definition 1.1.** Let \(\alpha \in (-\pi, \pi]\), \(\beta \in [0, 1)\) and let \(\mathcal{D}_q f(z)\) be as defined in (1.3), then a function \(f(z)\) analytic and univalent in the unit disk \(\mathbb{U}\) is said to be in \(\mathcal{A}_q(\alpha, \beta)\) if

\[
\Re \left\{ \mathcal{D}_q f(z) + \frac{1 + e^{i\alpha}}{2} z \mathcal{D}_q^2 f(z) \right\} > \beta \quad (z \in \mathbb{U}).
\]

**Remark 1.2.** The following are some subfamilies of \(\mathcal{A}_q(\alpha, \beta)\). Let \(q \uparrow 1\),

1. \(\alpha = 0 = \beta\), then \(\mathcal{A}_q(\alpha, \beta)\) becomes the family \(\mathcal{R}\) studied by Chichra [11].
(2) \( \alpha = 0 \), then \( \mathcal{A}_q(\alpha, \beta) \) becomes the family \( \mathcal{L}(\beta) \) studied by Silverman \[26\].
(3) \( \beta = 0 \), then \( \mathcal{A}_q(\alpha, \beta) \) becomes the family \( \mathcal{A}(\alpha, 0) \) studied by Silverman and Silvia \[27\].
(4) \( \alpha = \pi \) and \( \beta = 0 \), then \( \mathcal{A}_q(\alpha, \beta) \) becomes the family \( \mathcal{B} \) studied by MacGregor \[21\] and later by Babalola and Opoola \[9\].
(5) \( \alpha = \pi \), then \( \mathcal{A}_q(\alpha, \beta) \) becomes the family \( \mathcal{B}(\beta) \) studied by Krishna \textit{et al.} \[19\].
(6) then \( \mathcal{A}_q(\alpha, \beta) \) becomes the family \( \mathcal{R}(\alpha, \beta) \) studied by Mahzoon and Kargar \[22\].

2. Preliminary Lemmas

The following are the relevant lemmas that are applied in the course of proving some of our results. Let \( \mathcal{P} \) represent the family of functions of the form

\[
p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad (z \in U, \ p(0) = 1 \text{ and } \Re p(z) > 0).
\]

(2.1) \( \mathcal{P} \) is the familiar family of Carathéodory functions \[12\]. Now for \( p \in \mathcal{P} \), the following lemmas hold true.

**Lemma 2.1.** \[14\]. \(|p_k| \leq 2 \ (k \in \mathbb{N})\).

**Lemma 2.2.** \[28\]. \( |p_2 - \lambda p_1^2| \leq 2 \max \{1, |2\lambda - 1| \} \ (\lambda \in \mathbb{C})\).

**Lemma 2.3.** \[20\].

\[
2p_2 = p_1^2 + (4 - p_1^2)x,
4p_2 = p_1^4 + 2(4 - p_1^2)p_1^2x + (4 - p_1^2)^2x^2
4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z
\]

for some \( x, z \) such that \(|x|, |z| \leq 1\).

3. Main Results

The following are the established results.

**Theorem 3.1.** Let \( \alpha \in (-\pi, \pi] \) and \( \beta \in [0, 1) \). If \( f(z) \in \mathcal{A}_q(\alpha, \beta) \), then

\[
a_k = \frac{2(1 - \alpha)p_{k-1}}{[k]q \psi_k} \quad (\psi_k = 2 + (1 + e^{i\alpha})[k - 1]_q, \ k = \{2, 3, \ldots\}). \quad (3.1)
\]

**Proof.** Let \( f(z) \in \mathcal{A}_q(\alpha, \beta) \) so that in view of (2.1), (1.5) can be expressed as

\[
D_q f(z) + \frac{1 + e^{i\alpha}}{2}z D_q^2 f(z) = \beta + (1 - \beta)p(z)
\]

(3.2)

so putting (2.1) and (1.4) into (3.2) gives

\[
1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} + \sum_{k=2}^{\infty} \frac{1 + e^{i\alpha}}{2} [k]_q [k - 1]_q a_k z^{k-1} = \beta + (1 - \beta) + \sum_{k=1}^{\infty} (1 - \beta)p_k z^k
\]

\[
1 + \sum_{k=2}^{\infty} \left\{1 + \frac{1 + e^{i\delta}}{2} [k - 1]_q \right\} [k]_q a_k z^{k-1} = 1 + \sum_{k=2}^{\infty} (1 - \beta)p_{k-1} z^{k-1}
\]
which implies that
\[
\{2 + (1 + e^{i\delta})[k-1]_q\} \frac{[k]_q}{2} a_k = (1 - \beta)p_{k-1}
\]
so that
\[
a_k = \frac{2(1 - \beta)p_{k-1}}{[k]_q \{2 + (1 + e^{i\delta})[k-1]_q\}} = \frac{2(1 - \beta)p_{k-1}}{[k]_q \psi_k} \tag{3.3}
\]

**Remark 3.2.** Express \(|\psi_k|\) as follows. From (3.1),
\[
|\psi_k| = |2 + (1 + e^{i\alpha})[k-1]_q|
\]
\[
= \sqrt{2\{2 + [k-1]_q(2 + [k-1]_q)(1 + \cos \alpha)\}} \geq 2 \quad (k = \{2, 3, \ldots\}).
\]

**Corollary 3.3.** Let \(\alpha \in (-\pi, \pi)\) and \(\beta \in [0, 1)\). If \(f(z) \in \mathcal{A}_q(\alpha, \beta)\), then
\[
|a_k| \leq \frac{4(1 - \beta)}{[k]_q|\psi_k|} \quad (k = \{2, 3, \ldots\}) \tag{3.4}
\]
where \(|\psi_k|\) is as defined in Remark 3.2.

In [4], Ali et al. investigated the Fekete-Szego problem related to the \(t\)-th root transformation for some subfamilies of \(S\). For a function \(f \in S\) of the form (1.1), the \(t\)-th root transformation is defined by (see [14])
\[
F(z) = \sqrt[t]{f(z^t)} = z + \frac{1}{t} a_2 z^{t+1} + \left( \frac{1}{t} a_3 - \frac{1 - t}{2t^2} a_2^2 \right) z^{2t+1} + \cdots \tag{3.5}
\]
\[
\equiv z + d_{t+1} z^{t+1} + d_{2t+1} z^{2t+1} + d_{3t+1} z^{3t+1} + \cdots \quad (t \in \mathbb{N}). \tag{3.6}
\]
Thus our theorem follows.

**Theorem 3.4.** Let \(f(z) \in \mathcal{A}_q(\alpha, \beta)\). If \(\alpha \in (-\pi, \pi)\), \(\beta \in [0, 1)\) and \(\rho \in \mathbb{R}\), then
\[
|d_{2t+1} - \rho d_{t+1}^2| \leq \left\{ \begin{array}{ll}
\frac{8(1 - \beta)}{2t^2 q_2^2} & \text{for } \rho \leq \frac{1}{2} \left( \frac{t[2]_q^2 q_2^2}{2(1 - \beta)(1 + \beta) q_2^2} + 1 - t \right) \\
\frac{16(1 - \beta)^2(2\alpha + 1 - \beta)}{t^2 [2]_q^4 q_2^4} & \text{for } \rho \geq \frac{1}{2} \left( \frac{t[2]_q^2 q_2^2}{2(1 - \alpha)(1 + \beta) q_2^2} + 1 - t \right)
\end{array} \right.
\]
where \(t \in \mathbb{N}\), \(d_{kt+1}\) is as defined in (3.5) and \(|\psi_k|\) is as defined in Remark 3.2.

**Proof.** It follows from Theorem 3.1 that by substituting \(a_2\) and \(a_3\) into (3.5) gives
\[
d_{t+1} = \frac{1}{t} a_2 = \frac{2(1 - \alpha)p_1}{t[2]_q q_2} \tag{3.7}
\]
and
\[
d_{2t+1} = \frac{1}{t} a_3 - \frac{t - 1}{2t^2} a_2^2 = \frac{2(1 - \beta)p_2}{t[3]_q q_3} - \frac{2(t - 1)(1 - \beta)^2 p_1^2}{t^2 [2]_q^2 q_2^2} \tag{3.8}
\]
so that
\[
|d_{2t+1} - \rho d_{t+1}^2| \leq \left| \frac{2(1 - \beta) p_2}{t[3]_q q_3} \right| + \left| \frac{2(1 - \beta)^2(2\rho + t - 1)p_1^2}{t^2 [2]_q^2 q_2^2} \right|. \tag{3.9}
\]
This implies that

\[ |d_{2t+1} - \rho d_{t+1}^2| \leq 2 \left| \frac{2(1 - \beta)p_2}{t[3]_q \psi_3} \right| \]

if \[ \left| \frac{2(1 - \beta)^2(2\rho + t - 1)p_1^2}{t^2[2]_q^2 \psi_2^2} \right| \leq \left| \frac{2(1 - \beta)p_2}{t[3]_q \psi_3} \right| \] (3.10)

and

\[ |d_{2t+1} - \rho d_{t+1}^2| \leq 2 \left| \frac{2(1 - \beta)^2(2\rho + t - 1)p_1^2}{t^2[2]_q^2 \psi_2^2} \right| \]

if \[ \left| \frac{2(1 - \beta)p_2}{t[3]_q \psi_3} \right| \leq \left| \frac{2(1 - \beta)^2(2\rho + t - 1)p_1^2}{t^2[2]_q^2 \psi_2^2} \right| \] (3.11)

Now from (3.10) and by using Lemma 2.1 we have

\[ |d_{2t+1} - \rho d_{t+1}^2| \leq \frac{8(1 - \beta)}{t[3]_q \psi_3} \]

for \( \rho \leq \frac{1}{2} \left\{ \frac{t[2]_q^2 \psi_2^2}{2(1 - \beta)[3]_q \psi_3} + 1 \right\} \)

and from (3.11) and by using Lemma 2.1 we have

\[ |d_{2t+1} - \rho d_{t+1}^2| \leq \frac{16(1 - \beta)^2(2\rho + t - 1)}{t^2[2]_q^2 \psi_2^2} \]

for \( \rho \geq \frac{1}{2} \left\{ \frac{t[2]_q^2 \psi_2^2}{2(1 - \beta)[3]_q \psi_3} + 1 \right\} \).

Thus putting the results together completes the proof.

\[ \square \]

**Theorem 3.5.** Let \( f(z) \in A_q(\alpha, \beta) \). If \( \alpha \in (-\pi, \pi], \beta \in [0, 1) \) and \( q \in \mathbb{C} \), then

\[ |d_{2t+1} - \rho d_{t+1}^2| \leq \frac{4(1 - \beta)(2\rho + t - 1)}{t[3]_q \psi_3} \max \left\{ 1, \left| \frac{2(1 - \beta)^2(2\rho + t - 1)[3]_q \psi_3}{t[2]_q^2 \psi_2^2} - 1 \right| \right\} \]

where \( t \in \mathbb{N} \), \( d_{kt+1} \) is as defined in (3.5) and \( |\psi_k| \) is as defined in Remark 3.2.

**Proof.** From (3.7) and (3.8),

\[ |d_{2t+1} - \rho d_{t+1}^2| = \frac{2(1 - \beta)}{t[3]_q \psi_3} \left\{ p_2 - \frac{(1 - \beta)(2\rho + t - 1)[3]_q \psi_3}{t[2]_q^2 \psi_2^2} p_1^2 \right\} \]

\[ \leq \frac{2(1 - \beta)}{t[3]_q \psi_3} \left| p_2 - \lambda p_1^2 \right| \]

where

\[ \lambda = \frac{(1 - \beta)(2\rho + t - 1)[3]_q \psi_3}{t[2]_q^2 \psi_2^2} \]

and by applying Lemma 2.2 completes the proof. \[ \square \]

**Theorem 3.6 (Hankel Determinant: \( H_2(2) \)).** Let \( f(z) \in A_q(\alpha, \beta) \). If \( \beta \in [0, 1) \), then

\[ |H_2(2)| = |a_2a_4 - a_3^2| \leq \frac{18(1 - \beta)^2}{L} + \frac{16(1 - \beta)^2}{M} \]

where

\[ L = \left| [2]_q [4]_q \psi_2 \psi_4 \right| = \left| [2]_q [4]_q \psi_2 \right| \left| \psi_4 \right| > 0 \]

\[ M = \left| [3]_q^2 \psi_3^2 \right| = \left| [3]_q^2 \psi_3 \right| > 0 \]

(3.12)
and $|\psi_k|$ is as defined in (3.1).

**Proof.** From Theorem 3.1,

$$a_2a_4 - a_3^2 = \left( \frac{2(1 - \beta)p_1}{[2]_q\psi_2} \right) \left( \frac{2(1 - \beta)p_3}{[4]_q\psi_4} \right) - \left( \frac{2(1 - \beta)p_2}{[3]_q\psi_3} \right)^2$$

$$= \frac{4(1 - \beta)^2p_1p_3}{[2]_q[4]_q\psi_2\psi_4} - \frac{4(1 - \beta)^2p_2^2}{[3]_q^2\psi_3^2}.$$

Applying Lemma 2.3 simplifying gives

$$|a_2a_4 - a_3^2| = \left| \frac{(1 - \beta)^2p_1^4}{[2]_q[4]_q\psi_2\psi_4} + \frac{2(1 - \beta)^2(4 - p_1^2)p_2^2x}{[2]_q[4]_q\psi_2\psi_4} \right| - \left| \frac{(1 - \beta)^2(4 - p_2^2)p_2^2x}{[3]_q^2\psi_3^2} \right|.$$

Now let $p = p_1$ so that $|p_1| = |p| \leq 2$ and without loss of generality, $p \in [0, 2]$. Applying triangle inequality, (3.12), for $|z| \leq 1$ and regrouping such that $\eta = |x| \leq 1$, then we have

$$|a_2a_4 - a_3^2| \leq \left\{ \begin{array}{l}
\frac{(1 - \beta)^2p_1^4}{L} + \frac{2(1 - \beta)^2(4 - p_1^2)p_2^2x}{L} + \frac{(1 - \beta)^2p_1^4}{M} \\
\frac{2(1 - \beta)^2(4 - p_1^2)p_2^2}{L} + \frac{2(1 - \beta)^2(4 - p_1^2)p_2^2}{M} \end{array} \right\} \eta^2 + \left\{ \begin{array}{l}
\frac{(1 - \beta)^2(4 - p_1^2)p_2^2}{L} - \frac{2(1 - \beta)^2(4 - p_1^2)p_2^2}{L} \\
\frac{(1 - \beta)^2(4 - p_1^2)p_2^2}{M} \end{array} \right\} \eta^2 = F_1(\eta, p).$$

Maximizing the function $F_1(\eta, p)$ in the closed interval $\eta \in [0, 1]$ gives

$$\frac{\partial F_1(\eta, p)}{\partial \eta} = \left\{ \begin{array}{l}
\frac{2(1 - \beta)^2(4 - p_1^2)p_2^2}{L} + \frac{2(1 - \beta)^2(4 - p_1^2)p_2^2}{M} \\
- \frac{2(1 - \beta)^2(4 - p_1^2)p_2^2}{L} + \frac{(1 - \beta)^2(4 - p_1^2)p_2^2}{M} \end{array} \right\} \eta^2.$$
Expanding and simplifying the terms for $p$ gives

$$F_1(1,p) \leq \frac{16(1-\beta)^2}{M} + \frac{12(1-\beta)^2}{L}p^2 - \frac{2(1-\beta)^2}{L}p^4 = F_2(p)$$

so that

$$F_2'(p) = \frac{24(1-\beta)^2}{L}p - \frac{8(1-\beta)^2}{L}p^3.$$  \hspace{1cm} (3.15)

Now at the critical points, $F_2'(p) = 0$ implies that

$$\frac{24(1-\beta)^2}{L}p - \frac{8(1-\beta)^2}{L}p^3 = \left\{ \frac{24(1-\beta)^2}{L} - \frac{8(1-\beta)^2}{L}p^2 \right\} p = 0$$

which means that $p_0 = 0$ and $p_1 = \sqrt{3}$, hence, from (3.15),

$$F_2''(p_0) = \frac{24(1-\beta)^2}{L} > 0 \quad \text{and} \quad F_2''(p_1) = -\frac{48(1-\beta)^2}{L} < 0.$$  

Now (3.14) attains maximum at

$$F_2(p) = F_2(\sqrt{3}) = \frac{16(1-\beta)^2}{M} + \frac{36(1-\beta)^2}{L} - \frac{18(1-\beta)^2}{L}$$

that is

$$|a_2a_4 - a_3^2| \leq \frac{16(1-\beta)^2}{M} + \frac{18(1-\beta)^2}{L}.$$  

and the proof is complete. \hspace{1cm} \Box

**Acknowledgement:** The authors thank the referees for their individual and collective valuable suggestions that helped to produce this quality paper.

**References**


---

1 Department of Mathematics, University of Ilorin, Ilorin, Nigeria. 
Email address: lasode_ayo@yahoo.com

2 Department of Mathematics, University of Ilorin, Ilorin, Nigeria. 
Email address: opoola_stc@yahoo.com