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FEKETE-SZEGO ESTIMATES AND SECOND HANKEL DETERMINANT FOR A GENERALIZED SUBFAMILY OF ANALYTIC FUNCTIONS DEFINED BY q-DIFFERENTIAL OPERATOR

AYOTUNDE O. LASODE[1*] AND TIMOTHY O. OPOOLA[2]

Abstract. In this paper, we considered a family of analytic and univalent functions having positive real parts in the unit disk and defined by a q-difference operator. The coefficients, the Fekete-Szegö estimates and the second Hankel determinant were established for the family of functions. Our family of functions generalized some earlier known ones and by varying some parameters, our results also generalized some known ones.

1. Introduction and preliminaries

Let the family of normalized analytic functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathscr{U} := \{z \in \mathbb{C} : |z| < 1\}) \tag{1.1}$$

be denoted by $\mathcal{A}$. Also, let $\mathcal{S} \subset \mathcal{A}$ also denote the family of functions analytic and univalent in $\mathscr{U}$.

The Fekete-Szegö functional $\lambda_\sigma(f) = |a_3 - \sigma a_2^2|$ for $f(z) \in \mathcal{S}$ is well-known for its role as a functional in determining the sharp upper bound for functions $f(z) \in \mathcal{S}$ in geometric function theory. It was established by Fekete and Szegö [13] when they disproved the conjecture of Littlewood and Parley that the modulus of coefficients of odd functions $f \in \mathcal{S}$ are less than or equal to 1. The functional has received great attention (see for instance [2, 5, 8, 10, 30]) particularly in many subfamilies of analytic and univalent functions. The establishment of sharp upper bound for functional $\lambda_\sigma(f)$ for any family of functions $\mathcal{V} \subset \mathcal{A}$ is what is known as the Fekete-Szegö problem of $\mathcal{V}$.

In geometric function theory, the rth-Hankel determinant denoted by $\mathcal{H}_r(k)$ $(r, k \in \mathbb{N})$, for each function $f \in \mathcal{S}$ was considered by Pommerenke [23] and defined by

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* Corresponding author.

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$$\mathcal{H}_r(k) = \begin{vmatrix} a_k & a_{k+1} & \dots & a_{k+r-1} \\ a_{k+1} & a_{k+2} & \dots & a_{k+r} \\ a_{k+2} & a_{k+3} & \dots & a_{k+r+1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+r-1} & a_{k+r} & \dots & a_{k+2(r-1)} \end{vmatrix}$$

where the $a_k s$ (with $a_1 = 1$ in f) are the corresponding coefficients of z^k ($k \in \mathbb{N}$) in $f \in \mathcal{S}$. For some specific values of r and k,

$$|\mathcal{H}_2(1)| = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_3 - {a_2}^2| \text{ and } |\mathcal{H}_2(2)| = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - {a_3}^2|. \quad (1.2)$$

We note that the determinant $|\mathcal{H}_2(1)|$ coincides with the Fekete-Szegö functional $\lambda_1(f)$, hence, $\lambda_\sigma(f)$ is a generalisation of $|\mathcal{H}_2(1)|$ in (1.2). Some recent investigations in this direction includes the works in [8, 10, 19, 25]. It is worth mentioning that Pommerenke [23] gave some areas of applications of Hankel determinants in the study of singularities and the power series with integral coefficients of analytic functions. Another area of its application is in the solution of problems of orthogonal polynomials (see Junod [17]).

Jackson [15, 16] (see also [6, 7, 18]), introduced the q-derivative as follows. For functions $f \in \mathcal{A}$ and $0 < q < 1$, the q-differentiation of f is defined by

$$\left.\begin{aligned} \mathcal{D}_q f(z) &= \begin{cases} f'(z) & \text{for } z = 0 \\ \frac{f(z)-f(qz)}{(1-q)z} & \text{for } z \neq 0 \end{cases} \\ \mathcal{D}_q^2 f(z) &= \mathcal{D}_q(\mathcal{D}_q f(z)) \end{aligned}\right\} \quad (1.3)$$

From (1.1) and (1.3), we can establish that

$$\left.\begin{aligned} \mathcal{D}_q f(z) &= 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} \\ \mathcal{D}_q^2 f(z) &= \sum_{k=2}^{\infty} [k-1]_q [k]_q a_k z^{k-2} \end{aligned}\right\} \quad (1.4)$$

where $[k]_q = \frac{1-q^k}{1-q}$, $[k-1]_q = \frac{1-q^{k-1}}{1-q}$, $\lim\limits_{q\uparrow 1}[k]_q = k$ and $\lim\limits_{q\uparrow 1}[k-1]_q = k-1$. Notably, some researchers (such as [1, 2, 3, 24, 29]) have used the q-difference operator in some ways to define some families of analytic functions.

Here, the q-difference operator is employed to define a new family of analytic functions and generalize some results. The investigated family of functions is defined as follows.

Definition 1.1. Let $\alpha \in (-\pi, \pi]$, $\beta \in [0, 1)$ and let $\mathcal{D}_q f(z)$ be as defined in (1.3), then a function $f(z)$ analytic and univalent in the unit disk $\mathscr{U}$ is said to be in $\mathcal{A}_q(\alpha, \beta)$ if

$$\Re\left\{\mathcal{D}_q f(z) + \frac{1 + e^{i\alpha}}{2} z \mathcal{D}_q^2 f(z)\right\} > \beta \quad (z \in \mathscr{U}). \quad (1.5)$$

Remark 1.2. The following are some subfamilies of $\mathcal{A}_q(\alpha, \beta)$. Let $q \uparrow 1$,

(1) $\alpha = 0 = \beta$, then $\mathcal{A}_q(\alpha, \beta)$ becomes the family $\mathcal{R}$ studied by Chichra [11].

(2) $\alpha = 0$, then $\mathcal{A}_q(\alpha, \beta)$ becomes the family $\mathcal{L}(\beta)$ studied by Silverman [26].
(3) $\beta = 0$, then $\mathcal{A}_q(\alpha, \beta)$ becomes the family $\mathcal{A}(\alpha, 0)$ studied by Silverman and Silvia [27].
(4) $\alpha = \pi$ and $\beta = 0$, then $\mathcal{A}_q(\alpha, \beta)$ becomes the family $\mathcal{B}$ studied by MacGregor [21] and later by Babalola and Opoola [9].
(5) $\alpha = \pi$, then $\mathcal{A}_q(\alpha, \beta)$ becomes the family $\mathcal{B}(\beta)$ studied by Krishna *et al.* [19].
(6) then $\mathcal{A}_q(\alpha, \beta)$ becomes the family $\mathcal{R}(\alpha, \beta)$ studied by Mahzoon and Kargar [22].

2. Preliminary Lemmas

The following are the relevant lemmas that are applied in the course of proving some of our results. Let $\mathcal{P}$ represent the family of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad (z \in \mathscr{U},\ p(0) = 1 \text{ and } \Re p(z) > 0). \tag{2.1}$$

$\mathcal{P}$ is the familiar family of Carathéodory functions [12]. Now for $p \in \mathcal{P}$, the following lemmas hold true.

Lemma 2.1. [14]. $|p_k| \leq 2$ $(k \in \mathbb{N})$.

Lemma 2.2. [28]. $\left|p_2 - \lambda p_1^2\right| \leq 2 \max\left\{1, \left|2\lambda - 1\right|\right\}$ $(\lambda \in \mathbb{C})$.

Lemma 2.3. [20].

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$
$$4p_2^2 = p_1^4 + 2(4 - p_1^2)p_1^2 x + (4 - p_1^2)^2 x^2$$
$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some x, z such that $|x|, |z| \leq 1$.

3. Main Results

The following are the established results.

Theorem 3.1. *Let $\alpha \in (-\pi, \pi]$ and $\beta \in [0, 1)$. If $f(z) \in \mathcal{A}_q(\alpha, \beta)$, then*

$$a_k = \frac{2(1 - \alpha)p_{k-1}}{[k]_q \psi_k} \qquad (\psi_k = 2 + (1 + e^{i\alpha})[k - 1]_q,\ k = \{2, 3, \ldots\}). \tag{3.1}$$

Proof. Let $f(z) \in \mathcal{A}_q(\alpha, \beta)$ so that in view of (2.1), (1.5) can be expressed as

$$\mathcal{D}_q f(z) + \frac{1 + e^{i\alpha}}{2} z \mathcal{D}_q^2 f(z) = \beta + (1 - \beta)p(z) \tag{3.2}$$

so putting (2.1) and (1.4) into (3.2) gives

$$1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} + \sum_{k=2}^{\infty} \frac{1 + e^{i\alpha}}{2} [k]_q [k - 1]_q a_k z^{k-1} = \beta + (1 - \beta) + \sum_{k=1}^{\infty} (1 - \beta) p_k z^k$$

$$1 + \sum_{k=2}^{\infty} \left\{1 + \frac{1 + e^{i\delta}}{2} [k - 1]_q\right\} [k]_q a_k z^{k-1} = 1 + \sum_{k=2}^{\infty} (1 - \beta) p_{k-1} z^{k-1}$$

which implies that

$$\{2+(1+e^{i\delta})[k-1]_q\}\frac{[k]_q}{2}a_k=(1-\beta)p_{k-1}$$

so that

$$a_k=\frac{2(1-\beta)p_{k-1}}{[k]_q\{2+(1+e^{i\delta})[k-1]_q\}}=\frac{2(1-\beta)p_{k-1}}{[k]_q\psi_k} \tag{3.3}$$

□

Remark 3.2. Express $|\psi_k|$ as follows. From (3.1),

$$\begin{aligned}|\psi_k| &= |2+(1+e^{i\alpha})[k-1]_q| \\ &= \sqrt{2\{2+[k-1]_q(2+[k-1]_q)(1+\cos\alpha)\}}\geq 2 \quad (k=\{2,3,\ldots\}).\end{aligned}$$

Corollary 3.3. *Let $\alpha\in(-\pi,\pi]$ and $\beta\in[0,1)$. If $f(z)\in\mathcal{A}_q(\alpha,\beta)$, then*

$$|a_k|\leq\frac{4(1-\beta)}{[k]_q|\psi_k|}\qquad(k=\{2,3,\ldots\}) \tag{3.4}$$

where $|\psi_k|$ is as defined in Remark 3.2.

In [4], Ali *et al.* investigated the Fekete-Szegö problem related to the t^{th}-root transformation for some subfamilies of $\mathcal{S}$. For a function $f\in\mathcal{S}$ of the form (1.1), the t^{th}-root transformation is defined by (see [14])

$$F(z)=\sqrt[t]{f(z^t)}=z+\frac{1}{t}a_2z^{t+1}+\left(\frac{1}{t}a_3-\frac{t-1}{2t^2}a_2^2\right)z^{2t+1}+\cdots \tag{3.5}$$

$$\equiv z+d_{t+1}z^{t+1}+d_{2t+1}z^{2t+1}+d_{3t+1}z^{3t+1}+\cdots\quad(t\in\mathbb{N}). \tag{3.6}$$

Thus our theorem follows.

Theorem 3.4. *Let $f(z)\in\mathcal{A}_q(\alpha,\beta)$. If $\alpha\in(-\pi,\pi]$, $\beta\in[0,1)$ and $\rho\in\mathbb{R}$, then*

$$|d_{2t+1}-\rho d_{t+1}^2|\leq\begin{cases}\frac{8(1-\beta)}{t[3]_q|\psi_3|} & for\quad \rho\leq\frac{1}{2}\left(\frac{t[2]_q^2|\psi_2^2|}{2(1-\beta)[3]_q|\psi_3|}+1-t\right)\\ \frac{16(1-\beta)^2(2\alpha+t-1)}{t^2[2]_q^2|\psi_2^2|} & for\quad \rho\geq\frac{1}{2}\left(\frac{t[2]_q^2|\psi_2^2|}{2(1-\alpha)[3]_q|\psi_3|}+1-t\right)\end{cases}$$

where $t\in\mathbb{N}$, d_{kt+1} is as defined in (3.5) and $|\psi_k|$ is as defined in Remark 3.2.

Proof. It follows from Theorem 3.1 that by substituting a_2 and a_3 into (3.5) gives

$$d_{t+1}=\frac{1}{t}a_2=\frac{2(1-\alpha)p_1}{t[2]_q\psi_2} \tag{3.7}$$

and

$$d_{2t+1}=\frac{1}{t}a_3-\frac{t-1}{2t^2}a_2^2=\frac{2(1-\beta)p_2}{t[3]_q\psi_3}-\frac{2(t-1)(1-\beta)^2p_1^2}{t^2[2]_q^2\psi_2^2} \tag{3.8}$$

so that

$$|d_{2t+1}-\rho d_{t+1}^2|\leq\left|\frac{2(1-\beta)p_2}{t[3]_q\psi_3}\right|+\left|\frac{2(1-\beta)^2(2\rho+t-1)p_1^2}{t^2[2]_q^2\psi_2^2}\right|. \tag{3.9}$$

This implies that

$$|d_{2t+1}-\rho d_{t+1}^2|\leq 2\left|\frac{2(1-\beta)p_2}{t[3]_q\psi_3}\right| \quad \text{if} \quad \left|\frac{2(1-\beta)^2(2\rho+t-1)p_1^2}{t^2[2]_q^2\psi_2^2}\right|\leq\left|\frac{2(1-\beta)p_2}{t[3]_q\psi_3}\right| \tag{3.10}$$

and

$$|d_{2t+1}-\rho d_{t+1}^2|\leq 2\left|\frac{2(1-\beta)^2(2\rho+t-1)p_1^2}{t^2[2]_q^2\psi_2^2}\right| \quad \text{if} \quad \left|\frac{2(1-\beta)p_2}{t[3]_q\psi_3}\right|\leq\left|\frac{2(1-\beta)^2(2\rho+t-1)p_1^2}{t^2[2]_q^2\psi_2^2}\right|. \tag{3.11}$$

Now from (3.10) and by using Lemma 2.1 we have

$$|d_{2t+1}-\rho d_{t+1}^2|\leq\frac{8(1-\beta)}{t[3]_q|\psi_3|} \quad \text{for} \quad \rho\leqq\frac{1}{2}\left\{\frac{t[2]_q^2|\psi_2^2|}{2(1-\beta)[3]_q|\psi_3|}+1-t\right\}$$

and from (3.11) and by using Lemma 2.1 we have

$$|d_{2t+1}-\rho d_{t+1}^2|\leq\frac{16(1-\beta)^2(2\rho+t-1)}{t^2[2]_q^2|\psi_2^2|} \text{ for } \rho\geq\frac{1}{2}\left\{\frac{t[2]_q^2|\psi_2^2|}{2(1-\beta)[3]_q|\psi_3|}+1-t\right\}.$$

Thus putting the results together completes the proof. □

Theorem 3.5. *Let $f(z)\in\mathcal{A}_q(\alpha,\beta)$. If $\alpha\in(-\pi,\pi]$, $\beta\in[0,1)$ and $\varrho\in\mathbb{C}$, then*

$$|d_{2t+1}-\varrho d_{t+1}^2|\leq\frac{4(1-\beta)}{t[3]_q|\psi_3|}\max\left\{1,\left|\frac{2(1-\beta)(2\varrho+t-1)[3]_q\psi_3}{t[2]_q^2\psi_2^2}-1\right|\right\}$$

where $t\in\mathbb{N}$, d_{kt+1} is as defined in (3.5) and $|\psi_k|$ is as defined in Remark 3.2.

Proof. From (3.7) and (3.8),

$$\begin{aligned}|d_{2t+1}-\varrho d_{t+1}^2|&=\left|\frac{2(1-\beta)}{t[3]_q\psi_3}\left\{p_2-\frac{(1-\beta)(2\varrho+t-1)[3]_q\psi_3}{t[2]_q^2\psi_2^2}p_1^2\right\}\right|\\&\leq\frac{2(1-\beta)}{t[3]_q|\psi_3|}\left|p_2-\lambda p_1^2\right|\end{aligned}$$

where

$$\lambda=\frac{(1-\beta)(2\varrho+t-1)[3]_q\psi_3}{t[2]_q^2\psi_2^2}.$$

and by applying Lemma 2.2 completes the proof. □

Theorem 3.6 (Hankel Determinant: $\mathcal{H}_2(2)$)**.** *Let $f(z)\in\mathcal{A}_q(\alpha,\beta)$. If $\beta\in[0,1)$, then*

$$|\mathcal{H}_2(2)|=|a_2a_4-{a_3}^2|\leq\frac{18(1-\beta)^2}{L}+\frac{16(1-\beta)^2}{M}$$

where

$$\left.\begin{aligned}L&=\left|[2]_q[4]_q\psi_2\psi_4\right|=[2]_q[4]_q|\psi_2|\left|\psi_4\right|>0\\M&=\left|[3]_q^2\psi_3^2\right|=[3]_q^2\left|\psi_3\right|^2>0\end{aligned}\right\} \tag{3.12}$$

and $|\psi_k|$ *is as defined in* (3.1).

Proof. From Theorem 3.1,

$$a_2a_4 - {a_3}^2 = \left(\frac{2(1-\beta)p_1}{[2]_q\psi_2}\right)\left(\frac{2(1-\beta)p_3}{[4]_q\psi_4}\right) - \left(\frac{2(1-\beta)p_2}{[3]_q\psi_3}\right)^2$$
$$= \frac{4(1-\beta)^2p_1p_3}{[2]_q[4]_q\psi_2\psi_4} - \frac{4(1-\beta)^2p_2^2}{[3]_q^2\psi_3^2}.$$

Applying Lemma 2.3 simplifying gives

$$|a_2a_4 - {a_3}^2| = \left|\frac{(1-\beta)^2p_1^4}{[2]_q[4]_q\psi_2\psi_4} + \frac{2(1-\beta)^2(4-p_1^2)p_1^2x}{[2]_q[4]_q\psi_2\psi_4} - \frac{(1-\beta)^2(4-p_1^2)p_1^2x^2}{[2]_q[4]_q\psi_2\psi_4}\right.$$
$$+ \frac{2(1-\beta)^2(4-p_1^2)(1-|x|^2)p_1z}{[2]_q[4]_q\psi_2\psi_4} - \frac{(1-\beta)^2p_1^4}{[3]_q^2\psi_3^2}$$
$$\left. - \frac{2(1-\beta)^2(4-p_1^2)p_1^2x}{[3]_q^2\psi_3^2} - \frac{(1-\beta)^2(4-p_1^2)^2x^2}{[3]_q^2\psi_3^2}\right|.$$

Now let $p = p_1$ so that $|p_1| = |p| \leq 2$ and without loss of generality, $p \in [0, 2]$. Applying triangle inequality, (3.12), for $|z| \leq 1$ and regrouping such that $\eta = |x| \leq 1$, then we have

$$|a_2a_4 - {a_3}^2| \leq \left\{\frac{(1-\beta)^2p^4}{L} + \frac{2(1-\beta)^2(4-p^2)p}{L} + \frac{(1-\beta)^2p^4}{M}\right\}$$
$$+ \left\{\frac{2(1-\beta)^2(4-p^2)p^2}{L} + \frac{2(1-\beta)^2(4-p^2)p^2}{M}\right\}\eta$$
$$+ \left\{\frac{(1-\beta)^2(4-p^2)p^2}{L} - \frac{2(1-\beta)^2(4-p^2)p}{L}\right.$$
$$\left. + \frac{(1-\beta)^2(4-p^2)^2}{M}\right\}\eta^2 = F_1(\eta, p). \tag{3.13}$$

Maximizing the function $F_1(\eta, p)$ in the closed interval $\eta \in [0, 1]$ gives

$$\frac{\partial F_1(\eta,p)}{\partial\eta} = \left\{\frac{2(1-\beta)^2(4-p^2)p^2}{L} + \frac{2(1-\beta)^2(4-p^2)p^2}{M}\right\}$$
$$+ 2\left\{\frac{(1-\beta)^2(4-p^2)p^2}{L} - \frac{2(1-\beta)^2(4-p^2)p}{L} + \frac{(1-\beta)^2(4-p^2)^2}{M}\right\}\eta$$

Observe that $\frac{\partial F_1(\eta,p)}{\partial\eta} > 0$ for $\eta \in [0, 1]$, therefore, $F_1(\eta, p)$ is an increasing function, thus, it has maximum point at $\eta = 1$. Hence from (3.13),

$$\max_{\eta\in[0,1]} F_1(\eta, p) = F_1(1, p) \leq \frac{(1-\beta)^2p^4}{L} + \frac{(1-\beta)^2p^4}{M} + \frac{2(1-\beta)^2(4-p^2)p^2}{L}$$
$$+ \frac{2(1-\beta)^2(4-p^2)p^2}{M} + \frac{(1-\beta)^2(4-p^2)p^2}{L} + \frac{(1-\beta)^2(4-p^2)^2}{M}.$$

Expanding and simplifying the terms for p gives

$$F_1(1,p) \leq \frac{16(1-\beta)^2}{M} + \frac{12(1-\beta)^2}{L}p^2 - \frac{2(1-\beta)^2}{L}p^4 = F_2(p) \tag{3.14}$$

so that

$$F_2'(p) = \frac{24(1-\beta)^2}{L}p - \frac{8(1-\beta)^2}{L}p^3. \tag{3.15}$$

Now at the critical points, $F_2'(p) = 0$ implies that

$$\frac{24(1-\beta)^2}{L}p - \frac{8(1-\beta)^2}{L}p^3 = \left\{\frac{24(1-\beta)^2}{L} - \frac{8(1-\beta)^2}{L}p^2\right\}p = 0$$

which means that $p_0 = 0$ and $p_1 = \sqrt{3}$, hence, from (3.15),

$$F_2''(p_0) = \frac{24(1-\beta)^2}{L} > 0 \quad \text{and} \quad F_2''(p_1) = -\frac{48(1-\beta)^2}{L} < 0.$$

Now (3.14) attains maximum at

$$F_2(p) = F_2(\sqrt{3}) = \frac{16(1-\beta)^2}{M} + \frac{36(1-\beta)^2}{L} - \frac{18(1-\beta)^2}{L}$$

that is

$$|a_2a_4 - {a_3}^2| \leq \frac{16(1-\beta)^2}{M} + \frac{18(1-\beta)^2}{L}.$$

and the proof is complete. □

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References

1. S. Agrawal and S.K. Sahoo, *A generalization of starlike functions of order alpha,* Hokkaido Mathematical Journal **46**(1) (2017), 15-27.
2. O.P. Ahuja, A. Çetinkaya and Y. Polatoğlu, *Bieberbach-de Branges and Fekete-Szegö inequalities for certain families of q-convex and q-close-to-convex functions,* Journal of Computational Analysis and Applications **26**(4) (2019), 639-649.
3. H. Aldweby and M. Darus, *Coefficient estimates of classes of q-starlike and q-convex functions,* Advanced Studies in Contemporary Mathematics **26**(1) (2016), 21-26.
4. R.M. Ali, S.K. Lee, V. Ravichandran and S. Supramanian, *The Fekete-Szegö coefficient functional for transforms of analytic functions,* Bulletin of the Iranian Mathematical Society **35**(2) (2009), 119-142.
5. S. Altinkaya and S. Yalçin, *On the Chebyshev polynomial coefficient problem of some subclasses of bi-univalent functions,* Gulf Journal of Mathematics **5**(3) (2017), 34-40.
6. M.H. Annaby and Z.S. Mansour, *q-Fractional Calculus and Equations,* Springer-Verlag Inc, New York, 2012.
7. A. Aral, V. Gupta and R.P. Agarwal, *Applications of q-Calculus in Operator Theory,* Springer-Verlag Inc, New York, 2013.
8. R.O. Ayinla and T.O. Opoola, *The Fekete Szegö functional and second Hankel determinant for a certain subclass of analytic functions,* Applied Mathematics **10**(1) (2019), 1071-1078.
9. K.O. Babalola, and T.O. Opoola, *On the coefficients of a certain class of analytic functions,* Advances in Inequalities for Series (1-13), S. S. Dragomir and A. Sofo (Eds.), Nova Science Publishers Inc, New York, 2008.

10. R.A. Bello and T.O. Opoola, *Upper bounds for Fekete-Szegö functions and the second Hankel determinants for a class of starlike functions,* International Journal of Mathematics **13**(2) (2017), 34-39.
11. P.N. Chichra, *New subclasses of the class of close-to-convex functions,* Proceedings of the American Mathematical Society **62**(1) (1977), 37-43.
12. O.A. Fadipe-Joseph, Y.O. Afolabi and B.O. Moses, *Coefficient inequalities for a class of harmonic univalent functions,* Gulf Journal of Mathematics **3**(4) (2015), 85-97.
13. M. Fekete and G. Szegö, *Eine bemerkung über ungerade schlichte funktionen,* Journal of London Mathematical Society **8** (1933), 85-89.
14. A.W. Goodman, *Univalent Functions Vol 1,* Mariner Publishing Company Inc, Tampa, Florida, 1983.
15. F.H. Jackson, *On q-functions and a certain difference operator,* Transactions of the Royal Society of Edinburgh **46**(2) (1908), 253-281.
16. F.H. Jackson, *On q-difference,* American Journal of Mathematics **32**(4) (1910), 305-314.
17. A. Junod, *Hankel determinants and orthogonal polynomials,* Expositiones Mathematicae **21** (2003), 63–74.
18. P. Kac and V. Cheung, *Quantum Calculus,* Springer-Verlag Inc, New York, 2002.
19. D.V. Krishna, B. Venkateswarlu and T. RamReddy, *Third Hankel determinant for bounded turning functions of order alpha,* Journal of the Nigerian Mathematical Society **34**(2) (2015), 121-127.
20. R.J. Libera and E.J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex function,* Proceedings of the American Mathematical Society **85**(2) (1982), 225-230.
21. T.H. MacGregor, *Functions whose derivative has a positive real part,* Transactions of the American Mathematical Society **104**(3) (1962), 532-537.
22. H. Mahzoon and R. Kargar, *Further results for two certain subclasses of close-to-convex functions,* Asian-European Journal of Mathematics Article ID 2150045 (2020), 12 pages.
23. C. Pommerenke, *On the coefficients and Hankel determinants of univalent functions,* Proceedings of the London Mathematical Society **41**(1) (1966), 111-122.
24. C. Ramachandran, T. Soupramanien and B.A. Frasin, *New subclasses of analytic functions associated with q-difference operator,* European Journal of Pure and Applied Mathematics **10**(2) (2017), 348-362.
25. P. Sahoo, *Third Hankel determinant for a class of analytic univalent functions,* Electronic Journal of Mathematical Analysis and Applications **6**(1) (2018), 322-329.
26. H. Silverman, *A class of bounded starlike functions,* International Journal of Mathematics and Mathematical Sciences **17**(2) (1994), 249-252.
27. H. Silverman and E.M. Silvia, *Characterizations for subclasses of univalent functions,* Scientiae Mathematicae Japonicae **50**(1) (1999), 103–109.
28. H.M. Srivastava, D. Răducanu and P. Zaprawa, *A certain subclass of analytic functions defined by means of differential subordination,* Faculty of Sciences and Mathematics, University of Niš, Serbia **30**(14) (2016), 3743-3757.
29. R. Srivastava and H.M. Zayed, *Subclasses of analytic functions of complex order defined by q-derivative operator,* Studia Universitatis Babeş-Bolyai Mathematica **64**(1) (2019), 71-80.
30. S. Varma and T. Rosy, *Properties of a subclass of harmonic mappings with restricted analytic part,* Gulf Journal of Mathematics **10**(2) (2021), 9-17.

[1] Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
Email address: lasode_ayo@yahoo.com

[2] Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
Email address: opoola_stc@yahoo.com