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n-FP-INJECTIVE AND n-FLAT COMPLEXES

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Abstract. In this paper, we extend the notions of n-FP-injective and n-flat modules to that of complexes which we call n-FP-injective and n-flat complexes, respectively and we show that the class of all n-FP-injective (resp., n-flat) complexes together with its left (resp., right) orthogonal class forms a complete cotorsion theory. Also we characterize n-coherent rings in terms of n-FP-injective and n-flat complexes.

1. Introduction

Throughout this paper, let R be an associative ring with identity and $\mathscr{C}$ be the abelian category of complexes of R-modules. Unless stated otherwise, a complex and an R-module will be understood to be a complex of left R-modules and a left R-module, respectively.

In 1970s, Stenstrom [8] extended the notion of injective modules to that of FP-injective modules and characterized coherent rings in terms of FP-injective modules. Lee [7] introduced and studied the notions of n-flat and n-FP-injective (or n-absolutely pure) modules. Further, Yang and Liu [9] studied these notions well and gave some nice characterizations of n-coherent rings in terms of n-flat and n-FP-injective modules. They also proved that the class of all n-flat modules together with its right orthogonal class forms a perfect cotorsion theory whereas that of n-FP-injective modules together with its left orthogonal class forms a complete cotorsion theory. Later, Wang and Liu [10] extended the notions of FP-injective modules to that of complexes and characterized such complexes. Inspired by the above works in this article, we extend the notions of n-flat and n-FP-injective modules to that of complexes and characterize n-coherent rings in terms of n-flat and n-FP-injective complexes.

In Section 2, we recall some known definitions and terminologies that are needed in the sequel.

In Section 3, we extend the notions of n-FP-injective and n-flat modules to that of complexes which we call n-FP-injective and n-flat complexes, respectively. Also we show that the class of all n-FP-injective (resp., n-flat) complexes together with its left (resp., right) orthogonal class forms a complete cotorsion theory. Moreover,

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we see that the principal results on n-flat and n-FP-injective modules [9] remain true for n-flat and n-FP-injective complexes. In the last section, we characterize n-coherent rings in terms of n-flat and n-FP-injective complexes.

2. Preliminaries

In this section, we first recall some known definitions and terminologies which will be needed in the sequel.

In this paper a complex

$$\cdots \to C^{-1} \stackrel{\delta^{-1}}{\to} C^0 \stackrel{\delta^0}{\to} C^1 \stackrel{\delta^1}{\to} \cdots$$

will be denoted by C or (C, δ). Given an R-module M and an integer n, we will denote by $D^n(M)$ the complex

$$\cdots 0 \to 0 \to M \stackrel{id}{\to} M \to 0 \to 0 \cdots$$

with M in the $(n-1)$st and nth degrees. Similarly, we denote by $S^n(M)$ the complex

$$\cdots 0 \to 0 \to M \to 0 \to 0 \cdots$$

with M in the nth degree and 0 in the other places.

For any R-module M, we use $\overline{M} = D^0(M)$. Note that an R-module M is flat (resp., projective, injective) if and only if the complex $D^0(M)$ is flat (resp., projective, injective).

Given a complex C and an integer m, $C[m]$ denotes the complex such that $C[m]^n = C^{m+n}$ and whose boundary operators are $(-1)^m\delta^{m+n}$. The nth cycle of a complex C is defined as Ker δ^n and is denoted by Z^nC. The nth boundary of C is defined as Im δ^{n-1} and is denoted by B^nC.

Let X be a complex of left R-modules (resp., of right R-modules) and let Y be a complex of left R-modules. We denote by $\mathrm{Hom}(X, Y)$ (resp., $X \otimes Y$) the usual homomorphism complex (resp., tensor product) of the complexes X and Y. The nth degree term of the complex $\mathrm{Hom}(X, Y)$ is given by

$$\mathrm{Hom}(X, Y)^n = \prod_{t\in\mathbb{Z}} \mathrm{Hom}(X^t, Y^{n+t})$$

and whose boundary operators are

$$(\delta^n f)^m = \delta_Y^{n+m} f^m - (-1)^n f^{m+1}\delta_X^m.$$

The nth degree term of $X \otimes Y$ is given by

$$(X \otimes Y)^n = \bigoplus_{t\in\mathbb{Z}} (X^t \otimes_R Y^{n-t})$$

and

$$\delta(x \otimes y) = \delta_X^t(x) \otimes y + (-1)^t x \otimes \delta_Y^{n-t}(y),$$

for $x \in X^t$ and $y \in Y^{n-t}$.

For a complex C of left R-modules, we have a functor $- \otimes \mathscr{C} : \mathscr{C}_R \to \mathscr{C}_{\mathbb{Z}}$, where $\mathscr{C}_R$ denotes the category of right R-modules. The functor $- \otimes \mathscr{C} : \mathscr{C}_R \to \mathscr{C}_{\mathbb{Z}}$ being right exact, we can construct the left derived functors which we denote by $Tor_i(-, C)$. Given two complexes X and Y of $\mathscr{C}$, we use $Ext^i(X, Y)$ for $i \geq 0$ to denote the groups we obtain from the right derived functors of Hom and we use C^+ to denote the complex $\underline{Hom}(C, \overline{\mathbb{Q}/\mathbb{Z}})$.

Recall that a complex C is flat (resp., injective, projective) if C is exact and Z^nC is a flat (resp., an injective, projective) R-module for each $i \in \mathbb{Z}$. A complex C is finitely presented if and only if C is bounded and C^n is a finitely presented R-module for all $n \in \mathbb{Z}$. For unexplained terminologies and notations we refer to [1, 5, 6, 9].

Let $\mathcal{C}$ be a class of complexes in $\mathscr{C}$. The left orthogonal class of $\mathcal{C}$ (denoted by ${}^{\perp}\mathcal{C}$) is the class of all complexes F such that $Ext^1(F, C) = 0$ for every $C \in \mathcal{C}$ and the right orthogonal class of $\mathcal{C}$ (denoted by $\mathcal{C}^{\perp}$) is the class of all complexes F such that $Ext^1(C, F) = 0$ for every $C \in \mathcal{C}$. A pair of classes of complexes $(\mathcal{F}, \mathcal{C})$ is called a cotorsion theory if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be complete if for every complex A, there is an exact sequence $0 \to C \to F \to A \to 0$ such that $F \in \mathcal{F}$ and $C \in \mathcal{C}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be hereditary if $0 \to F' \to F \to F'' \to 0$ is exact with $F, F'' \in \mathcal{F}$, then $F' \in \mathcal{F}$.

Let $\mathcal{C}$ be a class of complexes in $\mathscr{C}$. We say that $\mathcal{C}$ is projectively resolving if it contains all projective complexes and for any short exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{C}$, the conditions $A \in \mathcal{C}$ and $B \in \mathcal{C}$ are equivalent. Also we say that $\mathcal{C}$ is injectively resolving if it contains all injective complexes and for any short exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathcal{C}$, the conditions $B \in \mathcal{C}$ and $C \in \mathcal{C}$ are equivalent.

Definition 2.1. [2] Let C be a complex of R-modules. A homology isomorphism $P \xrightarrow{\simeq} C$ with P a DG-projective complex, is called a DG-projective resolution of C. Note that every complex has a DG-projective resolution.

Definition 2.2. [2] Let n be a non-negative integer. A complex C of R-modules is said to have projective dimension at most n (denoted by $pd\ C \leq n$) if there exists a homology isomorphism $P \xrightarrow{\simeq} C$, with P a DG-projective complex of R-modules such that $P^i = 0$ for $i > n$. We use $pd_R M$ to denote the projective dimension of an R-module M.

Definition 2.3. [7] Let n be a non-negative integer. An R-module M is called n-FP-injective if $Ext^1_R(N, M) = 0$ for all finitely presented R-modules N with $pd_R N \leq n$. A right R-module M is called n-flat if $Tor^R_1(M, N) = 0$ for all finitely presented R-modules N with $pd_R N \leq n$.

3. n-FP-injective and n-flat complexes

In this section we introduce and study the notions of n-FP-injective and n-flat complexes. Further we see that the principal results on n-FP-injective and n-flat modules [9] remain true for n-FP-injective and n-flat complexes.

Definition 3.1. Let n be a non-negative integer. A complex C of R-modules is called n-FP-injective if $Ext^1(N, C) = 0$ for all finitely presented complexes N with $pd\ N \leq n$. A complex C of right R-modules is called n-flat if $Tor_1(C, N) = 0$ for all finitely presented complexes N with $pd\ N \leq n$.

We denote by $\mathscr{FP}_n$ the class of all n-FP-injective complexes and by $\mathscr{F}_n$ the class of all n-flat complexes. It is clear that the class $\mathscr{F}_n$ includes all the flat complexes and $\mathscr{FP}_n$ includes all the FP-injective complexes.

Proposition 3.2. *The class $\mathscr{F}\mathscr{P}_n$ is closed under extensions, direct products and pure subcomplexes*

Proof. It is clear that $\mathscr{F}\mathscr{P}_n$ is closed under extensions and direct products. Now let C be a complex in $\mathscr{F}\mathscr{P}_n$ and D be a pure subcomplex of C. For any finitely presented complex N with $pd\ N \leq n$, we have an exact sequence of complexes

$$0 \to D \to C \to C/D \to 0$$

which induces the following exact sequence

$$0 \to Hom(N, D) \to Hom(N, C) \to Hom(N, C/D) \to 0$$

and so $Ext^1(N, D) = 0$, since $Ext^1(N, C) = 0$. Hence D is an n-FP-injective complex. □

Proposition 3.3. *A complex C of right R-modules is n-flat if and only if C^+ is an n-FP-injective complex.*

Proof. It follows from [6, Lemma 5.4.2 (b)]. □

Proposition 3.4. *The class $\mathscr{F}_n$ is closed under extensions, direct sums and pure subcomplexes.*

Proof. It is easy to prove that $\mathscr{F}_n$ is closed under extensions. Since the tensor product commutes with direct sums, $\mathscr{F}_n$ is closed under direct sums. Now let C be any complex in $\mathscr{F}_n$ and D be a pure subcomplex of C. Then we get an exact sequence of complexes

$$0 \to D \to C \to C/D \to 0$$

which induces the following exact sequence of complexes

$$0 \to (C/D)^+ \to C^+ \to D^+ \to 0,$$

which is split and so D^+ is n-FP-injective by Proposition 3.3 and therefore D is n-flat by Proposition 3.3. □

Proposition 3.5. *Let C be a complex. Then the following hold.*

(1) *C is an n-flat complex of right R-modules if and only if $Tor_1(C, \overline{R/I}) = 0$ for any finitely generated left ideal I with $pd_R\ I \leq n-1$.*
(2) *C is n-FP-injective if and only if $Ext^1(\overline{R/I}, C) = 0$ for any finitely generated right ideal I with $pd_R\ I \leq n-1$.*

Proof. (1) Suppose C is an n-flat complex. Then $Tor_1(C, N) = 0$ for all finitely presented complexes N with $pd\ N \leq n$. Since I is finitely generated left ideal in R, R/I is finitely presented with $pd_R(R/I) \leq n$ and hence $\overline{R/I}$ is finitely presented with $pd\ \overline{R/I} \leq n$. Therefore $Tor_1(C, \overline{R/I}) = 0$ for any finitely generated left ideal I with $pd_R\ I \leq n-1$. Conversely, let L be a finitely presented complex with $pd\ L \leq n$. Then for each $m \in \mathbb{Z}$, there exists an exact sequence of R-modules $0 \to J \to R^n \to L^m \to 0$ for some $n \geq 0$ and $J \subseteq R^n$ finitely generated with $pd_R\ J \leq n-1$. Consider the pullback of $J \to R^n$ and $R \to R^n$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & J & \longrightarrow & R^{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_m} & & \downarrow & & \| & & \\
0 & \longrightarrow & R & \longrightarrow & R^n & \longrightarrow & R^{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L^m & = & L^m & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Then we have the following commutative diagram of complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \overline{K} & \longrightarrow & \overline{J} & \longrightarrow & \overline{R^{n-1}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\
0 & \longrightarrow & \overline{R} & \longrightarrow & \overline{R^n} & \longrightarrow & \overline{R^{n-1}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Thus $L \cong \text{Coker} f = \overline{R}/Imf$ and $Imf \cong \overline{K}$ is finitely generated with $pd\ (Imf) \leq n-1$, since K is finitely generated with $pd_R\ K \leq n-1$. Hence $Tor_1(C, L) \cong Tor_1(C, \overline{R}/Imf) = 0$ which proves that C is an n-flat complex.

(2) The proof is similar to (1). □

Theorem 3.6. *Let n be a non-negative integer and R be a ring. Then*

(1) $(\mathscr{F}_n, {\mathscr{F}_n}^{\perp})$ *is a complete cotorsion theory.*
(2) $({}^{\perp}\mathscr{F}\mathscr{P}_n, \mathscr{F}\mathscr{P}_n)$ *is a complete cotorsion theory.*

Proof. (1) Let $Card(R) \leq \aleph_\beta$ and $F \in \mathscr{F}_n$. Then we can write F as a union of a continuous chain $(F_\alpha)_{\alpha<\lambda}$ of pure subcomplexes of F such that $Card(F_0) \leq \aleph_\beta$ and $Card(F_{\alpha+1}/F_\alpha) \leq \aleph_\beta$ whenever $\alpha + 1 < \lambda$. If N is a complex such that $Ext^1(F_0, N) = 0$ and $Ext^1(F_{\alpha+1}/F_\alpha, N) = 0$ whenever $\alpha + 1 < \lambda$, then $Ext^1(F, N) = 0$ by [4, Theorem 7.3.4]. Since F_α is a pure subcomplex of F for any $\alpha < \lambda$, we have $F_\alpha \in \mathscr{F}_n$ by Proposition 3.3. Further F_α is a pure subcomplex of $F_{\alpha+1}$ whenever $\alpha + 1 < \lambda$, and so $F_{\alpha+1}/F_\alpha \in \mathscr{F}_n$ by Proposition 3.3. Let X be a set of representatives of all complexes $C \in \mathscr{F}_n$ with $Card(C) \leq \aleph_\beta$. Then $X^{\perp} = {\mathscr{F}_n}^{\perp}$. Hence $(\mathscr{F}_n, {\mathscr{F}_n}^{\perp})$ is a cotorsion theory. Since $(\mathscr{F}_n, {\mathscr{F}_n}^{\perp})$

is cogenerated by the set X, $(\mathscr{F}_n, \mathscr{F}_n{}^{\perp})$ is a complete cotorsion theory by [4, Theorem 7.4.1].

(2) Let Y be a complex in $({}^{\perp}\mathscr{F}\mathscr{P}_n)^{\perp}$ and let N be a finitely presented complex with $pd\ N \leq n$. Since the left orthogonal class of n-FP-injective complexes contains all finitely presented complexes whose projective dimension is at most n, we have $Ext^1(N, Y) = 0$, which gives that $Y \in \mathscr{F}\mathscr{P}_n$ and hence $({}^{\perp}\mathscr{F}\mathscr{P}_n)^{\perp} = \mathscr{F}\mathscr{P}_n$. Therefore ${}^{\perp}\mathscr{F}\mathscr{P}_n$ is a cotorsion theory. Then by Proposition 3.5, a complex C is n-FP-injective $\Leftrightarrow Ext^1(\overline{R/I}, C) = 0$ for any finitely generated ideal $I \subseteq R$ with $pd_R\ I \leq n-1$. Set $Y = \oplus(\overline{R/I})$, where the sum runs over all finitely generated left ideals I of R with $pd_R\ I \leq n-1$. Then $Y^{\perp} = \mathscr{F}\mathscr{P}_n$. Hence $({}^{\perp}\mathscr{F}\mathscr{P}_n, \mathscr{F}\mathscr{P}_n)$ is cogenerated by the set Y and therefore $({}^{\perp}\mathscr{F}\mathscr{P}_n, \mathscr{F}\mathscr{P}_n)$ is a complete cotorsion theory by [4, Theorem 7.4.1]. □

We obtain the following results due to Wang and Liu [10].

Proposition 3.7. *Let C be a complex. Then C is n-FP-injective if and only if C^n is an n-FP-injective R-module for all $n \in \mathbb{Z}$ and $Hom(N, C)$ is exact for all finitely presented complexes N with $pd\ N \leq n$.*

Proof. The proof is analogous to [10, Proposition 2.6]. □

Proposition 3.8. *If C is an exact complex and $Z^n(C)$ is an n-FP-injective R-module for all $n \in \mathbb{Z}$, then $Hom(N, C)$ is exact for each finitely presented complex N with $pd\ N \leq n$.*

Proof. The proof is analogous to [10, Proposition 2.7]. □

Proposition 3.9. *If C is an exact complex and $Z^n(C)$ is an n-FP-injective R-module for all $n \in \mathbb{Z}$, then C is an n-FP-injective complex.*

Proof. Suppose C is an exact complex. Then we have an exact sequence of R-modules

$$0 \to Z^n(C) \to C^n \to Z^{n+1}(C) \to 0$$

for each $n \in \mathbb{Z}$. Since the class of all n-FP-injective R-modules is closed under extensions, we get that C^n is an n-FP-injective R-module. Hence the result follows from Proposition 3.7 and Proposition 3.8. □

Proposition 3.10. *If C is an n-FP-injective complex, then C is exact and $Z^n(C)$ is an n-FP-injective R-module for all $n \in \mathbb{Z}$.*

Proof. Suppose that C is an n-FP-injective complex. Since $Ext^1(S^{1-i}(R), C) \cong H^i(C)$ for all $i \in \mathbb{Z}$ and $S^{1-i}(R)$ is finitely presented with $pd\ S^{1-i}(R) \leq n$, then C is exact. Next we prove that for any finitely presented R-module D with $pd_R\ D \leq n$, $Ext^1(D, Z^n(C)) = 0$.

Consider the exact sequence $0 \to K \to P \to D \to 0$ with P a finitely generated projective module. It yields an exact sequence of complexes

$$0 \to S^m(K) \to S^m(P) \to S^m(D) \to 0.$$

By hypothesis, $Ext^1(S^m(D), C) = 0$. So $Hom(S^m(P), C) \to Hom(S^m(K), C) \to 0$ is exact. By using the similar arguments as in [10, Proposition 2.9], we obtain the following exact sequence $Hom(P, Z^n(C)) \to Hom(K, Z^n(C)) \to 0$. On the other hand, we have an exact sequence

$$Hom(P, Z^n(C)) \to Hom(K, Z^n(C)) \to Ext^1(D, Z^n(C)) \to 0$$

which yields that $Ext^1(D, Z^n(C)) = 0$. Therefore $Z^n(C)$ is an n-FP-injective R-module. □

Theorem 3.11. *Let C be a complex. Then the following are equivalent:*

(1) *C is n-FP-injective.*
(2) *C is exact and $Z^n(C)$ is an n-FP-injective R-module for all $n \in \mathbb{Z}$.*
(3) *C^n is an n-FP-injective R-module for all $n \in \mathbb{Z}$ and $Hom(N, C)$ is exact for each finitely presented complex N with pd $N \leq n$.*

Proof. (1) $\Leftrightarrow$ (2) follows from Proposition 3.9 and Proposition 3.10.
(1) $\Leftrightarrow$ (3) follows from Proposition 3.7. □

Corollary 3.12. *An R-module M is n-FP-injective if and only if $D^i(M)$ is an n-FP-injective complex for every $i \in \mathbb{Z}$.*

Proof. By Theorem 3.11. □

Proposition 3.13. *A complex C of right R-modules is n-flat if and only if C^i is an n-flat right R-module for every $i \in \mathbb{Z}$ and $N \otimes C$ is exact for all finitely presented complexes N with pd $N \leq n$.*

Proof. Suppose C is n-flat. Then by Proposition 3.3, C^+ is n-FP-injective and so $(C^i)^+$ is an n-FP-injective R-module for $i \in \mathbb{Z}$ by Proposition 3.7. Hence C^i is n-flat by Proposition 3.3. On the other hand, let N be any finitely presented complex with *pd* $N \leq n$. Then $(N \otimes C)^+ \cong Hom(N, C^+)$ is exact and hence $N \otimes C$ is exact.

Conversely, suppose C^i is an n-flat right R-module for every $i \in \mathbb{Z}$ and $N \otimes C$ is exact for all finitely presented complexes N with *pd* $N \leq n$. Then $(C^i)^+$ is an n-FP-injective R-module for $i \in \mathbb{Z}$ by Proposition 3.7 and $(N \otimes C)^+$ is exact. Thus $Hom(N, C^+) \cong (N \otimes C)^+$ is exact and hence C^+ is n-FP-injective by Proposition 3.7. Therefore C is n-flat. □

Proposition 3.14. *A complex C of right R-modules is n-flat if and only if C is exact and C^i is an n-flat right R-module for every $i \in \mathbb{Z}$.*

Proof. It follows from Theorem 3.11 and Proposition 3.3. □

Corollary 3.15. *A right R-module M is n-flat if and only if $D^i(M)$ is an n-flat complex for every $i \in \mathbb{Z}$.*

Proof. It follows from Proposition 3.14. □

4. n-Coherent rings

Recall that a ring R is left n-coherent (for integers $n > 0$) if every finitely generated submodule of a free left R-module whose projective dimension at most $n-1$ is finitely presented. In this section we characterize n-coherent rings in terms of n-flat and n-FP-injective complexes.

Proposition 4.1. *The following are equivalent for a ring R:*

(1) *R is a left n-coherent ring;*
(2) *$Ext^1(\overline{I}, C) = 0$ for any FP-injective complex C and any finitely generated left ideal I with $pd_R\ I \leq n-1$;*
(3) *$Ext^2(\overline{R/I}, C) = 0$ for any FP-injective complex C and any finitely generated left ideal I with $pd_R\ I \leq n-1$;*
(4) *if $0 \to C \to B \to A \to 0$ is an exact sequence of complexes with C FP-injective and B n-FP-injective, then A is n-FP-injective.*

Proof. (1) $\Rightarrow$ (2) If I is any finitely generated ideal of R with $pd_R\ I \leq n-1$, then it is finitely presented with $pd_R\ I \leq n-1$. Then the complex $\overline{I}$ is finitely presented with $pd\ \overline{I} \leq n-1$. Therefore $Ext_1(\overline{I}, C) = 0$ for any FP-injective complex C and any finitely generated left ideal I with $pd_R\ I \leq n-1$.

(2) $\Rightarrow$ (3) Let C be an FP-injective complex and I be a finitely generated left ideal with $pd_R\ I \leq n-1$. Then $0 \to I \to R \to R/I \to 0$ is an exact sequence of modules if and only if $0 \to \overline{I} \to \overline{R} \to \overline{R/I} \to 0$ is an exact sequence of complexes, which gives rise to the exact sequence

$$0 = Ext^1(\overline{I}, C) \to Ext^2(\overline{R/I}, C) \to Ext^2(\overline{R}, C) = 0$$

by hypothesis. Therefore $Ext^2(\overline{R/I}, C) = 0$.

(3) $\Rightarrow$ (4) Let I be a finitely generated left ideal with $pd_R\ I \leq n-1$. Then there exists an exact sequence of complexes $0 \to C \to B \to A \to 0$ which induces the exactness of

$$0 = Ext^1(\overline{R/I}, B) \to Ext^1(\overline{R/I}, A) \to Ext^2(\overline{R/I}, C) = 0$$

by hypothesis and hence $Ext^1(\overline{R/I}, A) = 0$. Therefore A is n-FP-injective by Proposition 3.5.

(4) $\Rightarrow$ (1) Let I be a finitely generated left ideal with $pd_R\ I \leq n-1$. For any FP-injective complex C, there is an exact sequence $0 \to C \to E \to E/C \to 0$ with E injective. Then E/C is n-FP-injective by (4). Hence we have the following exact sequence

$$0 = Ext^1(\overline{R/I}, E/C) \to Ext^2(\overline{R/I}, C) \to Ext^2(\overline{R/I}, E) = 0$$

and so $Ext^1(\overline{I}, C) \cong Ext^2(\overline{R/I}, C) = 0$. It follows that $\overline{I}$ is finitely presented which implies that I is finitely presented. Therefore R is left n-coherent. □

We use $\mathscr{F}$ (resp., $\mathscr{FP}$) to denote the class of all flat (resp., FP-injective) complexes. Recall that a complex C is called cotorsion if it belongs to $\mathscr{F}^{\perp}$.

Proposition 4.2. *Let R be an n-coherent ring. Then the following are equivalent:*

(1) *every n-flat complex of right R-modules is flat;*
(2) *every cotorsion complex of right R-modules belongs to $\mathscr{F}_n^{\perp}$;*
(3) *every n-FP-injective complex of left R-modules is FP-injective;*
(4) *every finitely presented complex of left R-modules belongs to ${}^{\perp}\mathscr{F}\mathscr{P}_n$.*

Proof. (1) $\Leftrightarrow$ (2) Suppose every n-flat complex is flat. Let C be a cotorsion complex. Then $Ext^1(F, C) = 0$ for all $F \in \mathscr{F}$ and hence $Ext^1(F, C) = 0$ for all $F \in \mathscr{F}_n$. Thus $C \in \mathscr{F}_n^{\perp}$. Conversely, let C be a complex in $\mathscr{F}^{\perp}$. Then C is a cotorsion complex and hence $C \in \mathscr{F}_n^{\perp}$ by hypothesis. Thus $\mathscr{F}^{\perp} \subset \mathscr{F}_n^{\perp}$. Since $\mathscr{F} \subset \mathscr{F}_n$, we have that $\mathscr{F}_n^{\perp} \subset \mathscr{F}^{\perp}$ and hence $\mathscr{F}_n^{\perp} = \mathscr{F}^{\perp}$ which proves that every n-flat complex is a flat complex.

(1) $\Rightarrow$ (3) Let M be any n-FP-injective complex. Then M^+ is n-flat, and so M^+ is flat by (1). On the other hand, for any finitely presented complex N, there is an exact sequence of complexes

$$Tor_1(M^+, N) \to Ext^1(N, M)^+ \to 0$$

by [3, Lemma 2.7 (1)]. Thus $Ext^1(N, M) = 0$ which makes that M is FP-injective.

(3) $\Rightarrow$ (1) Let M be an n-flat complex of right R-modules. Then M^+ is n-FP-injective and so M^+ is FP-injective by (3). Hence M is a flat complex.

(3) $\Leftrightarrow$ (4) Suppose every n-FP-injective complex is FP-injective. Let N be a finitely presented complex. Then $Ext^1(N, F) = 0$ for all $F \in \mathscr{F}\mathscr{P}$ and hence $Ext^1(N, F) = 0$ for all $F \in \mathscr{F}\mathscr{P}_n$ by the hypothesis. Therefore $N \in {}^{\perp}\mathscr{F}\mathscr{P}$. Conversely, let N be a complex in ${}^{\perp}\mathscr{F}\mathscr{P}$. Then N is a finitely presented complex and hence $N \in {}^{\perp}\mathcal{FP}_n$ by hypothesis. Thus ${}^{\perp}\mathscr{F}\mathscr{P} \subset {}^{\perp}\mathscr{F}\mathscr{P}_n$. On the other hand, $\mathscr{F}\mathscr{P} \subset \mathscr{F}\mathscr{P}_n$ implies that ${}^{\perp}\mathscr{F}\mathscr{P}_n \subset {}^{\perp}\mathscr{F}\mathscr{P}$ and hence ${}^{\perp}\mathscr{F}\mathscr{P}_n = {}^{\perp}\mathscr{F}\mathscr{P}$ which shows that every n-FP-injective complex is FP-injective. □

Remark 4.3. If R is a left n-coherent ring and C is an n-FP-injective complex, then by [7, Theorem 1], we have $Ext^i(N, C) = 0$ for any finitely presented complex N with $pd\ N \leq n$ and $i \geq 1$.

Proposition 4.4. *For a ring R, the following are equivalent:*

(1) *R is left n-coherent;*
(2) *if C is an n-FP-injective complex, then C/D is n-FP-injective for any pure subcomplex D of C;*
(3) *if C is any injective complex, then C/D is n-FP-injective for any pure subcomplex D of C.*

Proof. (1) $\Rightarrow$ (2) Let C be an n-FP-injective complex and D be any pure subcomplex of C. For any finitely presented complex N with $pd\ N \leq n$, we have an exact sequence of complexes $0 \to D \to C \to C/D \to 0$ which induces an exact sequence

$$0 = Ext^1(N, C) \to Ext^1(N, C/D) \to Ext^2(N, D).$$

Since $Ext^2(N, D) = 0$ by Remark 4.3, we have $Ext^1(N, C/D) = 0$. Therefore C/D is n-FP-injective.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let C be an injective complex and D be any pure subcomplex of C. Thus we have an exact sequence $0 \to D \to C \to C/D \to 0$, which implies the exactness of

$$Ext^1(N, C/D) \to Ext^2(N, D) \to Ext^2(N, C) = 0$$

for any finitely presented complex N with $pd\ N \leq n$. Then $Ext^2(N, D) = 0$ since $Ext^1(N, C/D) = 0$ by (3). Therefore R is n-coherent by [7, Theorem 1]. □

Proposition 4.5. *Let R be a left n-coherent ring and $0 \to A \to B \to C \to 0$ be an exact sequence of complexes with A n-FP-injective. Then B is n-FP-injective if and only if C is n-FP-injective.*

Proof. Suppose $0 \to A \to B \to C \to 0$ is an exact complex with A and B n-FP-injective complexes. For any finitely presented complex N with $pd\ N \leq n$, we have the following exact sequence

$$0 \to Ext(N, C) \to Ext^2(N, A).$$

By Remark 4.3, $Ext^2(N, A) = 0$ and hence C is n-FP-injective. The converse follows from the fact that the class $\mathscr{F}\mathscr{P}_n$ is closed under extensions. □

Corollary 4.6. *Let R be a left n-coherent ring. Then the class of all n-FP-injective complexes is injectively resolving.*

Proof. It follows from Proposition 3.2 and Proposition 4.5. □

Proposition 4.7. *Let R be a left n-coherent ring and $0 \to A \to B \to C \to 0$ be an exact sequence of complexes of right R-modules with C n-flat. Then A is n-flat if and only if B is n-flat.*

Proof. Suppose $0 \to A \to B \to C \to 0$ is an exact sequence of complexes with C n-flat. Then we have the following exact sequence

$$0 \to C^+ \to B^+ \to A^+ \to 0$$

with C^+ n-FP-injective. Thus the result follows from the fact that

$$A \text{ is } n\text{-flat} \Leftrightarrow A^+ \text{ is } n\text{-FP-injective} \Leftrightarrow B^+ \text{ is } n\text{-FP-injective} \Leftrightarrow B \text{ is } n\text{-flat}$$

by Proposition 3.3 and Proposition 4.5. □

Corollary 4.8. *Let R be a left n-coherent ring. Then the class of all n-flat complexes is projectively resolving.*

Proof. It follows from Proposition 3.4 and Proposition 4.7. □

Proposition 4.9. *Let R be an n-coherent ring and C be a complex. Then C is n-FP-injective if and only if C^+ is an n-flat complex of right R-modules.*

Proof. Suppose C is n-FP-injective. Let N be any finitely presented complex with $pd\ N \leq n$. Then $Tor_1(C^+, N) \cong Ext^1(N, C)^+ = 0$ which proves that C^+ is n-flat. Conversely, suppose C^+ is an n-flat complex of right R-modules. Then for any finitely presented complex N with $pd\ N \leq n$, we have $Ext^1(N, C)^+ \cong Tor_1(C^+, N) = 0$. Therefore C is n-FP-injective. □

Corollary 4.10. *Let R be an n-coherent ring. Then C is an n-flat complex of right R-modules if and only if C^{++} is n-flat.*

Proof. It follows from Proposition 3.3 and Proposition 4.9. □

Theorem 4.11. *The following are equivalent for a ring R and a complex C:*

(1) *R is left n-coherent.*
(2) *C is n-FP-injective if and only if C^+ is an n-flat complex of right R-modules.*
(3) *C is n-FP-injective if and only if C^{++} is n-FP-injective.*

Proof. (1) $\Rightarrow$ (2) follows from Proposition 4.9 and (2) $\Leftrightarrow$ (3) follows from Proposition 3.3.

(2) $\Rightarrow$ (1). It suffices to show that for every injective complex E and any pure subcomplex A of E, E/A is n-FP-injective. Let A be any pure subcomplex of an injective complex E. Then the pure exact sequence $0 \to A \to E \to E/A \to 0$ induces the pure exact sequence $0 \to (E/A)^+ \to E^+ \to A^+ \to 0$. By (2), E^+ is an n-flat complex. Since $(E/A)^+$ is a pure subcomplex of E^+, $(E/A)^+$ is n-flat by Proposition 3.4. Thus E/A is n-FP-injective by (2). Therefore R is left n-coherent by Proposition 4.4. □

Proposition 4.12. *The following conditions are equivalent:*

(1) *every complex of left R-modules is n-FP-injective;*
(2) *every cotorsion complex of left R-modules is n-FP-injective;*
(3) *every complex of right R-modules is n-flat;*
(4) *every cotorsion complex of right R-modules is n-flat;*
(5) *$({}^{\perp}\mathscr{F}\mathscr{P}_n, \mathscr{F}\mathscr{P}_n)$ is a hereditary cotorsion theory and every complex of left R-modules in ${}^{\perp}\mathscr{F}\mathscr{P}_n$ is n-FP-injective.*

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) are obvious.

(2) $\Rightarrow$ (3) Let C be any complex of R-modules. Then C^+ is n-FP-injective by (2), and so C is n-flat by Proposition 3.3.

(4) $\Rightarrow$ (1) Let C be any complex of left R-modules. Then C^+ is n-flat by (4), and so C^{++} is n-FP-injective by Proposition 3.3. Since M is a pure submodule of M^{++} for any R-module M by [4, Proposition 5.3.9], similarly we have that C is a pure subcomplex of C^{++} for any complex C. Hence C is n-FP-injective by Proposition 3.2.

(5) $\Rightarrow$ (1) Let C be any complex of left R-modules. Since $({}^{\perp}\mathscr{F}\mathscr{P}_n, \mathscr{F}\mathscr{P}_n)$ is complete by Theorem 3.6, there is an exact sequence $0 \to K \to F \to C \to 0$ with $F \in {}^{\perp}\mathscr{F}\mathscr{P}_n, K \in \mathscr{F}\mathscr{P}_n$. Then F is n-FP-injective by Proposition 3.2 and hence C is n-FP-injective by (5). □

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