Gulf Journal of Mathematics ISSN: 2309-4966
Vol 13, Issue 2 (2022) 42-66 https://doi.org/10.56947/gjom.v13i2.590

EXISTENCE AND UNIQUENESS OF RENORMALIZED SOLUTION TO MULTIVALUED HOMOGENEOUS NEUMANN PROBLEM WITH L^1-DATA

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Abstract. In this paper, we discuss the existence and uniqueness of renormalized solution to nonlinear multivalued elliptic problem $\beta(u) - \operatorname{div} a(x, Du) \ni f$ in Ω, with homogeneous Neumann boundary conditions and L^1-data. The functional setting involves Lebesgue and Sobolev spaces with variable exponent. Some a-priori estimates are used to obtain our results.

1. Introduction

Neumann problems with L^1-data have been widely studied in the literature. Several of them concern the homogeneous Neumann condition on the boundary. In this manuscript, we will discuss the existence and uniqueness of renormalized solutions of a problem with Neumann boundary conditions. More precisely, let us consider the following problem:

$$P(\beta, f) \quad \begin{cases} \beta(u) - \operatorname{div} a(x, Du) \ni f & \text{in} \quad \Omega, \\ a(x, Du).\eta = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where f is just a summable function. Here $\beta = \partial j$ is a maximal monotone graph on $\mathbb{R}^2$ with $0 \in \beta(0)$. Ω is a smooth bounded open set of $\mathbb{R}^N, (N \geq 2)$ with boundary $\partial\Omega$ and η the outer unit normal vector. The vector field a satisfies the following standard Leray-Lions conditions.

(H_1) Carathéodory condition: the function $a(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ has the following two properties:
$x \mapsto a(x, \xi)$ is measurable on $x \in \Omega$ for all $\xi \in \mathbb{R}^N$ and $\xi \mapsto a(x, \xi)$ is continuous on $\mathbb{R}^N$ for all $x \in \Omega$ with $a(., 0) = 0$.

(H_2) Coercivity condition: for almost every $x \in \Omega$, for all $\xi \in \mathbb{R}^N$,

$$a(x, \xi).\xi \geq \alpha |\xi|^{p(x)} \text{ for some } \alpha > 0.$$

(H_3) Monotonicity condition: for almost $x \in \Omega$, for all $\xi \neq \eta \in \mathbb{R}^N$,

$$(a(x, \xi) - a(x, \eta)).(\xi - \eta) > 0.$$

Date: Received: Oct 27, 2022; Accepted: May 16, 2022.
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2010 *Mathematics Subject Classification.* Primary 35J60.
Key words and phrases. Elliptic equation, variable exponent, maximal monotone graph, renormalized solution, L^1-data.

(H_4) Growth condition: for almost every $x \in \Omega$, for all $\xi \in \mathbb{R}^N$,

$$|a(x,\xi)| \leq \Lambda(j(x) + |\xi|^{p(x)-1}),$$

where Λ is a positive constant and $j \in L^{p'(x)}(\Omega)$, $1/p(.) + 1/p'(.) = 1$.
These assumptions are classic for the study of nonlinear operators in divergent form (see Leray-Lions [9]). It appears in (H_2) and (H_4) an exponent p depending of the variable x, meaning that we work in Lebesgue and Sobolev spaces with variable exponent. Equations with non-standard growth and L^1- data in Lebesgue and Sobolev space with variable exponent are investigated by several authors. For more details, one can refer to [3, 6, 8, 12, 15] and references therein.The need to work in these spaces is motivated by its use in modeling electrorheological and thermorheological fluids (cf.[12]), as well as for image restoration [3]. For this problem, the right-hand side is L^1, we can use the notion of renormalized solution of Di Perna-Lions (cf.[4]) in the case of constant exponent p and for p depending on x, we borrow the one of Wittbold and Zimmermann (cf.[15]) to adapt it to our case.
In [15], the authors study the stationary problem:

$$(s,f)\begin{cases}\beta(u) - \operatorname{div} a(x,Du) - \operatorname{div} F \ni f & \text{in} \quad \Omega,\\ u = 0 & \text{on} \quad \partial\Omega,\end{cases}$$

with right-hand side $f \in L^1(\Omega)$, $F : \mathbb{R} \longrightarrow \mathbb{R}^N$ locally Lipschitz continuous and $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ a set-valued, maximal monotone mapping such that $0 \in \beta(0)$, $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the same assumptions of the problem $P(\beta, f)$ and which is a version of (s,f), for $F = 0$ where the Dirichlet boundary condition is replaced by the Neumann one. Note that Ouaro and Ouédraogo (cf.[11]) prove the existence and uniqueness of entropy solution of $P(\beta, f)$ under the additional hypothese, $\overline{dom(\beta)} = [m, M] \subset \mathbb{R}$ with $m \leq 0 \leq M$.
We prove the existence of renormalized solutions when the data are smooth, by penalizing the problem, i.e. by adding a strongly monotone term and the uniqueness is obtained by comparison principle. When the data are not smooth, the compactness argument in L^1 is obtained by doubling the variables. The data of the problem being no smooth, and the associated boundary conditions being of Neumann type some difficulty appear in the study of $P(\beta, f)$ because contrary to the homogeneous Dirichlet case, the Poincaré's inequality and even the Poincaré-Wirtinger's inequality cannot be used.
To carry out this work, we organize in the following way: In the section 2, we recall the basic proprieties of Lebesgue and Sobolev spaces with variables exponent. In section 3, we define the renormalized solution and state our main result of $P(\beta, f)$ (see Theorem 3.6 below). Finally, section 4 is devoted to the proof of the main result.

2. Mathematical preliminaries

We recall in this section, some basic properties of the generalized Lebesgue Sobolev spaces with variable exponents. Given a measurable function $p(.) : \Omega \to [1, +\infty)$ such that $1 < p^- \leq p^+ < +\infty$, we will use the following notation throughout the paper:

$$p^- = ess \inf_{\Omega} p(x) \quad and \quad p^+ = ess \sup_{\Omega} p(x).$$

We define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable functions $f : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(f) = \int_{\Omega} |f(x)|^{p(.)} dx$$

is finite. If the exponent is bounded; i.e., if $p^+ < \infty$, then the expression

$$||f||_{p(.)} = \inf\{\lambda > 0 : \rho_{p(.)}(\frac{f}{\lambda}) \leq 1\},$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm. The space $\left(L^{p(.)}(\Omega), ||.||_{p(.)}\right)$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $1/p(.) + 1/p'(.) = 1$. For any $f \in L^{p(.)}(\Omega)$ and $g \in L^{p'(.)}(\Omega)$ the Hölder-type inequality

$$\left|\int_{\Omega} fg dx\right| \leq (\frac{1}{p^-} + \frac{1}{p'^-})||f||_{p(.)}||g||_{p'(.)}. \tag{2.1}$$

Next, we define the generalized Sobolev space $W^{1,p(.)}(\Omega)$, also called Sobolev space with variable exponent

$$W^{1,p(.)}(\Omega) = \{|f| \in L^{p(.)}(\Omega) : |Df| \in L^{p(.)}(\Omega)\},$$

which is a Banach space equipped with the norm

$$||f||_{1,p(.)} = ||f||_{p(.)} + ||Df||_{p(.)}.$$

The space $\left(W^{1,p(.)}(\Omega), \| \cdot \|_{1,p(.)}\right)$ is a separable and reflexive Banach space.
An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(x)}(\Omega)$. We have the following result (see [5]).

Proposition 2.1. *If $u_n, u \in L^{p(x)}(\Omega)$ and $p^+ < +\infty$, then the following properties hold:*

(i) $||u||_{p(x)} > 1 \Longrightarrow ||u||^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq ||u||^{p^+}_{p(x)}$;

(ii) $||u||_{p(x)} < 1 \Longrightarrow ||u||^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq ||u||^{p^-}_{p(x)}$;

(iii) $||u||_{p(x)} < 1$ *(resp.* $= 1; > 1) \Longleftrightarrow \rho_{p(x)}(u) < 1$ *(resp.* $= 1; > 1)$*;*

(iv) $||u_n||_{p(x)} \to 0$ *(resp.* $\to +\infty) \Longleftrightarrow \rho_{p(x)}(u_n) \to 0$*(resp.* $\to +\infty)$*;*

(v) $\rho_{p(x)}(\frac{u}{||u||_{p(x)}}) = 1.$

For a measurable function $u : \Omega \to \mathbb{R}$, we introduce the function

$$\rho_{1,p(.)}(u) := \int_\Omega |u|^{p(x)}\,dx + \int_\Omega |\nabla u|^{p(x)}\,dx$$

and we have the following proposition (see [14, 16]):

Proposition 2.2. *If $u \in W^{1,p(.)}(\Omega)$, then the following properties hold:*

(i) $|u|_{1,p(.)} > 1 \Longrightarrow |u|^{p_-}_{1,p(.)} \leq \rho_{1,p(.)}(u) \leq |u|^{p_+}_{1,p(.)}$;

(ii) $|u|_{1,p(.)} < 1 \Longrightarrow |u|^{p_+}_{1,p(.)} \leq \rho_{1,p(.)}(u) \leq |u|^{p_-}_{1,p(.)}$;

(iii) $|u|_{1,p(.)} < 1$ *(resp.* $= 1; > 1$) $\Longleftrightarrow \rho_{1,p(.)}(u) < 1$ *(resp.* $= 1; > 1$*).*

We refer to Kovacik and Rakosnik in [7] for further properties of variable exponent Lebesgue-Sobolev spaces. Let $meas(A) = |A|$ be the Lebesgue measure of the part $A \subset \mathbb{R}^N$ and χ_A its characteristic function. For $r \in \mathbb{R}$, let $r^+ := \max(r, 0)$ and $\mathrm{sign}_0^+(r)$ be the function defined by $\mathrm{sign}_0^+(r) = 1$ if $r > 0$, $\mathrm{sign}_0^+(r) = 0$ if $r \leq 0$. Moreover, for any f, g we write $f \wedge g = \min(f, g)$ and $f \vee g = \max(f, g)$.

3. Renormalized solution

In this section, we fix the notations, give the concept of renormalized solution for Problem $P(\beta, f)$. To begin, we note $Dom(\beta)$ to designate the domain of β and T_k to denote the truncation function at height $k > 0$, defined by $T_k(r) = \max\{-k, \min(k, r)\}$, for all $r \in \mathbb{R}$.

Let γ be a maximal monotone graph defined on $\mathbb{R}\times\mathbb{R}$. We recall the main section γ_0 of γ defined by:

$$\gamma_0(s) = \begin{cases} \text{the element of minimal absolute value of } \gamma(s) & \text{if } \gamma(s) \neq \emptyset, \\ +\infty & \text{if } [s, +\infty) \cap Dom(\gamma) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap Dom(\gamma) = \emptyset. \end{cases}$$

In addition, we need the following associated functions

$$S_k = T_{k+1} - T_k \quad \textit{and} \quad h_l(r) = \min((l + 1 - |r|)^+, 1).$$

Note that T_k and S_k are Lipschitz continuous functions satisfying $|T_k(r)| \leq k$, $|S_k| \leq 1$ and likewise that $|h_l| \leq 1$. In [1], the authors introduce the set

$$\mathcal{T}^{1,p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \quad \text{measurable} \quad \text{such that}, T_k(u) \in W^{1,p(x)}(\Omega), \forall k > 0\}.$$

Lemma 3.1. *(See[1]) For every $u \in \mathcal{T}^{1,1}_{loc}(\Omega)$ there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$DT_k(u) = v\chi_{\{|u|<k\}} \ \textit{a.e. in } \Omega.$$

Furthermore, $u \in W^{1,1}_{loc}(\Omega)$ if and only if $v \in L^1_{loc}(\Omega)$, and then $v \equiv Du$ in the usual weak sense.

Let us give the definition of renormalized solutions for the problem $P(\beta, f)$.

Definition 3.2. A renormalized solutions of $P(\beta, f)$ is a couple of functions (u, b) satisfying the following conditions:
(i) $u : \Omega \to \mathbb{R}$ is measurable, $b \in L^1(\Omega)$, $u(x) \in Dom(\beta(x))$ and $b(x) \in \beta(u(x))$ for a.e. x in Ω,
(ii) for all $k > 0$, $T_k(u) \in W^{1,p(x)}(\Omega)$,
(iii)

$$\int_\Omega bh(u)\psi dx + \int_\Omega a(x, Du).D(h(u)\psi)dx = \int_\Omega fh(u)\psi dx \tag{3.1}$$

holds for every $h \in C_c^1(\mathbb{R})$ and $\psi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.
Moreover,

$$\lim_{k\to\infty} \int_{\{k<|u|<k+1\}} a(x, Du).Dudx = 0. \tag{3.2}$$

We also introduce the notion of entropy solutions for the problem $P(\beta, f)$.

Definition 3.3. For $f \in L^1(\Omega)$, a entropy solution of $P(\beta, f)$ is a pair of functions $(u, b) \in \mathcal{T}^{1,p(x)}(\Omega) \times L^1(\Omega)$, $u(x) \in Dom(\beta(x))$ and $b(x) \in \beta(u(x))$ for a.e. x in Ω such that

$$\int_\Omega b\,T_k(u-\psi)\,dx + \int_\Omega a(x, Du).DT_k(u-\psi)\,dx \le \int_\Omega f\,T_k(u-\psi)\,dx, \tag{3.3}$$

for all $\psi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ such that $\psi(x) \in \beta(u(x))$ for a.e. x in Ω.

Remark 3.4. It is clear that each term in (3.1) is well defined. Condition (3.2) is classical in the framework of renormalized solutions and gives additional information on Du.
Relation between the notions of renormalized and entropy solution is given through the Proposition 3.5 below. This result allows us to prove uniqueness of the renormalized solution of problem $P(\beta, f)$.

Proposition 3.5. *(see [11]) For $f \in L^1(\Omega)$ and under assumptions $(H_1)-(H_4)$, renormalized solution and entropy solution of problem $P(\beta, f)$ are equivalent.*

The main result of this section is the following.

Theorem 3.6. *Assume that $(H_1)-(H_4)$ hold. Then, there exists a unique renormalized solution u to problem $P(\beta, f)$. The uniqueness is understood in the sense of $b(u)$.*

4. Proof of Theorem 3.6

We use approximate methods for the multi-step proof. For a given f in $L^\infty(\Omega)$, we prove the existence and uniqueness of a renormalized solution of the penalized approximation problem of $P(\beta, f)$. The uniqueness can be obtained by a comparison principle. Then, for $f \in L^1(\Omega)$ and for $m, n \in \mathbb{N}$, we consider the bimonotone sequence $f_{m,n}$ given by $f_{m,n} = (f \wedge m) \vee (-n)$. Let us note that for any m, n in $\mathbb{N}$, $|f_{m,n}| \le |f|$ a.e. in Ω. We complete the proof of this theorem i.e. the uniqueness of solution of renormalized solution in the end of the section.

4.1. **Existence and uniqueness results for $L^\infty(\Omega)$-data.**

Proposition 4.1. *For $f \in L^\infty(\Omega)$, there exists at least one renormalized solution (u, b) of $P(\beta, f)$.*

Proof. We proceed by approximation of problem by introduce the quantity $|s|^{p(x)-2}s$ to exploit a minimization method. We will make some a priori estimates and using some convergence results to pass to the limit.

Step 1: Approximate solution for L^∞-data.
Let $f \in L^\infty(\Omega)$, we consider the approached penalized problem of $P(\beta, f)$, for $\varepsilon > 0$ by

$$P(\beta_\varepsilon, f)\begin{cases} \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + \varepsilon|u_\varepsilon|^{p(x)-2}u_\varepsilon - \operatorname{div} a(x, Du_\varepsilon) = f & \text{in} \quad \Omega, \\ a(x, Du_\varepsilon).\eta = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where β_ε is the Yosida approximation of β (see [2]).

Proposition 4.2. *For every $f \in L^\infty(\Omega)$ there exists at least one weak solution $u_\varepsilon \in W^{1,p(x)}(\Omega)$ of the problem $P(\beta_\varepsilon, f)$.*

Proof. A function $u_\varepsilon \in W^{1,p(x)}(\Omega)$ is called a weak solution of problem $P(\beta_\varepsilon, f)$ if

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\psi dx + \varepsilon \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon\psi dx + \int_\Omega a(x, Du_\varepsilon).D\psi dx = \langle f, \psi\rangle, \quad (4.1)$$

for any $\psi \in W^{1,p(x)}(\Omega)$, where $\langle ., .\rangle$ denotes the duality pairing between $W^{1,p(x)}(\Omega)$ and $\left(W^{1,p(x)}(\Omega)\right)^*$.
For $\varepsilon > 0$ we define the operator $A_\varepsilon : W^{1,p(x)}(\Omega) \to \left(W^{1,p(x)}(\Omega)\right)^*$ by

$$\langle A_\varepsilon u_\varepsilon, \psi\rangle = \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\psi dx + \varepsilon \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon\psi dx + \int_\Omega a(x, Du_\varepsilon).D\psi dx.$$

The operator A_ε satisfies the following properties:

Lemma 4.3. *A_ε is of type (M), bounded and coercive.*

Proof. (of Lemma 4.3.) This proof is subdivided into three assertions.

Assertion 1: The operator A_ε is bounded for any $\varepsilon > 0$.
Let choose $\psi = u_\varepsilon$ as a test function in (4.1), we get

$$\langle A_\varepsilon u_\varepsilon, u_\varepsilon\rangle = \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))u_\varepsilon dx + \varepsilon \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon u_\varepsilon dx + \int_\Omega a(x, Du_\varepsilon).Du_\varepsilon dx.$$

Then,

$$\begin{aligned} |\langle A_\varepsilon u_\varepsilon, u_\varepsilon\rangle| \leq \int_\Omega |\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))u_\varepsilon| dx &+ \varepsilon \int_\Omega ||u_\varepsilon|^{p(x)-2}u_\varepsilon u_\varepsilon| dx \\ &+ \int_\Omega |a(x, Du_\varepsilon).Du_\varepsilon| dx. \end{aligned} \quad (4.2)$$

As $\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))$ is bounded in $L^{p'(\cdot)}(\Omega)$, then there exist a constant $C_1 > 0$ such that by using Hölder's inequality, we get

$$\int_\Omega |\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))u_\varepsilon|dx \leq ||\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))||_{p'(x)}||u_\varepsilon||_{p(x)} \leq C_1||u_\varepsilon||_{1,p(x)}. \tag{4.3}$$

Thanks to Hölder's type inequality, we have

$$\varepsilon \int_\Omega |u_\varepsilon|^{p(x)}dx \leq \varepsilon \left(\frac{1}{p-} + \frac{1}{p'-}\right) \Omega^{\frac{1}{p'-}} \|u_\varepsilon\|_{p(x)} \leq C\|u_\varepsilon\|_{1,p(x)}. \tag{4.4}$$

Moreover, using Hölder's inequality and the growth condition (H_4), the last term of the inequality (4.2) leads to

$$\begin{aligned}
\int_\Omega |a(x, Du_\varepsilon).Du_\varepsilon|dx &\leq \left(\frac{1}{p-} + \frac{1}{p'-}\right) ||a(x, Du_\varepsilon)||_{p'(x)}||Du_\varepsilon||_{p(x)} \\
&\leq \underbrace{C_1'\left(||j||_{p'(x)} + ||Du_\varepsilon||_{p(x)}^{p(x)-1}\right)}_{C_2} ||Du_\varepsilon||_{p(x)} \\
&\leq C_2||Du_\varepsilon||_{p(x)} \\
&\leq C_2||u_\varepsilon||_{1,p(x)}.
\end{aligned} \tag{4.5}$$

Gathering (4.3)-(4.5) in (4.2), we deduce that there is a constant $C > 0$ depending on ε, C_1, C_2 such that

$$|\langle A_\varepsilon u_\varepsilon, u_\varepsilon\rangle| \leq C||u_\varepsilon||_{1,p(x)},$$

so that A_ε is bounded.

Assertion 2: The operator A_ε is coercive.
We recall that

$$\langle A_\varepsilon u_\varepsilon, u_\varepsilon\rangle = \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))u_\varepsilon dx + \varepsilon \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon u_\varepsilon dx + \int_\Omega a(x, Du_\varepsilon).Du_\varepsilon dx.$$

Using the monotonicity of β_ε and (H_2) hypothesis, we deduce that

$$\begin{aligned}
\langle A_\varepsilon u_\varepsilon, u_\varepsilon\rangle &\geq \int_\Omega a(x, Du_\varepsilon).Du_\varepsilon dx + \varepsilon \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon u_\varepsilon dx \\
&\geq \alpha \int_\Omega |Du_\varepsilon|^{p(x)}dx + \varepsilon \int_\Omega |u_\varepsilon|^{p(x)}dx \\
&\geq \min(\alpha, \varepsilon)\left(\int_\Omega |Du_\varepsilon|^{p(x)}dx + \int_\Omega |u_\varepsilon|^{p(x)}dx\right) \\
&\geq \min(\alpha, \varepsilon)\rho_{1,p(x)}(u_\varepsilon).
\end{aligned} \tag{4.6}$$

Letting $||u_\varepsilon||_{1,p(x)}$ tend to infinity in (4.6), we deduce from Proposition 2.2 that, $\dfrac{\langle A_\varepsilon u_\varepsilon, u_\varepsilon\rangle}{||u_\varepsilon||_{1,p(x)}} \to +\infty$. Thus, A_ε is coercive.

Assertion 3: The operator A_ε is of type (M).
According to [13], if $\mathcal{A}_{\varepsilon,1}$ is of type (M) and if $\mathcal{A}_{\varepsilon,2}$ is monotone and weakly continuous, then $\mathcal{A}_{\varepsilon,1}+\mathcal{A}_{\varepsilon,2}$ is of type (M).
We have

$$\begin{aligned}\langle A_\varepsilon u,\psi\rangle &= \int_\Omega a(x,Du).D\psi dx+\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u))\psi dx+\varepsilon\int_\Omega |u|^{p(x)-2}u\psi dx\\ &= \int_\Omega a(x,Du).D\psi dx+\langle \mathcal{A}_{\varepsilon,2}u,\psi\rangle.\end{aligned}$$

Let us show that $\mathcal{A}_{\varepsilon,2}$ is monotone.
We have

$$\langle \mathcal{A}_{\varepsilon,2}u,\psi\rangle=\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u))\psi dx+\varepsilon\int_\Omega |u|^{p(x)-2}u\psi dx.$$

For $u,v\in W^{1,p(x)}(\Omega)$, we have

$$\begin{aligned}\langle \mathcal{A}_{\varepsilon,2}u-\mathcal{A}_{\varepsilon,2}v,u-v\rangle &= \langle \mathcal{A}_{\varepsilon,2}u,u-v\rangle-\langle \mathcal{A}_{\varepsilon,2}v,u-v\rangle\\ &= \int_\Omega(\beta_\varepsilon(T_{1/\varepsilon}(u))-\beta_\varepsilon(T_{1/\varepsilon}(v)))(u-v)dx\\ &+ \varepsilon\int_\Omega(|u|^{p(x)-2}u-|v|^{p(x)-2}v)(u-v)dx. \qquad (4.7)\end{aligned}$$

By the monotonicity of $u\mapsto |u|^{p(x)-2}u$, we deduce that

$$\varepsilon\int_\Omega(|u|^{p(x)-2}u-|v|^{p(x)-2}v)(u-v)dx\geq 0.$$

β_ε being monotone, obviously

$$\langle \mathcal{A}_{\varepsilon,2}u-\mathcal{A}_{\varepsilon,2}v,u-v\rangle\geq 0. \qquad (4.8)$$

Let us show that $\mathcal{A}_{\varepsilon,2}$ is weakly continuous, i.e. for all sequence $(u_n)_{n\in\mathbb{N}}\subset W^{1,p(.)}(\Omega)$ converging weakly to $u\in W^{1,p(.)}(\Omega)$, we get $\mathcal{A}_{\varepsilon,2}u_n$ which converges to $\mathcal{A}_{\varepsilon,2}u$.
For all $\psi\in W^{1,p(.)}(\Omega)$, we have

$$\langle A_{\varepsilon,2}u_n,\psi\rangle = \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_n))\psi dx+\varepsilon\int_\Omega |u_n|^{p(x)-2}u_n\psi dx. \qquad (4.9)$$

We have $|\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\psi|\leq \max(|\beta_\varepsilon(1/\varepsilon)|,|\beta_\varepsilon(-1/\varepsilon)|)|\psi|\in L^{p(.)}(\Omega)$.
Let $(u_n)\subset W^{1,p(.)}(\Omega)$ be converging weakly to some $u\in W^{1,p(.)}(\Omega)$. Then $u_n\to u$ strongly in $L^{p(.)}(\Omega)$. Thus, $\exists M>0,|u_n|\leq M$, so $|u_n|^{p(x)-1}|\psi|\leq \max(M^{p_--1},M^{p_+-1})|\psi|\in L^{p(.)}(\Omega)$.
The generalized Lebesgue convergence theorem allows us to pass to the limit in (4.9) as $n\to+\infty$ to get $\lim_{n\to+\infty}\langle A_{\varepsilon,2}u_n,\psi\rangle=\langle A_{\varepsilon,2}u,\psi\rangle$ i.e., $\mathcal{A}_{\varepsilon,2}u_n\rightharpoonup \mathcal{A}_{\varepsilon,2}u$.
For all $u,v\in W^{1,p(.)}(\Omega)$ we have

$$\langle \mathcal{A}_{\varepsilon,1}u-\mathcal{A}_{\varepsilon,1}v,u-v\rangle=\int_\Omega\big(a(x,Du)-a(x,Dv)\big)(u-v)dx.$$

Since the integral is non decreasing, we observe that the monotone character of a implies that $A_{\varepsilon,1}$ is monotone. Moreover, by (H_4) it follows that $A_{\varepsilon,1}$ is

hemicontinuous. We conclude that $A_{\varepsilon,1}$ is pseudo-monotone thus is of type (M). $A_{\varepsilon,1}$ being of type (M) and $A_{\varepsilon,2}$ is monotone, weakly continuous, then the operator A_ε is of type (M).
Hence the proof of Lemma 4.3 is complete. □

According to the classical theorem of Lions [see [10], Theorem 2.7] there exists at least a weak solution $u_\varepsilon \in W^{1,p(x)}(\Omega)$ of $P(\beta_\varepsilon, f)$ which ends the proof of Proposition 4.2. □

We establish uniqueness of solutions u_ε of $P(\beta_\varepsilon, f)$ with right-hand sides $f \in L^\infty(\Omega)$ through a comparison principle that will play an important role in the next.

Proposition 4.4. *For $\varepsilon > 0$ fixed, $f, \tilde{f} \in L^\infty(\Omega)$ let $u_\varepsilon, \tilde{u_\varepsilon} \in W^{1,p(.)}(\Omega)$ be two weak solutions of $P(\beta_\varepsilon, f)$ and $P(\beta_\varepsilon, \tilde{f})$ respectively. Then the following comparison principle holds.*

$$\varepsilon \int_\Omega \left(|u_\varepsilon|^{p(x)-2}u_\varepsilon - |\tilde{u}_\varepsilon|^{p(x)-2}\tilde{u}_\varepsilon\right)^+ \leq \int_\Omega \left(f - \tilde{f}\right) sign_0^+(u_\varepsilon - \tilde{u}_\varepsilon). \tag{4.10}$$

Proof. Let u_ε and $\tilde{u_\varepsilon}$ be two weak solutions of $P(\beta_\varepsilon, f)$ and $P(\beta_\varepsilon, \tilde{f})$ respectively. For $k > 0$, taking $\psi = \frac{1}{k}T_k(u_\varepsilon - \tilde{u_\varepsilon})^+$ as a test function in (4.1), and by differentiating the equations written in u_ε and $\tilde{u_\varepsilon}$, we obtain:

$$J_1 + J_2 + J_3 = J_4, \tag{4.11}$$

with

$$\begin{aligned}
J_1 &= \int_\Omega \left(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) - \beta_\varepsilon(T_{1/\varepsilon}(\tilde{u_\varepsilon}))\right)\frac{1}{k}T_k(u_\varepsilon - \tilde{u_\varepsilon})^+ dx,\\
J_2 &= \varepsilon \int_\Omega \left(|u_\varepsilon|^{p(x)-2}u_\varepsilon - |\tilde{u_\varepsilon}|^{p(x)-2}\tilde{u_\varepsilon}\right)\frac{1}{k}T_k(u_\varepsilon - \tilde{u_\varepsilon})^+ dx,\\
J_3 &= \frac{1}{k}\int_A \left(a(x, Du_\varepsilon) - a(x, D\tilde{u_\varepsilon})\right)D(u_\varepsilon - \tilde{u_\varepsilon})dx,\\
J_4 &= \int_\Omega \left(f - \tilde{f}\right)\frac{1}{k}T_k(u_\varepsilon - \tilde{u_\varepsilon})^+ dx,
\end{aligned}$$

where $A = \{0 < u_\varepsilon - \tilde{u_\varepsilon} < k\}$.
Letting k tend to zero and taking account the monotonicity of β_ε and a, we infer that

$$\varepsilon \int_\Omega \left(|u_\varepsilon|^{p(x)-2}u_\varepsilon - |\tilde{u_\varepsilon}|^{p(x)-2}\tilde{u_\varepsilon}\right)^+ dx \leq \int_\Omega \left(f - \tilde{f}\right) sign_0^+(u_\varepsilon - \tilde{u_\varepsilon})dx.$$

□

Remark 4.5. An immediate consequence of Proposition 4.4 is that for two second members f and $\tilde{f}$ in $L^\infty(\Omega)$, with $f \leq \tilde{f}$, then $u_\varepsilon \leq \tilde{u_\varepsilon}$ a.e. in Ω. Moreover, β_ε being monotone it follows that

$$\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \leq \beta_\varepsilon(T_{1/\varepsilon}(\tilde{u_\varepsilon})) \qquad a.e.\ in\ \Omega.$$

Step 2: A priori estimates.

Lemma 4.6. *For $0 < \varepsilon \leq 1$, let $u_\varepsilon \in W^{1,p(x)}(\Omega)$ be a solution of $P(\beta_\varepsilon, f)$. Then, there exists a constant $C > 0$, not depending on ε such that*

$$\|\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\|_\infty \leq ||f||_\infty \tag{4.12}$$

and

$$\|D(u_\varepsilon)\|_{p(.)} \leq C. \tag{4.13}$$

Moreover,

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon).Du_\varepsilon dx \leq k \int_{\{|u_\varepsilon|>l\}} |f| dx \tag{4.14}$$

holds for all $0 < \varepsilon \leq 1$ and all $k, l > 0$.

Proof. In order to prove (4.13), we choose $\psi = u_\varepsilon$ as test function in (4.1) to obtain

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))u_\varepsilon dx + \varepsilon \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon u_\varepsilon dx$$
$$+ \int_\Omega a(x, Du_\varepsilon).Du_\varepsilon dx = \int_\Omega f u_\varepsilon dx. \tag{4.15}$$

The two first terms of (4.15) are non-negatives. Therefore, the coercivity of a leads to the estimate

$$\alpha \int_\Omega |Du_\varepsilon|^{p(x)} dx \leq \int_\Omega |f u_\varepsilon| dx. \tag{4.16}$$

Using Hölder inequality and Proposition 2.2, we obtain from (4.16)

$$||Du_\varepsilon||_{p(.)} \leq \left(\frac{1}{\alpha}||f||_1\,||u_\varepsilon||_\infty\right)^{\frac{1}{p^-(.)}} \tag{4.17}$$

or

$$||Du_\varepsilon||_{p(.)} \leq \left(\frac{1}{\alpha}||f||_1\,||u_\varepsilon||_\infty\right)^{\frac{1}{p^+(.)}}. \tag{4.18}$$

Setting $C = \max\left\{\left(\frac{1}{\alpha}||f||_1\,||u_\varepsilon||_\infty\right)^{\frac{1}{p^-(.)}},\ \left(\frac{1}{\alpha}||f||_1\,||u_\varepsilon||_\infty\right)^{\frac{1}{p^+(.)}}\right\}$, we get (4.13).
In order to obtain (4.12), we take $\theta_h^\varepsilon = \frac{1}{h}(T_{k+h}(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))) - T_k(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))$ as test function in (4.1), with $\varepsilon, h > 0$ to get

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\theta_h^\varepsilon dx + \varepsilon \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon \theta_h^\varepsilon dx$$
$$+ \int_\Omega a(x, Du_\varepsilon).D\theta_h^\varepsilon dx = \int_\Omega f\,\theta_h^\varepsilon dx. \tag{4.19}$$

Since $|u_\varepsilon|^{p(x)-2}u_\varepsilon,\ \beta_\varepsilon$ are monotonous and increasing, with $\beta_\varepsilon(0) = 0$, we use (H_2) and the fact that $a(x, Du_\varepsilon).Du_\varepsilon \geq 0$ to obtain

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\theta_h^\varepsilon dx \leq \int_\Omega f\,\theta_h^\varepsilon dx.$$

The previous inequality is written

$$h\int_{\{|u_\varepsilon|>h+k\}}\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))sign(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))dx \le h\int_{\{|u_\varepsilon|>h+k\}} f\, sign(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))dx$$
$$+\int_{\{k\le|u_\varepsilon|\le h+k\}} f\big[\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))) - ksign(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))\big]dx,$$

which leads to

$$h\int_{\{|u_\varepsilon|>h+k\}}\big[\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) - f\big]sign(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))$$
$$\le \int_{\{k\le|u_\varepsilon|\le h+k\}} f\big[\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))) - ksign(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))\big]. \tag{4.20}$$

Dividing (4.20) by h, and passing to the limit with $h \to 0$, yields

$$\int_{\{|u_\varepsilon|>k\}}\big[\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) - f\big] \le 0. \tag{4.21}$$

From (4.21), we deduce for $k > 0$,

$$\int_\Omega \big(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) - k\big) \le \int_\Omega \big(f - k\big). \tag{4.22}$$

Choosing $k > ||f||_\infty$, we obtain from (4.22) the claim $||\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))||_\infty \le ||f||_\infty$. To end the proof of Lemma 4.6 , we now prove (4.14). We take $\psi = T_k(u_\varepsilon - T_l(u_\varepsilon))$ in (4.1) to obtain

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))T_k(u_\varepsilon - T_l(u_\varepsilon))dx + \varepsilon\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon\, T_k(u_\varepsilon - T_l(u_\varepsilon))dx$$
$$+\int_\Omega a(x, Du_\varepsilon).DT_k(u_\varepsilon - T_l(u_\varepsilon))dx = \int_\Omega f\, T_k(u_\varepsilon - T_l(u_\varepsilon))dx. \tag{4.23}$$

Since $T_k(u_\varepsilon - T_l(u_\varepsilon))$ and u_ε have the same sign, it is clear that the first two terms on the left-hand side of (4.23) are non-negative. Then, we deduce from (4.23) that

$$\int_\Omega a(x, Du_\varepsilon).DT_k(u_\varepsilon - T_l(u_\varepsilon))\, dx \le \int_\Omega fT_k(u_\varepsilon - T_l(u_\varepsilon))\, dx.$$

Since $DT_k(u_\varepsilon - T_l(u_\varepsilon)) = Du_\varepsilon\chi_{\{l<|u|<l+k\}}$, the previous inequality is written

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon).Du_\varepsilon\, dx \le \int_\Omega f\, T_k(u_\varepsilon - T_l(u_\varepsilon))dx \le k\int_{\{|u_\varepsilon|>l\}} |f|dx.$$

□

Next, we get a priori estimates on u and Du through the following Lemma.

Lemma 4.7. *Assume that* $(H_1) - (H_4)$ *hold true and* $f \in L^\infty(\Omega)$*. Let* u_ε *be a renormalized solution of* $P(\beta_\varepsilon, f)$*. For* k *large enough, we have*

$$meas\{|u_\varepsilon| > k\} \le \frac{||f||_\infty}{\min\Big(\beta_\varepsilon(k), -\beta_\varepsilon(-k)\Big)}, \tag{4.24}$$

and

$$meas\{|Du_\varepsilon| > k\} \leq \frac{C'}{k^{p-}} + \frac{||f||_\infty}{\min\Big(\beta_\varepsilon(k), -\beta_\varepsilon(-k)\Big)}. \tag{4.25}$$

Proof. Let's start by proving (4.24).

$$\int_\Omega |\beta_\varepsilon(T_k(u_\varepsilon))|dx \;=\; \int_{\{|u_\varepsilon|>k\}} |\beta_\varepsilon(T_k(u_\varepsilon))|dx + \int_{\{|u_\varepsilon|\leq k\}} |\beta_\varepsilon(T_k(u_\varepsilon))|dx,$$

by dropping the last term of right-hand side, we have

$$\int_{\{|u_\varepsilon|>k\}} |\beta_\varepsilon(T_k(u_\varepsilon))|dx \leq \int_\Omega |\beta_\varepsilon(T_k(u_\varepsilon))|dx. \tag{4.26}$$

From inequality (4.12) and (4.26), we deduce that

$$\int_{\{|u_\varepsilon|>k\}} |\beta_\varepsilon(T_k(u_\varepsilon))|dx \leq ||f||_\infty. \tag{4.27}$$

Since β_ε is monotone increasing with $\beta_\varepsilon(0) = 0$, for

$$u_\varepsilon > k \;\Rightarrow \beta_\varepsilon(k) \leq \beta_\varepsilon(u_\varepsilon) \;\Rightarrow \quad \beta_\varepsilon(k) \leq |\beta_\varepsilon(u_\varepsilon)|,$$

$$u_\varepsilon < -k \Rightarrow -\beta_\varepsilon(-k) \leq -\beta_\varepsilon(u_\varepsilon) \;\Rightarrow -\beta_\varepsilon(-k) \leq |\beta_\varepsilon(u_\varepsilon)|$$

a result,

$$\min\big(\beta_\varepsilon(k), -\beta_\varepsilon(-k)\big) \leq |\beta_\varepsilon(u_\varepsilon)| \tag{4.28}$$

Combining (4.27) with (4.29), we get

$$\min\big(\beta_\varepsilon(k), -\beta_\varepsilon(-k)\big)\, meas(\{|u_\varepsilon|>k\}) \leq ||f||_\infty.$$

Hence the desired result. It remains to prove the estimate (4.25). For $k, \lambda > 0$ we take

$$\Phi(k,\lambda) = meas\{|Du_\varepsilon|^{p^-} > \lambda, |u_\varepsilon| > k\}.$$

Thanks to (4.24), we have

$$\Phi(k,0) = meas\{|u_\varepsilon| > k\}.$$

As function $\lambda \longmapsto \Phi(k,\lambda)$ is non increasing, we get for $k, \lambda > 0$ and for $0 \leq s \leq \lambda$, $\Phi(0,\lambda) \leq \Phi(0,s)$.

$$\begin{aligned}
&\Phi(0,\lambda) = meas\{|Du_\varepsilon|^{p^-} > \lambda\} = \frac{1}{\lambda}\int_0^\lambda \Phi(0,\lambda)ds \leq \frac{1}{\lambda}\int_0^\lambda \Phi(0,s)ds,\\
&\leq \frac{1}{\lambda}\int_0^\lambda \Phi(k,s)ds + \frac{1}{\lambda}\int_0^\lambda \big(\Phi(0,s) - \Phi(k,s)\big)ds,\\
&\leq \frac{1}{\lambda}\int_0^\lambda \Phi(k,0)ds + \frac{1}{\lambda}\int_0^\lambda \big(\Phi(0,s) - \Phi(k,s)\big)ds,\\
&\leq \Phi(k,0) + \frac{1}{\lambda}\int_0^\lambda \big(\Phi(0,s) - \Phi(k,s)\big)ds.
\end{aligned} \tag{4.29}$$

Hence,

$$meas\{|Du_\varepsilon|^{p^-} > \lambda\} \leq meas\{|u_\varepsilon| > k\} + \frac{1}{\lambda}\int_0^\lambda \big(\Phi(0,s) - \Phi(k,s)\big)ds. \quad (4.30)$$

To end, we remark that $\Phi(0,s) - \Phi(k,s) = meas\{|Du_\varepsilon|^{p^-} > s, |u_\varepsilon| \leq k\}$ and we get

$$\int_0^\infty \big(\Phi(0,s) - \Phi(k,s)\big)ds = \int_{\{|u_\varepsilon| \leq k\}} |Du_\varepsilon|^{p^-} dx. \quad (4.31)$$

Since

$$\int_{\{|u_\varepsilon|\leq k\}} |Du_\varepsilon|^{p^-} dx = \int_{\{|u_\varepsilon|\leq k,\, |Du_\varepsilon|>1\}} |Du_\varepsilon|^{p^-} dx + \int_{\{|u_\varepsilon|\leq k,\, |Du_\varepsilon|\leq 1\}} |Du_\varepsilon|^{p^-} dx$$
$$\leq \int_{\{|u_\varepsilon|\leq k,\, |Du_\varepsilon|>1\}} |Du_\varepsilon|^{p^-} dx + meas(\Omega)$$
$$\leq \int_{\{|u_\varepsilon|\leq k\}} |Du_\varepsilon|^{p(.)} dx + meas(\Omega).$$

we have thanks above inequality and (4.13),

$$\int_{\{|u_\varepsilon|\leq k\}} |Du_\varepsilon|^{p^-} dx \leq C + meas(\Omega) \equiv C'. \quad (4.32)$$

Combining (4.31) and (4.32), it follows that

$$\int_0^\infty \big(\Phi(0,s) - \Phi(k,s)\big)ds \leq C' \quad (4.33)$$

Thanks to (4.30) and (4.33), we deduce that

$$meas\{|Du_\varepsilon|^{p^-} > \lambda\} \leq \frac{C'}{\lambda} + \frac{||f||_\infty}{\min\Big(\beta_\varepsilon(k), -\beta_\varepsilon(-k)\Big)}. \quad (4.34)$$

Minimizing this inequality on λ, we see that an optimal choice is, up to a multiplicative constant $\lambda = k^{p^-}$, which leads to (4.25). □

Step 3: Basic convergence results.

The a priori estimates in Lemma 4.6 imply the following convergence results.

Lemma 4.8. *For $k > 0$, as ε tends to zero, we have:*

(i) $T_k(u_\varepsilon) \to T_k(u)$ *in* $L^{p-}(\Omega)$ *and* $DT_k(u_\varepsilon) \rightharpoonup DT_k(u)$ *in* $(L^{p(.)}(\Omega))^N$,

(ii) $\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \overset{\star}{\rightharpoonup} b$ *in* $L^\infty(\Omega)$,

(iii) $a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u))$ *in* $(L^{p'(x)}(\Omega))^N$.

Proof. (**i**) For $k > 0$, the sequence $(DT_k(u_\varepsilon))_{\varepsilon>0}$ is bounded in $L^{p(.)}(\Omega)$, thus, the sequence $(T_k(u_\varepsilon))_{\varepsilon>0}$ is bounded in $W^{1,p(.)}(\Omega)$. Therefore, we can extract a subsequence, still denoted $(T_k(u_\varepsilon))_{\varepsilon>0}$ such that for all $k > 0$, $(T_k(u_\varepsilon))_{\varepsilon>0}$ converges weakly to σ_k in $W^{1,p(.)}(\Omega)$ and also that $(T_k(u_\varepsilon))_{\varepsilon>0}$ converges strongly to σ_k in $L^{p^-}(\Omega)$.

We show that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is Cauchy in measure. Let $s > 0$ and $k > 0$ be fixed.

Define $E_n = \{|u_n| > k\}$, $E_m = \{|u_m| > k\}$ and $E_{n,m} = \{|T_k(u_n) - T_k(u_m)| > s\}$. Note that $\{|u_n - u_m| > s\} \subset E_n \cup E_m \cup E_{n,m}$, and hence,

$$meas(\{|u_n - u_m| > s\}) \leq meas(E_n) + meas(E_m) + meas(E_{n,m}). \tag{4.35}$$

Let $\eta > 0$, using the previous inequality, we choose $k = k(\eta)$ such that

$$meas(E_n) \leq \frac{\eta}{3} \quad \text{and} \quad meas(E_m) \leq \frac{\eta}{3}. \tag{4.36}$$

Since $T_k(u_\varepsilon)$ converge strongly in $L^{p-}(\Omega)$, then it is a Cauchy sequence in $L^{p-}(\Omega)$, therefore

$$\forall s > 0,\, \eta > 0,\, \exists n_0 = n_0(s,\eta) \quad such \quad that \quad \forall\, n, m \geq n_0(s,\eta),$$
$$\Big(\int_\Omega |T_k(u_n) - T_k(u_m)|^{p(x)}\Big)^{\frac{1}{p-}} \leq \Big(\frac{\eta s^{p-}}{3}\Big)^{\frac{1}{p-}}.$$

We deduce that,

$$\begin{aligned} \forall n \geq n_0,\, \forall m \geq n_0,\, meas(E_{n,m}) &\leq \frac{1}{s^{p-}} \int_\Omega (|T_k(u_n) - T_k(u_m)|)^{p(x)} dx \\ &\leq \frac{\eta}{3}. \end{aligned} \tag{4.37}$$

From (4.35)-(4.37) we deduce that

$$meas(\{|u_n - u_m| > s\}) \leq \eta, \tag{4.38}$$

for all $n, m \geq n_0(s,\eta)$. Relation (4.38) implies that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in measure and there exists a measurable function u such that $u_\varepsilon \to u$ in measure. Then, we can extract a subsequence still denoted $(u_\varepsilon)_{\varepsilon>0}$, such that $u_\varepsilon \to u$ a.e. in Ω.
According to (4.13), the sequence $(DT_k(u_\varepsilon))_{\varepsilon>0}$ is bounded in $(L^{p(.)}(\Omega))^N$. We can extract a subsequence still denoted $(DT_k(u_\varepsilon))_{\varepsilon>0}$ which converges weakly to $DT_k(u)$ as ε tends to zero.
On the other hand, by (4.12) we get that $\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))$ is bounded in $L^\infty(\Omega)$ and therefore it is converges weakly-$\star$ to b in $L^\infty(\Omega)$. Hence (**ii**).
(**iii**) According to (H_4), we have $\|(a(x, DT_k(u_\varepsilon)))\|_{p'(.)} \leq C'$ (with $C' > 0$). We can extract a subsequence still denoted $(a(x, DT_k(u_\varepsilon)))_{\varepsilon>0}$ which converges weakly to Φ_k in $(L^{p'(.)}(\Omega)^N)$ as ε tends to zero. Using the pseudo-monotone argument, we show that $\Phi_k = a(x, DT_k(u))$ almost everywhere in Ω. First of all, we prove that for all $k > 0$, we have

$$\limsup_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)).D(T_k(u_\varepsilon) - T_k(u))dx \leq 0. \tag{4.39}$$

Using $\psi = h_l(u_\varepsilon)\big(T_k(u_\varepsilon) - T_k(u)\big)$ as test function in (4.1), we have

$$\int_\Omega h_l(u_\varepsilon) a(x, DT_k(u_\varepsilon)).D(T_k(u_\varepsilon) - T_k(u))dx = B_{k,l,\varepsilon} + C_{k,l,\varepsilon} + D_{k,l,\varepsilon} + E_{k,l,\varepsilon}, \tag{4.40}$$

where

$$\begin{aligned} B_{k,l,\varepsilon} &= \int_\Omega f h_l(u_\varepsilon)\big(T_k(u_\varepsilon) - T_k(u)\big)dx, \\ C_{k,l,\varepsilon} &= -\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)h_l(u_\varepsilon)\big(T_k(u_\varepsilon) - T_k(u)\big), \\ D_{k,l,\varepsilon} &= -\varepsilon \int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon(T_k(u_\varepsilon) - T_k(u))h(u_\varepsilon)dx, \\ E_{k,l,\varepsilon} &= -\int_\Omega h_l'(u_\varepsilon)a(x, DT_k(u_\varepsilon)).Du_\varepsilon\big(T_k(u_\varepsilon) - T_k(u)\big)dx. \end{aligned}$$

Now we pass to the limit in (4.40) with $l \to \infty$ and $\varepsilon \to 0$ respectively. Using the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\varepsilon\to 0}\lim_{l\to\infty} B_{k,l,\varepsilon} = \lim_{\varepsilon\to 0}\lim_{l\to\infty} \int_\Omega f h_l(u_\varepsilon)\big(T_k(u_\varepsilon) - T_k(u)\big)dx = 0 \tag{4.41}$$

and

$$\lim_{\varepsilon\to 0}\lim_{l\to\infty} C_{k,l,\varepsilon} = \lim_{\varepsilon\to 0}\lim_{l\to\infty} \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)h_l(u_\varepsilon)\big(T_k(u_\varepsilon) - T_k(u)\big)dx = 0. \tag{4.42}$$

For $D_{k,l,\varepsilon}$, as $\Big||u_\varepsilon|^{p(x)-2}u_\varepsilon(T_k(u_\varepsilon) - T_k(u))h_l(u_\varepsilon)\Big| \leq 2kC$, it's clear that

$$\lim_{\varepsilon\to 0}\lim_{l\to\infty} D_{k,l,\varepsilon} = -\lim_{\varepsilon\to 0}\lim_{l\to\infty} \varepsilon\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon(T_k(u_\varepsilon) - T_k(u))h_l(u_\varepsilon)dx = 0. \tag{4.43}$$

It remains to treat $E_{k,l,\varepsilon}$. We have

$$|E_{k,l,\varepsilon}| \leq 2k\int_{\{l<|u_\varepsilon|<l+1\}} a(x, Du_\varepsilon)Du_\varepsilon dx,$$

which implies by Lebesgue's dominated convergence theorem,

$$\limsup_{\varepsilon\to 0}\limsup_{l\to\infty} E_{k,l,\varepsilon} \leq 0. \tag{4.44}$$

Combining (4.41)-(4.44) and the fact that $h_l \to 1$ as $l \to \infty$, we can pass to the limit in (4.40) as $l \to \infty$ and as $\varepsilon \to 0$ respectively, to obtain (4.39).
Now, we use the Minty's arguments based on the monotonicity property of a to show that, for all $k > 0$, $\Phi_k = a(x, DT_k(u))$ a.e. in Ω. By assumption (H_4), the sequence $(a(x, DT_k(u_\varepsilon)))_{\varepsilon>0}$ is bounded in $(L^{p'(x)}(\Omega))^N$ which is reflexive. Thus, there exists $\Phi_k \in (L^{p'(x)}(\Omega))^N$ such that, up to a subsequence, $a(x, DT_k(u_\varepsilon)) \rightharpoonup \Phi_k$ in $(L^{p'(x)}(\Omega))^N$.

Let $\varphi \in \mathcal{D}(\Omega)$ and $\lambda \in \mathbb{R}^\star$. Using (4.39) and assumption (H_3) we get

$$\begin{aligned}
\lambda \int_\Omega \Phi_k D\varphi dx &= \lim_{\varepsilon\to 0} \int_\Omega \lambda a(x, DT_k(u_\varepsilon))D\varphi dx \\
&\geq \limsup_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon))D(T_k(u_\varepsilon) - T_k(u) + \lambda\varphi)dx, \\
&\geq \limsup_{\varepsilon\to 0} \int_\Omega a(x, D[T_k(u) - \lambda\varphi])D(T_k(u_\varepsilon) - T_k(u) + \lambda\varphi)dx, \\
&\geq \lambda \int_\Omega a(x, D[T_k(u) - \lambda\varphi])D\varphi. \qquad (4.45)
\end{aligned}$$

Dividing by $\lambda < 0$ and by $\lambda > 0$, and passing to the limit with $\lambda \to 0$, we obtain

$$\int_\Omega \Phi_k D\varphi dx = \int_\Omega a(x, DT_k(u))D\varphi dx, \quad \forall\varphi \in \mathcal{D}(\Omega).$$

Hence $a(x, DT_k(u)) = \Phi_k$ almost everywhere in Ω. We conclude that

$$a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u)) \quad \text{weakly in} \quad (L^{p'(x)}(\Omega))^N.$$

□

Remark 4.9. From (4.39) and (H_3) we deduce that

$$\lim_{\varepsilon\to 0} \int_\Omega \big(a(x, DT_k(u_\varepsilon)) - a(x, DT_k(u))\big)(DT_k(u_\varepsilon) - DT_k(u)) = 0. \qquad (4.46)$$

Lemma 4.10. *For all $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$,*

$$D[h(u_\varepsilon)\varphi] \to D[h(u_\varepsilon)\varphi] \quad \textit{strongly in} \quad (L^{p(\cdot)}(\Omega))^N, \quad \textit{as} \quad \varepsilon \to 0.$$

Proof. For any $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned}
D[h(u_\varepsilon)\varphi] - D[h(u)\varphi] &= (h(u_\varepsilon) - h(u))D\varphi + h'(u_\varepsilon)\varphi[Du_\varepsilon - Du] \\
&\quad + \big(h'(u_\varepsilon) - h'(u)\big)\varphi Du \\
&:= \psi_1^\varepsilon + \psi_2^\varepsilon + \psi_3^\varepsilon. \qquad (4.47)
\end{aligned}$$

For the term ψ_1^ε, we consider $\rho_{p(.)}(\psi_1^\varepsilon) = \displaystyle\int_\Omega |(h(u_\varepsilon) - h(u))D\varphi|^{p(x)}\, dx$.
Set $\Theta_1^\varepsilon(x) = |(h(u_\varepsilon) - h(u))D\varphi|^{p(x)}$.
We have $\Theta_1^\varepsilon(x) \to 0$ a.e. $x \in \Omega$ as $\varepsilon \to 0$ and

$$|\Theta_1^\varepsilon(x)| \leq C(h, p_-, p_+)|D\varphi|^{p(x)} \in L^1(\Omega).$$

Then, by the Lebesgue dominated convergence theorem, we get that $\lim_{\varepsilon\to 0} \rho_{p(.)}(\psi_1^\varepsilon) = 0$.
Hence,

$$\|\psi_1^\varepsilon\|_{L^{p(\cdot)}(\Omega)} \to 0 \text{ as } \varepsilon \to 0. \qquad (4.48)$$

For the term ψ_2^ε we consider $\rho_{p(.)}(\psi_2^\varepsilon) = \displaystyle\int_\Omega |h'(u_\varepsilon)\varphi(DT_l(u_\varepsilon) - DT_l(u))|^{p(x)}\, dx$ for some $l > 0$ such that $\operatorname{supp}(h) \subset [-l, l]$.

Set $\Theta_2^\varepsilon(x) = |h'(u_\varepsilon)\varphi(DT_l(u_\varepsilon) - DT_l(u))|^{p(x)}$.
We have $\Theta_2^\varepsilon(x) \to 0$ a.e. $x \in \Omega$ as $\varepsilon \to 0$ and

$$|\Theta_2^\varepsilon(x)| \leq C(h, p_-, p_+, \|\varphi\|_\infty)|DT_l(u_\varepsilon) - DT_l(u)|^{p(x)}.$$

Since $DT_l(u_\varepsilon) \to DT_l(u)$ strongly in $\left(L^{p(.)}(\Omega)\right)^N$, we get

$$\rho_{p(.)}(DT_l(u_\varepsilon) - DT_l(u)) \to 0 \text{ as } \varepsilon \to 0,$$

which is equivalent to say

$$\lim_{\varepsilon\to 0} \int_\Omega |DT_l(u_\varepsilon) - DT_l(u)|^{p(x)}\, dx = 0.$$

Then $|DT_l(u_\varepsilon) - DT_l(u)|^{p(.)} \to 0$ strongly in $L^1(\Omega)$.
By the Lebesgue generalized convergence theorem, one has

$$\lim_{\varepsilon\to 0} \int_\Omega \Theta_2^\varepsilon(x)\, dx = \lim_{\varepsilon\to 0} \rho_{p(.)}(\psi_2^\varepsilon) = 0.$$

Hence,

$$\|\psi_2^\varepsilon\|_{L^{p(.)}(\Omega)} \to 0 \text{ as } \varepsilon \to 0. \tag{4.49}$$

For the term ψ_3^ε we consider $\rho_{p(.)}(\psi_3^\varepsilon) = \displaystyle\int_\Omega |(h'(u_\varepsilon) - h'(u))\varphi Du|^{p(x)}\, dx$.
Set $\Theta_3^\varepsilon(x) = |(h'(u_\varepsilon) - h'(u))\varphi Du|^{p(x)}$.
We have $\Theta_3^\varepsilon(x) \to 0$ a.e. $x \in \Omega$ as $\varepsilon \to 0$ and

$$|\Theta_3^\varepsilon(x)| \leq C(h, p_-, p_+, \|\varphi\|_\infty)|DT_l(u)|^{p(x)} \in L^1(\Omega),$$

with some $l > 0$ such that $\operatorname{supp}(h) \subset [-l, l]$.
Then, by the Lebesgue dominated convergence theorem, we get $\lim\limits_{\varepsilon\to 0} \rho_{p(.)}(\psi_3^\varepsilon) = 0$.
Hence,

$$\|\psi_3^\varepsilon\|_{L^{p(.)}(\Omega)} \to 0 \text{ as } \varepsilon \to 0. \tag{4.50}$$

Thanks to (4.48)-(4.50), we get

$$\left\|\psi_1^\varepsilon + \psi_2^\varepsilon + \psi_3^\varepsilon\right\|_{L^{p(.)}(\Omega)} \to 0 \text{ as } \varepsilon \to 0.$$

□

Step 4: Passage to the limit in equation (4.1).
We want to pass to the limit in the approached problem as ε tends to zero. Taking $h_l(u_\varepsilon)h(u)\psi$ as test function in (4.1), where $h \in C_c^1(\mathbb{R})$ and $\psi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, we get

$$I_\varepsilon^1 + I_\varepsilon^2 + I_\varepsilon^3 = I_\varepsilon^4 \tag{4.51}$$

where

$$\begin{aligned} I_\varepsilon^1 &= \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))h_l(u_\varepsilon)h(u)\psi dx,\\ I_\varepsilon^2 &= \varepsilon\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon h_l(u_\varepsilon)h(u)\psi dx\\ I_\varepsilon^3 &= \int_\Omega a(x,Du_\varepsilon).D[h_l(u_\varepsilon)h(u)\psi]dx,\\ I_\varepsilon^4 &= \int_\Omega f h_l(u_\varepsilon)h(u)\psi dx. \end{aligned}$$

Item 1: Passing to the limit as $\varepsilon\to 0$

By Lebesgue dominated convergence Theorem, we see that

$$\lim_{\varepsilon\to 0} I_\varepsilon^2 = \lim_{\varepsilon\to 0}\varepsilon\int_\Omega |u_\varepsilon|^{p(x)-2}u_\varepsilon h_l(u_\varepsilon)h(u)\psi dx = 0, \tag{4.52}$$

and

$$\lim_{\varepsilon\to 0} I_\varepsilon^4 = \lim_{\varepsilon\to 0}\int_\Omega f h_l(u_\varepsilon)h(u)\psi dx = \int_\Omega f h_l(u)h(u)\psi dx. \tag{4.53}$$

Using Lemmas 4.10 and 4.8, we obtain respectively

$$D[h_l(u_\varepsilon)h(u)\psi] \longrightarrow D[h_l(u)h(u)\psi] \quad \text{strongly in} \quad (L^{p(x)}(\Omega))^N$$

and

$$a(x,Du_\varepsilon) \rightharpoonup a(x,Du) \quad \text{weakly in} \quad (L^{p'(x)}(\Omega))^N.$$

Therefore,

$$\lim_{\varepsilon\to 0} I_\varepsilon^3 = \lim_{\varepsilon\to 0}\int_\Omega a(x,Du_\varepsilon).D[h_l(u_\varepsilon)h(u)\psi]dx = \int_\Omega a(x,Du).D[h_l(u)h(u)\psi]dx. \tag{4.54}$$

Now, we are concerning with the term I_ε^1.

h_l is continuous in $supp\, h_l \subset [-1;1]$ means that the sequence $\big(h_l(u_\varepsilon)\big)_{\varepsilon>0}$ is bounded and therefore equi-integrable. Hence $h_l(u_\varepsilon)$ converge strongly in $L^1(\Omega)$ to $h_l(u)$ as ε tends to zero. Furthermore $\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))$ converges weakly-$\star$ to b in $L^\infty(\Omega)$, it follows that

$$\lim_{\varepsilon\to 0} I_\varepsilon^1 = \lim_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))h_l(u_\varepsilon)h(u)\psi dx = \int_\Omega b h_l(u)h(u)\psi dx. \tag{4.55}$$

Item 2: Passing to the limit as $l\to +\infty$

By combining (4.52),(4.53) and (4.54) we have

$$I_l^1 + I_l^2 = I_l^3, \tag{4.56}$$

where

$$\begin{aligned} I_l^1 &= \int_\Omega b h_l(u_\varepsilon)h(u)\psi dx,\\ I_l^2 &= \int_\Omega a(x,Du).D[h_l(u)h(u)\psi]dx,\\ I_l^3 &= \int_\Omega f h_l(u)h(u)\psi dx. \end{aligned}$$

Choosing $m > 0$ such that supp $h \subset [-m, m]$, we can replace u by $T_m(u)$ in I_l^1, I_l^2 and I_l^3. Therefore, it follows that

$$\lim_{l\to+\infty} I_l^1 = \int_\Omega b\, h(u)\, \psi\, dx, \tag{4.57}$$

$$\lim_{l\to+\infty} I_l^2 = \int_\Omega a(x, Du).D[h(u)\psi]\, dx, \tag{4.58}$$

$$\lim_{l\to+\infty} I_l^3 = \int_\Omega f\, h(u)\, \psi\, dx. \tag{4.59}$$

Combining (4.57) with (4.58)-(4.59) we obtain

$$\int_\Omega b\, h(u)\, \psi\, dx + \int_\Omega a(x, Du).D[h(u)\psi]\, dx = \int_\Omega f\, h(u)\, \psi\, dx, \tag{4.60}$$

for all $h \in C_c^1(\mathbb{R})$ and $\psi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.
Now, we prove that u satisfied (3.2). Since $u_\varepsilon \to u$ as $\varepsilon \to 0$ and $meas(\{|u_\varepsilon|>l\}) \to 0$ uniformly as l tends to ∞, we pass to the limit in (4.14) as l tends to $+\infty$, to deduce that

$$\lim_{l\to+\infty} \int_{\{l<|u|<l+1\}} a(x, Du).Dudx = 0.$$

Item 3: subdifferential argument
For any given maximal monotonic graph β, it exists a function $j : \mathbb{R} \to [0, \infty]$, convex, lower semi-continuous, proper such that $\beta(r) = \partial j(r)$ for all $r \in \mathbb{R}$, a.e. in Ω. j_ε has the following properties:
(i) for $\varepsilon > 0$, j_ε is convex and differentiable. Moreover $\beta_\varepsilon(r) = \partial j_\varepsilon(r)$ for $r \in \mathbb{R}$ and a.e. in Ω.
(ii) $\lim_{\varepsilon\to 0} j_\varepsilon(r) = j(r)$. From (i), it follows that

$$j_\varepsilon(r) \geq j_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + (r - T_{1/\varepsilon}(u_\varepsilon))\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \tag{4.61}$$

holds for all $r \in \mathbb{R}$ and a.e. in Ω. Let $E \subset \Omega$ be a measurable set and χ_E its characteristic function. Let us fix $\varepsilon_0 > 0$ and multiply (4.61) by the function $h_l(u_\varepsilon)\chi_E$ then integrate this last quantity on E. Using (ii), we obtain

$$\begin{aligned}\int_E j_\varepsilon(r) h_l(u_\varepsilon) dx \geq \int_E & j_{\varepsilon_0}(T_{l+1}(u_\varepsilon)) h_l(u_\varepsilon) dx \\ & + (r - T_{l+1}(u_\varepsilon)) h_l(u_\varepsilon) \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\end{aligned} \tag{4.62}$$

for all $r \in \mathbb{R}$ and $0 < \varepsilon < \varepsilon_0$. By stretching $\varepsilon_0 \to 0$ and $l \to \infty$, we obtain

$$\int_E j(r) dx \geq \int_E j(u) dx + b(r - u).$$

Since E is arbitrarily chosen, we deduce from preceding inequality that

$$j(r) \geq j(u)) + b(r - u), \tag{4.63}$$

for $r \in \mathbb{R}$ and for $x \in \Omega$, $u(x) \in D(\beta(u(x)))$ and $b(x) \in \beta(u(x))$ a.e. in Ω. The proof of the Proposition 4.1 is then complete. □

In the following proposition, we show that for the right-hand side $f \in L^\infty(\Omega)$, the renormalized solution of $P(\beta, f)$ is an extension to the weak solution concept.

Proposition 4.11. *Let (u, b) be a renormalized solution to $P(\beta, f)$ for $f \in L^\infty(\Omega)$. Then $u \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and thus, in particular u is a weak solution to $P(\beta, f)$.*

Proof. The proof of Proposition 4.11 follows the same lines as the proof of Proposition 5.2 in [15]. □

4.2. **Approximate solutions for L^1-data.**

For $f \in L^1(\Omega)$ and for $m, n \in \mathbb{N}$, let $f_{m,n} \in L^\infty(\Omega)$ as defined at the beginning of section 4, there exists $u_{m,n} \in W^{1,p(.)}(\Omega)$, $b_{m,n} \in L^\infty(\Omega)$, such that $(u_{m,n}, b_{m,n})$ is renormalized solution of $P(b_{m,n}, f_{m,n})$. Therefore, for $h \in C_c^1(\mathbb{R})$ and $\psi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega b_{m,n} h(u_{m,n})\psi dx + \int_\Omega a(x, Du_{m,n}).D(h(u_{m,n})\psi)dx = \int_\Omega f_{m,n} h(u_{m,n})\psi dx. \tag{4.64}$$

In the following, we give a priori estimates necessary for the rest of the work.

Lemma 4.12. *For $m, n \in \mathbb{N}$ let $(u_{m,n}, b_{m,n})$ be a renormalized solutions of $P(b_{m,n}, f_{m,n})$. Then, for any $k > 0$ and $m, n \in \mathbb{N}$, we have*

$$\int_\Omega |DT_k(u_{m,n})|^{p(x)} \leq \frac{k}{\alpha}\|f\|_1 \tag{4.65}$$

and there exists a constant $C_2(k) > 0$, not depending on m, n, such that

$$\|DT_k(u_{m,n})\|_{p(.)} \leq C_2(k). \tag{4.66}$$

Moreover,

$$\|b_{m,n})\|_1 \leq \|f\|_1 \tag{4.67}$$

holds for all $m, n \in \mathbb{N}$.

Proof. Choosing for all $l, k > 0$, $h_l(u_{m,n})\, T_k(u_{m,n})$ as a test function in (4.64), we get

$$\begin{aligned}\int_\Omega b_{m,n}\, h_l(u_{m,n})\, T_k(u_{m,n})dx \quad &+ \int_\Omega a(x, DT_k(u_{m,n})).D(h_l(u_{m,n})\, T_k(u_{m,n}))dx \\ &= \int_\Omega f_{m,n}\, h_l(u_{m,n})\, T_k(u_{m,n})dx.\end{aligned}$$

Since all the terms of the left hand side of the previous equality are non-negative and

$$\int_\Omega |f_{m,n}\, h_l(u_{m,n})\, T_k(u_{m,n})|\, dx \leq k\|f_{m,n}\|_\infty \leq k\|f\|_1, \tag{4.68}$$

using assumption (H_2), we obtain

$$\int_\Omega |DT_k(u_{m,n})|^{p(x)} dx \leq \frac{k}{\alpha}\|f\|_1 \equiv C_2(k). \tag{4.69}$$

Moreover, we have

$$\int_\Omega b_{m,n} T_k(u_{m,n})dx \le k\|f\|_1.$$

Dividing the above inequality by $k > 0$, we get

$$\int_\Omega b_{m,n} \frac{1}{k} T_k(u_{m,n})dx \quad \le \quad \|f\|_1. \tag{4.70}$$

As $\lim\limits_{k\to\infty} \dfrac{1}{k} T_k(u_{m,n}) = sign_0(u_{m,n})$ and $b_{m,n} \in \beta_{m,n}(u_{m,n})$, passing to the limit as $k \to +\infty$ in (4.70), we obtain

$$\int_\Omega |b_{m,n}|dx \quad \le \quad \|f\|_1. \tag{4.71}$$

□

In order to pass to the limit in the problem $P(b_{m,n}, f_{m,n})$, a strong convergence of $u_{m,n}$ in $L^1(\Omega)$ is necessary. The comparison principle in L^1 plays a important role in this part. By the result of Proposition 4.11, we deduce that for every $m, n \in \mathbb{N}$, we have

$$u^\varepsilon_{m,n+1} \le u^\varepsilon_{m,n} \le u^\varepsilon_{m+1,n}, \tag{4.72}$$

a.e. in Ω and therefore, passing to the limit in (4.72) as ε tends to zero, we obtain

$$u_{m,n+1} \le u_{m,n} \le u_{m+1,n}, \tag{4.73}$$

a.e. in Ω.
Setting $b_\varepsilon := \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))$, from (4.72) and Remark 4.5 we can write that

$$b^\varepsilon_{m,n+1} \le b^\varepsilon_{m,n} \le b^\varepsilon_{m+1,n}, \tag{4.74}$$

a.e. in Ω. Moreover, using the fact that $b^\varepsilon_{m,n} \overset{\star}{\rightharpoonup} b_{m,n}$ in $L^\infty(\Omega)$ and this convergence preserves order, we get,

$$b_{m,n+1} \le b_{m,n} \le b_{m+1,n}, \tag{4.75}$$

a.e. in Ω and for all $m, n \in \mathbb{N}$.
By (4.75) and (4.67), for any $n \in \mathbb{N}$ there exists $b^n \in L^1(\Omega)$ such that $b_{m,n} \to b^n$ as $m \to +\infty$ in $L^1(\Omega)$ and almost everywhere in Ω and $b \in L^1(\Omega)$, such that $b^n \to b$ as $n \to +\infty$ in $L^1(\Omega)$ and almost everywhere in Ω.
From the inequality (4.73), we deduce that the sequence $(u_{m,n})_m$ is monotone increasing, hence, for any $n \in \mathbb{N}, u_{m,n} \to u^n$ almost everywhere in Ω, where $u^n : \Omega \to \overline{\mathbb{R}}$ is a measurable function. Using (4.73) again, we conclude that the sequence $(u_n)_n$ is monotone decreasing, hence, $u^n \to u$ where $u : \Omega \to \overline{\mathbb{R}}$ is a measurable function. In consequence, we can write:

$$u_{m,n} \uparrow_m u_n \downarrow_n u \qquad u_{m,n} \downarrow_n u_m \uparrow_m u \quad in \quad L^1(\Omega). \tag{4.76}$$

Moreover, from the inequality (4.69), the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ is bounded in $W^{1,p(x)}(\Omega)$. We can extract from this sequence a subsequence still denoted $(T_k(u_n))_{n\in\mathbb{N}}$ which converges weakly to v_k in $W^{1,p(x)}(\Omega)$ and strongly in $L^{p(x)}(\Omega)$ when $n \to +\infty$. Moreover $DT_k(u_n) \rightharpoonup g_k$ in $(L^{p(x)}(\Omega))^N$ when $n \to +\infty$. Through convergence (4.76), we conclude that $v_k = T_k(u)$ and $g_k = DT_k(u)$.

According to Lemma 4.8, we conclude that $u \in \mathcal{T}^{1,p(x)}(\Omega)$. So the properties (i) and (ii) of Definition 3.2 are satisfied.
In order to show that u is finite almost everywhere, we give an estimate on the level sets of $u_{m,n}$ in the following.

Lemma 4.13. *For $m, n \in \mathbb{N}$, let $(u_{m,n}, b_{m,n})$ be a renormalized solutions of $P(b_{m,n}, f_{m,n})$. Then, there exists a constant $C > 0$, not depending on $m, n \in \mathbb{N}$ such that*

$$|\{|u_{m,n}| \geq l\}| \leq C\, l^{-(p^- -1)} \tag{4.77}$$

and

$$\lim_{n\to+\infty} \lim_{m\to+\infty} |\{|u_{m,n}| \geq l\}| = |\{|u| \geq l\}| \leq C\, l^{-(p^- -1)} \leq C\, l^{-(p^- -1)}, \tag{4.78}$$

for all $l \geq 1$.

Proof. The proof of Lemma 4.13 follows the same lines as the proof of Lemma 6.2 in [15]. □

Lemma 4.14. *For $m, n \in \mathbb{N}$, let $(u_{m,n}, b_{m,n})$ be a renormalized solutions of $P(b_{m,n}, f_{m,n})$. There exists a subsequence $(m(n))_n$ such that setting*

$$b_n := b_{m(n),n},\ f_n := f_{m(n),n} \text{ and } u_n := u_{m(n),n},$$

we have

$$u_n \to u \text{ a.e. in } \Omega. \tag{4.79}$$

Moreover, for any $k > 0$,

$$T_k(u_n) \to T_k(u) \text{ in } L^{p(\cdot)}(\Omega) \text{ and a. e. in } \Omega, \tag{4.80}$$

$$DT_k(u_n) \rightharpoonup DT_k(u) \text{ in } \left(L^{p(\cdot)}(\Omega)\right)^N, \tag{4.81}$$

$$a(x, DT_k(u_n)) \rightharpoonup a(x, DT_k(u)) \text{ in } \left(L^{p'(\cdot)}(\Omega)\right)^N, \tag{4.82}$$

as $n \to +\infty$.

Proof. The proof of Lemma 4.14 follows the same lines as the proof of Lemma 6.4 in [15]. □

4.3. **Passing to the limit in (4.64).**

Choosing $h_l(u_n)\, h(u)\, \psi$ as test function in (4.64), where $h \in C_c^1(\mathbb{R})$ and $\psi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, and using Lemma 4.14 we obtain

$$\int_\Omega b_n\, h_l(u_n)\, h(u)\, \psi dx + \int_\Omega a(x, Du_n).D[h_l(u_n)\, h(u)\, \psi] dx = \int_\Omega f_n\, h_l(u_n)\, h(u)\, \psi dx,$$

that we write

$$I_n^1 + I_n^2 = I_n^3, \tag{4.83}$$

with

$$\begin{aligned} I_n^1 &= \int_\Omega b_n\, h_l(u_n)\, h(u)\, \psi dx, \\ I_n^2 &= \int_\Omega a(x, Du_n).D[h_l(u_n)\, h(u)\, \psi] dx, \\ I_n^3 &= \int_\Omega f_n\, h_l(u_n)\, h(u)\, \psi dx. \end{aligned}$$

We want to pass to the limit in (4.83) as n tends to ∞ and l tends to ∞ respectively.

Step 1: passing to the limit with $n \to +\infty$
The convergence results of Lemma 4.14 allow us to conclude that

$$\lim_{n\to\infty} I_n^1 = \int_\Omega b\, h_l(u)\, h(u)\, \psi\, dx, \tag{4.84}$$

$$\lim_{n\to\infty} I_n^3 = \int_\Omega f\, h_l(u)\, h(u)\, \psi\, dx. \tag{4.85}$$

With similar arguments as in the proof of (4.54), it follows that

$$\lim_{n\to\infty} I_n^2 = \int_\Omega a(x, Du).D[h_l(u_n)\, h(u)\, \psi]\, dx. \tag{4.86}$$

Step 2: passage to the limit with $l \to +\infty$
Combining (4.84)-(4.86) we get for all $l \geq 1$,

$$I_l^1 + I_l^2 = I_l^3, \tag{4.87}$$

where

$$\begin{aligned} I_l^1 &= \int_\Omega b\, h_l(u)\, h(u)\, \psi dx, \\ I_l^2 &= \int_\Omega a(x, Du).D[h_l(u)\, h(u)\, \psi] dx, \\ I_l^3 &= \int_\Omega f\, h_l(u)\, h(u)\, \psi dx. \end{aligned}$$

Choosing $k > 0$ such that $supp(h) \subset [-k, k]$, then $T_k(u) = u$ on $supp(h)$ and we can replace u by $T_k(u)$ in I_l^1, I_l^2 and I_l^3. Hence

$$\lim_{l\to\infty} I_l^1 = \int_\Omega b\, h(u)\, \psi\, dx, \tag{4.88}$$

$$\lim_{l\to\infty} I_l^2 = \int_\Omega a(x, Du).D[h(u)\, \psi]\, dx, \tag{4.89}$$

$$\lim_{l\to\infty} I_l^3 = \int_\Omega f\, h(u)\, \psi\, dx. \tag{4.90}$$

Combining (4.88)-(4.90) we finally obtain (3.1), which completes the proof of the existence of renormalized solutions of the problem $P(\beta, f)$.

To end the proof of Theorem 3.6, we now show the uniqueness of the renormalized solutions of the problem $P(\beta, f)$.

4.4. **Proof of uniqueness.**

To complete the proof of Theorem 3.6, we show the uniqueness of renormalized solutions (in the sense of $b(u)$) of the problem $P(\beta, f)$ with $f \in L^1(\Omega)$.
For $f \in L^1(\Omega)$, let (u_1, b_1) and (u_2, b_2) be two renormalized solutions of $P(\beta, f)$.
For (u_1, b_1), we choose $\psi = u_2$ as test function in (3.3) to get

$$\int_\Omega b_1 T_k(u_1 - u_2)dx + \int_\Omega a(x, Du_1).DT_k(u_1 - u_2)\, dx \leq \int_\Omega f\, T_k(u_1 - u_2)dx. \tag{4.91}$$

Similarly, we get for (u_2, b_2)

$$\int_\Omega b_2 T_k(u_2 - u_1)dx + \int_\Omega a(x, Du_2).DT_k(u_2 - u_1)\, dx \leq \int_\Omega f\, T_k(u_2 - u_1)dx. \tag{4.92}$$

Adding equations (4.91) and (4.92) yields

$$\int_\Omega (b_1 - b_2)\, T_k(u_1 - u_2)dx + \int_\Omega \big(a(x, Du_1) - a(x, Du_2)\big).DT_k(u_1 - u_2)\, dx \leq 0. \tag{4.93}$$

Note that the sub-gradient of a convex function is monotone i.e
$\langle \partial j(u_1) - \partial j(u_2); u_1 - u_2\rangle \geq 0$. Since $b_1 \in \partial j(u_1)$ and $b_2 \in \partial j(u_2)$, we deduce that

$$\int_\Omega \big(b_1 - b_2\big) T_k(u_1 - u_2)dx \geq 0. \tag{4.94}$$

Thanks to assumption (H_3), we have

$$\int_\Omega \big(a(x, Du_1) - a(x, Du_2)\big) DT_k(u_1 - u_2)dx \geq 0. \tag{4.95}$$

Therefore, we deduce from (4.93)

$$\int_\Omega \big(b_1 - b_2\big) T_k(u_1 - u_2)dx = 0 \tag{4.96}$$

and

$$\int_\Omega \big(a(x, Du_1) - a(x, Du_2)\big) DT_k(u_1 - u_2)dx = 0. \tag{4.97}$$

As a is strictly monotone, we get from (4.97) that $DT_k(u_1) = DT_k(u_2)$ a.e. in Ω.
Therefore, there exists a constant c such that $u_1 - u_2 = c$ a.e. in Ω.
At last let us see that $b_1 = b_2$. Indeed, passing to the limit as $k \to +\infty$ in (4.96), we obtain

$$\begin{aligned} \lim_{k \to 0} \int_\Omega \big(b_1 - b_2\big) \frac{1}{k} T_k(u_1 - u_2)dx &= \int_\Omega \big(b_1 - b_2\big) sign_0(u_1 - u_2)dx \\ &= \int_\Omega |b_1 - b_2| dx = 0. \end{aligned} \tag{4.98}$$

From (4.98), we deduce that $b_1 = b_2$ a.e. in Ω.
In summary,

$$u_1 - u_2 = c \quad \text{and} \quad b_1 = b_2 \quad \text{a.e. in} \quad \Omega. \tag{4.99}$$

This concludes the proof of Theorem 3.6.

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