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ON VON NEUMANN-JORDAN TYPE CONSTANT AND SUFFICIENT CONDITIONS FOR FIXED POINTS OF MULTIVALUED NONEXPANSIVE MAPPINGS

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Abstract. In the present paper, we give some geometric conditions in terms of the von Neumann-Jordan type constant and the Domínguez-Benavides' coefficient which in turn are sufficient for the existence of fixed points of multivalued nonexpansive and $1-\chi$-contractive nonself mappings satisfying the inwardness condition and normal structure in a reflexive Banach space. Our main results improve some well known results in the recent literature.

1. Introduction

Fixed point theory is one of the most powerful tools of modern mathematics. Not only is it used on a daily basis in pure and applied mathematics, but it also serves as a bridge between analysis and topology, and provides a very fruitful area of interaction between the two. In particular, fixed point theory for multivalued mappings has many useful applications in applied sciences, such as, game theory and mathematical economics. Thus, it is natural to try of extending the known fixed point results for singlevalued mappings to the setting of multivalued mappings.

In 1969, S. B. Nadler [21] extended the Banach Contraction Principle to multivalued contractive mappings in complete metric spaces. Since then, the metric fixed point theory of multivalued mappings has been rapidly developed. Some classical fixed point theorems for singlevalued nonexpansive mappings have been extended to multivalued nonexpansive mappings. Nevertheless, the fixed point theory of multivalued nonexpansive mappings is much more complicated and difficult than the corresponding theory of singlevalued nonexpansive mappings and many problems remain unsolved in it, for instance, the possibility of extending the well known Kirk's theorem [17], that is, do Banach spaces with weak normal structure have the fixed point property for multivalued nonexpansive mappings?

The concept of normal structure plays an important role in metric fixed point theory for nonexpansive mappings. Since under various geometric properties of a Banach space often measured by different geometric constants, normal structure

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of the space is guaranteed, it is natural to study if those properties imply the fixed point property for multivalued mappings. Recently, many geometric constants for a Banach space have been investigated. Both the James constant $J(X)$ and the von Neumann-Jordan constant $C_{NJ}(X)$ play an important role in the description of various geometric structures. Therefore, many recent studies have focused on these constants (see [3, 4, 5, 6, 7, 8, 10, 11, 15, 16, 20, 23, 25, 26]). S. Dhompongsa et al. [5] introduced the Domínguez-Lorenzo condition ((DL)-condition, in short) which implies weak normal structure of a Banach space and the fixed point property for multivalued nonexpansive mappings. A possible approach to the above problem is to look for geometric conditions in a Banach space X which imply the (DL)-condition. In this setting, the following results have been obtained:

(i) B. Gavira [11, Theorem 3] showed that a Banach space X with

$$J(X) < 1 + \frac{1}{R(1, X)}$$

satisfies the (DL)-condition.

(ii) J. Zhang and Y. Cui [25, Theorem 3.1] proved that a Banach space X with

$$C_{NJ}(X) < \frac{\left(1 + \frac{1}{R(1,X)}\right)^2}{2}$$

implies the (DL)-condition.

Following the above, we show some geometric conditions on a Banach space X concerning the von Neumann-Jordan type constant and the coefficient $R(1, X)$, which imply the existence of fixed points for multivalued nonexpansive and $1-\chi$-contractive nonself mappings satisfying the inwardness condition and normal structure in a reflexive Banach space. The results obtained in this work, improve some previous results in the recent papers.

2. Preliminaries

At first, we recall some concepts and notations which will be needed in this paper.

We shall assume throughout this paper that X be a Banach space with the closed unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ and the unit sphere $S_X = \{x \in X : \|x\| = 1\}$. Recall that a Banach space X is uniformly nonsquare [14] if and only if there exists $\delta > 0$ such that $\|x+y\| \leq 2-\delta$ or $\|x-y\| \leq 2-\delta$ for all $x, y \in B_X$.

Recently, Y. Takahashi [23] has introduced the James and von Neumann-Jordan type constants of Banach spaces. For $p \in [-\infty, +\infty)$ and $t \geq 0$, the James type constant is defined by

$$J_{X,p}(t) = \begin{cases} \sup\left\{\left(\frac{\|x+ty\|^p + \|x-ty\|^p}{2}\right)^{\frac{1}{p}} : x, y \in S_X\right\}, & p \neq -\infty, \\ \sup\left\{\min\left(\|x+ty\|, \|x-ty\|\right) : x, y \in S_X\right\}, & p = -\infty. \end{cases}$$

As mentioned in [23], we could define the von Neumann-Jordan type constant $C_p(X)$ by using the James type constant $J_{X,p}(t)$ as

$$C_p(X) = \sup\left\{\frac{J^2_{X,p}(t)}{1+t^2} \,:\, 0 \le t \le 1\right\}.$$

In particular, if $t = 1$ or $t = 1$, $p = -\infty$, then we get $C'_p(X)$ and $C_{-\infty}(X)$, respectively. It is obvious that $C'_p(X) \le C_p(X)$. Note that the von Neumann-Jordan type constant includes some well known geometrical constants, such as the von Neumann-Jordan constant $C_{NJ}(X)$ (see [3]), and the Zbăganu constant $C_Z(X)$ (see [24]). These constants are defined by $C_{NJ}(X) = C_2(X)$ and $C_Z(X) = C_0(X)$.

The coefficient $R(1, X)$ is defined by T. Domínguez-Benavides [9] as

$$R(1, X) = \sup\left\{\liminf_{n\to\infty} \|x_n + x\|\right\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \le 1$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$\limsup_{n\to\infty}\big(\limsup_{m\to\infty} \|x_n - x_m\|\big) \le 1.$$

Obviously, $1 \le R(1, X) \le 2$.

We next review some concepts and results of multivalued mappings. For more details on this topic, the reader is referred to [1, 13].

Let E be a nonempty subset of X. We shall denote by $FB(E)$ the family of all nonempty bounded closed subsets of E, by $F(E)$ the family of all nonempty closed subsets of E, by $FC(E)$ the family of all nonempty closed convex subsets of E, and by $KC(E)$ the family of all nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}, \qquad A, B \in FB(X).$$

A multivalued mapping $T : E \to F(X)$ is said to be a contraction if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \le k\|x - y\|, \qquad x, y \in E. \tag{2.1}$$

In this case, we also say that T is k-contractive.

If (2.1) is valid when $k = 1$, then T is called nonexpansive. A point x is a fixed point for a multivalued mapping T if $x \in Tx$.

Recall that the Hausdorff measure of noncompactness of a nonempty bounded subset B of X are defined as the number

$$\chi(B) = \inf\big\{d > 0 \,:\, B \text{ can be covered by finitely many balls of radii } \le d\big\}.$$

A multivalued mapping $T : E \to 2^X$ is called χ-condensing (respectively, $1-\chi$-contractive) if, for each bounded subset B of E with $\chi(B) > 0$, there holds the inequality

$$\chi(T(B)) < \chi(B) \quad \Big(\text{respectively, } \chi(T(B)) \le \chi(B)\Big).$$

Here $T(B) = \bigcup_{x\in B} Tx$.

Definition 2.1. ([10]) Let E be a nonempty closed subset of a Banach space X. The inward set of E at $x \in E$ is given by

$$I_E(X) = \Big\{x + \lambda(y - x) \;:\; \lambda \geq 1,\, y \in E\Big\}.$$

In case E is a nonempty closed convex subset of a Banach space X, we have

$$I_E(X) = \Big\{x + \lambda(y - x) \;:\; \lambda \geq 0,\, y \in E\Big\}.$$

A multivalued mapping $T : E \to 2^X$ is said to be inward (respectively, weakly inward) on E if

$$Tx \subset I_E(X) \quad \Big(\text{respectively, } Tx \subset \overline{I_E(X)}\Big) \qquad \text{for all } x \in E.$$

Definition 2.2. ([2]) A nonempty bounded and convex subset K of a Banach space X is said to have normal structure if for every convex subset E of K that contains more than one point there is a point $x_0 \in E$ such that

$$\sup\big\{\|x_0 - y\| \;:\; y \in E\big\} < \sup\big\{\|x - y\| \;:\; x, y \in E\big\}.$$

A Banach space X is said to have normal structure if every bounded convex subset of X has normal structure. A Banach space X is said to have weak normal structure if for each weakly compact convex set K of X that contains more than one point has normal structure.

It is clear that for a reflexive Banach space, normal structure and weak normal structure coincide. It was proved by W. A. Kirk [17] that if a weakly compact convex subset E of X has normal structure, then any nonexpansive mapping on E has a fixed point. Since then, much attention has been focused on normal structure. Whether or not a Banach space X has normal structure depends on the geometry of the unit sphere S_X, or closed unit ball B_X.

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r\big(E, \{x_n\}\big)$ and the asymptotic center $A\big(E, \{x_n\}\big)$ of $\{x_n\}$ in E are defined by

$$r\big(E, \{x_n\}\big) = \inf\Big\{\limsup_{n\to\infty} \|x_n - x\| \;:\; x \in E\Big\}$$

and

$$A\big(E, \{x_n\}\big) = \Big\{x \in E \;:\; \limsup_{n\to\infty} \|x_n - x\| = r\big(E, \{x_n\}\big)\Big\},$$

respectively. It is well known that $A\big(E, \{x_n\}\big)$ is a nonempty weakly compact convex set whenever E is (see [13]).

The sequence $\{x_n\}$ is called regular with respect to E if $r\big(E, \{x_n\}\big) = r\big(E, \{x_{n_i}\}\big)$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to E if $A\big(E, \{x_n\}\big) = A\big(E, \{x_{n_i}\}\big)$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. Furthermore, $\{x_n\}$ is called regular asymptotically uniform with respect to E if $\{x_n\}$ is regular and asymptotically uniform with respect to E.

Lemma 2.3. *Let $\{x_n\}$ and E be as above. Then the following assertions hold.*

(i) (Goebel [12], Lim [19]) *There always exists a subsequence of $\{x_n\}$ which is regular with respect to E.*

(ii) (Kirk [18]) *If E is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform with respect to E.*

Let C be a nonempty bounded subset of X. The Chebyshev radius of C relative to E is defined by

$$r_E(C) = \inf_{x\in E} \sup_{y\in C} \|x - y\|.$$

In 2006, S. Dhompongsa et al. [5] introduced the Domínguez-Lorenzo condition ((DL)-condition, in short) as follows.

Definition 2.4. ([5]) A Banach space X is said to satisfy the (DL)-condition if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular with respect to E,

$$r_E\big(A\big(E, \{x_n\}\big)\big) \leq \lambda\, r\big(E, \{x_n\}\big).$$

The next results show that the (DL)-condition is stronger than weak normal structure and also implies the existence of fixed points for multivalued nonexpansive and $1-\chi$-contractive nonself mappings satisfying the inwardness condition in a Banach space.

Theorem 2.5. ([5]) *Let X be a Banach space satisfying the (DL)-condition. Then X has weak normal structure.*

Theorem 2.6. ([5]) *Let X be a reflexive Banach space satisfying the (DL)-condition and suppose that E be a nonempty bounded closed convex separable subset of X. Assume that $T : E \to KC(X)$ be a nonexpansive and $1-\chi$-contractive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition:*

$$Tx \subset I_E(X) \qquad \text{for all } x \in E.$$

Then T has a fixed point.

Theorem 2.7. ([4, 5]) *Let E be a nonempty weakly compact convex subset of a Banach space X which satisfies the (DL)-condition and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.*

In the sequel, we recall some basic facts and concepts about ultrapowers of Banach spaces which are the main ingredient of our result. Ultrapowers are proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. For an in-depth discussion on the Banach space ultrapower construction, the reader is directed to [13, 22].

Let $\mathcal{F}$ be a filter on $\mathbb{N}$ and let X be a Banach space. A sequence $\{x_n\}$ in X converges to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in \mathbb{N} : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on $\mathbb{N}$ is called an ulterafilter if it is maximal with respect to set inclusion. An ultrafilter is called trivial if it is of the form $\{A \subset \mathbb{N} : i_0 \in A\}$ for some fixed $i_0 \in \mathbb{N}$, otherwise, it is called nontrivial. Let $\ell_\infty(X)$ denotes the subspace of the product space $\prod_{n\in\mathbb{N}} X$ equipped with the norm

$$\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty.$$

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathcal{U}} = \big\{(x_n) \in \ell_\infty(X) \ : \ \lim_{\mathcal{U}} \|x_n\| = 0\big\}.$$

The ultrapower of X, denoted by $\widetilde{X}$, is the quotient space $\frac{\ell_\infty(X)}{N_{\mathcal{U}}}$ equipped with the quotient norm. Write $(x_n)_{\mathcal{U}}$ to denote the elements of the ultrapower. It follows from the definition of the quotient norm that

$$\big\|(x_n)_{\mathcal{U}}\big\| = \lim_{\mathcal{U}} \|x_n\|.$$

Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically.

3. Main results

We are ready to state and prove our main results.

Theorem 3.1. *Let E be a nonempty weakly compact convex subset of a Banach space X and suppose that $\{x_n\}$ be a bounded sequence in E regular with respect to E. Then*

$$r_E\big(A\big(E, \{x_n\}\big)\big) \leq \frac{R(1,X)\sqrt{2C_p'(X)}}{R(1,X)+1}\, r\big(E, \{x_n\}\big).$$

Proof. For convenience, we denote $r = r\big(E, \{x_n\}\big)$ and $A = A\big(E, \{x_n\}\big)$. We can assume $r > 0$. Since $\{x_n\} \subset E$ is bounded and E is a weakly compact set, we can also assume, by passing through a subsequence if necessary, that $\{x_n\}$ is weakly convergent to a point $x \in E$ and $d := \lim_{n \neq m} \|x_n - x_m\|$ exists. We note that since $\{x_n\}$ is regular with respect to E, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. Since the norm is weak lower semicontinuity, it follows that

$$\liminf_n \|x_n - x\| \leq \liminf_n \liminf_m \|x_n - x_m\| = \lim_{n \neq m} \|x_n - x_m\| = d.$$

Let $\varepsilon > 0$. Taking a subsequence if necessary, we can assume that $\|x_n - x\| < d + \varepsilon$ for all n.

If $z \in A$, then $\limsup_n \|x_n - z\| = r$ and $\|x - z\| \leq \liminf_n \|x_n - z\| \leq r$. Denote $R = R(1,X)$. Since $(x_n - x) \xrightarrow{w} 0$ and by using the definition of R, we have

$$R \geq \liminf_n \left\| \frac{x_n - x}{d+\varepsilon} + \frac{z - x}{r} \right\| = \liminf_n \left\| \frac{x_n - x}{d+\varepsilon} - \frac{x - z}{r} \right\|.$$

On the other hand, observe that the convexity of E implies that $\frac{R-1}{R+1}x+\frac{2}{R+1}z\in E$ and by the weak lower semicontinuity of the norm, we have

$$\begin{aligned}
&\liminf_n\left\|\frac{x_n-z}{r}+\frac{1}{R}\left(\frac{x_n-x}{d+\varepsilon}-\frac{x-z}{r}\right)\right\|\\
&=\liminf_n\left\|\left(\frac{1}{r}+\frac{1}{R(d+\varepsilon)}\right)x_n-\left(\frac{1}{R(d+\varepsilon)}+\frac{1}{Rr}\right)x-\left(\frac{1}{r}-\frac{1}{Rr}\right)z\right\|\\
&\geq\left\|\left(\frac{1}{r}-\frac{1}{Rr}\right)x+\frac{2}{Rr}z-\left(\frac{1}{r}+\frac{1}{Rr}\right)z\right\|\\
&=\left(\frac{1}{r}+\frac{1}{Rr}\right)\left\|\frac{R-1}{R+1}x+\frac{2}{R+1}z-z\right\|\\
&\geq\left(1+\frac{1}{R}\right)\left(\frac{r_E(A)}{r}\right),\\
&\liminf_n\left\|\frac{x_n-z}{r}-\frac{1}{R}\left(\frac{x_n-x}{d+\varepsilon}-\frac{x-z}{r}\right)\right\|\\
&=\liminf_n\left\|\left(\frac{1}{r}-\frac{1}{R(d+\varepsilon)}\right)(x_n-x)-\left(\frac{1}{r}+\frac{1}{Rr}\right)(z-x)\right\|\\
&\geq\left(\frac{1}{r}+\frac{1}{Rr}\right)\|z-x\|\geq\left(1+\frac{1}{R}\right)\left(\frac{r_E(A)}{r}\right).
\end{aligned}$$

For every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that

(1) $\|x_N-z\|\leq r+\varepsilon.$

(2) $\left\|\dfrac{r(x_N-x)}{d+\varepsilon}-(x-z)\right\|\leq R(r+\varepsilon).$

(3) $\dfrac{1}{Rr}\left\|R(x_N-z)+\dfrac{r(x_N-x)}{d+\varepsilon}-(x-z)\right\|\geq\left(1+\dfrac{1}{R}\right)\left(\dfrac{r_E(A)}{r}\right)\left(\dfrac{r-\varepsilon}{r}\right).$

(4) $\dfrac{1}{Rr}\left\|R(x_N-z)-\left(\dfrac{r(x_N-x)}{d+\varepsilon}-(x-z)\right)\right\|\geq\left(1+\dfrac{1}{R}\right)\left(\dfrac{r_E(A)}{r}\right)\left(\dfrac{r-\varepsilon}{r}\right).$

In the ultrapower $\widetilde{X}$ of X, we consider

$$\widetilde{u}=\left(\frac{x_N-z}{r+\varepsilon}\right)_{\mathcal{U}}\in B_X,\qquad \widetilde{v}=\frac{1}{R(r+\varepsilon)}\left(\frac{r(x_N-x)}{d+\varepsilon}-(x-z)\right)_{\mathcal{U}}\in B_X.$$

By applying the above estimates, we obtain

$$\begin{aligned}
\|\widetilde{u}+\widetilde{v}\|&=\frac{1}{R(r+\varepsilon)}\left\|R(x_N-z)+\frac{r(x_N-x)}{d+\varepsilon}-(x-z)\right\|\\
&\geq\left(1+\frac{1}{R}\right)\left(\frac{r_E(A)}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right),\\
\|\widetilde{u}-\widetilde{v}\|&=\frac{1}{R(r+\varepsilon)}\left\|R(x_N-z)-\left(\frac{r(x_N-x)}{d+\varepsilon}-(x-z)\right)\right\|\\
&\geq\left(1+\frac{1}{R}\right)\left(\frac{r_E(A)}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right).
\end{aligned}$$

Therefore, by using the definition of $C'_p(\widetilde{X})$, we have

$$C'_p(\widetilde{X}) \geq \frac{\left(\frac{\|\widetilde{u}+\widetilde{v}\|^p+\|\widetilde{u}-\widetilde{v}\|^p}{2}\right)^{\frac{2}{p}}}{2}$$
$$\geq \frac{\left(\left(1+\frac{1}{R}\right)^p\left(\frac{r_E(A)}{r}\right)^p\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^p\right)^{\frac{2}{p}}}{2}$$
$$\geq \frac{\left(1+\frac{1}{R}\right)^2\left(\frac{r_E(A)}{r}\right)^2\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^2}{2}.$$

Since the above inequality is true for every $\varepsilon > 0$ and $C'_p(X) = C'_p(\widetilde{X})$, we obtain

$$r_E(A) \leq \frac{R\sqrt{2C'_p(X)}}{R+1}\, r.$$

□

Corollary 3.2. *Let E be a nonempty bounded closed convex separable subset of a Banach space X and suppose that $T : E \to KC(X)$ be a nonexpansive and $1-\chi$-contractive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition:*

$$Tx \subset I_E(X) \qquad \text{for all } x \in E.$$

Assume that

$$C'_p(X) < \frac{\left(1+\frac{1}{R(1,X)}\right)^2}{2}.$$

Then T has a fixed point.

Proof. Observe that $C'_p(X) < 2$ since $R(1,X) \geq 1$. This implies that X is uniformly nonsquare, and consequently, X is reflexive. So every bounded closed convex set is weakly compact. Now, since $C'_p(X) < \frac{(1+\frac{1}{R(1,X)})^2}{2}$, it follows that X satisfies the (DL)-condition by Theorem 3.1. Therefore, T has a fixed point by Theorem 2.6. □

By applying Theorems 2.7 and 3.1, we immediately obtain the following result.

Corollary 3.3. *Let E be a nonempty bounded closed convex subset of a Banach space X such that*

$$C'_p(X) < \frac{\left(1+\frac{1}{R(1,X)}\right)^2}{2}$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 3.4. *Let X be a Banach space such that*

$$C_p'(X) < \frac{\left(1+\frac{1}{R(1,X)}\right)^2}{2}.$$

Then X has normal structure.

Proof. Because X is reflexive, it then implies that normal structure and weak normal structure are the same. Now, since $C_p'(X) < \frac{(1+\frac{1}{R(1,X)})^2}{2}$, it follows that X satisfies the (DL)-condition by Theorem 3.1. Hence, X has normal structure by Theorem 2.5. □

Remark 3.5. Corollaries 3.2, 3.3, and 3.4 are sharp in the sense that there is a Banach space X such that $C_p'(X) = \frac{(1+\frac{1}{R(1,X)})^2}{2}$ and X does not satisfy the (DL)-condition. Consider the Bynum space $\ell_{2,\infty}$ defined as $\ell_{2,\infty} := (\ell_2, \|\cdot\|_{2,\infty})$ where $\|x\|_{2,\infty} := \max\{\|x^+\|_2, \|x^-\|_2\}$ with $x^+(i) = \max\{x(i), 0\}$ for each $i \geq 1$ and $x^- = x^+ - x$. We use the computation to conclude that the space $\ell_{2,\infty}$ is a limiting space for Corollary 3.2, Corollary 3.3, and Corollary 3.4, i.e., that corollaries are sharp. It is known that $J(\ell_{2,\infty}) = 1 + \frac{1}{\sqrt{2}}$ (see [15]) and $C_{NJ}(\ell_{2,\infty}) = \frac{3}{2}$ (see [15]). Repeating the arguments in [15], we can get that $C_{NJ}'(\ell_{2,\infty}) = \frac{3+2\sqrt{2}}{4}$. By the inequalities $\frac{J^2(X)}{2} \leq C_p'(X) \leq C_{NJ}'(X)$ (see [23]), we have $C_p'(\ell_{2,\infty}) = \frac{3+2\sqrt{2}}{4}$ $(-\infty \leq p \leq 2)$. It is easy to see that $R(1, \ell_{2,\infty}) = \sqrt{2}$ (see [9]). Thus, we have

$$C_p'(\ell_{2,\infty}) = \frac{3+2\sqrt{2}}{4} = \frac{(1+\frac{1}{R(1,\ell_{2,\infty})})^2}{2}$$

and $\ell_{2,\infty}$ fails to have weak normal structure. So $\ell_{2,\infty}$ does not satisfy the (DL)-condition.

Theorem 3.6. *Let E be a nonempty weakly compact convex subset of a Banach space X and suppose that $\{x_n\}$ be a bounded sequence in E regular with respect to E. Then*

$$r_E\big(A\big(E,\{x_n\}\big)\big) \leq \frac{R(1,X)\sqrt{2C_{-\infty}(X)}}{R(1,X)+1}\, r\big(E,\{x_n\}\big).$$

Proof. Let r, A, $\{x_n\}$, x, z and R be as in the proof of Theorem 3.1. Repeating the arguments in the proof of Theorem 3.1, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| \leq r + \varepsilon.$

(2) $\left\|\dfrac{r(x_N - x)}{d+\varepsilon} - (x - z)\right\| \leq R(r+\varepsilon).$

(3) $\left\|R(x_N - z) + \left(\dfrac{r(x_N - x)}{d+\varepsilon} - (x - z)\right)\right\| \geq (R+1)r_E(A)\left(\dfrac{r-\varepsilon}{r}\right).$

(4) $\left\|R(x_N - z) - \left(\dfrac{r(x_N - x)}{d+\varepsilon} - (x - z)\right)\right\| \geq (R+1)r_E(A)\left(\dfrac{r-\varepsilon}{r}\right).$

In the ultrapower $\widetilde{X}$ of X, we consider

$$\widetilde{u} = R(x_N - z)_{\mathcal{U}}, \qquad \widetilde{v} = \Big(\frac{r(x_N - x)}{d+\varepsilon} - (x - z)\Big)_{\mathcal{U}}.$$

By applying the above estimates, we obtain $\|\widetilde{u}\| \leq R(r+\varepsilon)$, $\|\widetilde{v}\| \leq R(r+\varepsilon)$, and

$$\begin{aligned}
\|\widetilde{u} + \widetilde{v}\| &= \left\| R(x_N - z) + \frac{r(x_N - x)}{d+\varepsilon} - (x - z) \right\| \\
&\geq (R+1) r_E(A) \left(\frac{r-\varepsilon}{r} \right), \\
\|\widetilde{u} - \widetilde{v}\| &= \left\| R(x_N - z) - \left(\frac{r(x_N - x)}{d+\varepsilon} - (x - z) \right) \right\| \\
&\geq (R+1) r_E(A) \left(\frac{r-\varepsilon}{r} \right).
\end{aligned}$$

Therefore, by using the definition of $C'_{-\infty}(\widetilde{X})$, we have

$$\begin{aligned}
C_{-\infty}(\widetilde{X}) &\geq \frac{\min\left\{\|\widetilde{u}+\widetilde{v}\|^2, \|\widetilde{u}-\widetilde{v}\|^2\right\}}{\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2} \\
&\geq \frac{(R+1)^2 \left(r_E(A)\right)^2 \left(\frac{r-\varepsilon}{r}\right)^2}{2R^2 (r+\varepsilon)^2}.
\end{aligned}$$

Since the above inequality is true for every $\varepsilon > 0$ and $C_{-\infty}(X) = C_{-\infty}(\widetilde{X})$, we obtain

$$r_E(A) \leq \frac{R\sqrt{2C_{-\infty}(X)}}{R+1}\, r.$$

□

Corollary 3.7. *Let E be a nonempty bounded closed convex separable subset of a Banach space X and suppose that $T : E \to KC(X)$ be a nonexpansive and $1-\chi$-contractive mapping such that $T(E)$ is a bounded set which satisfies the inwardness condition:*

$$Tx \subset I_E(X) \qquad \text{for all } x \in E.$$

Assume that

$$C_{-\infty}(X) < \frac{\left(1 + \frac{1}{R(1,X)}\right)^2}{2}.$$

Then T has a fixed point.

Proof. Observe that $C_{-\infty}(X) < 2$ since $R(1, X) \geq 1$. This implies that X is uniformly nonsquare, and consequently, X is reflexive. So every bounded closed convex set is weakly compact. Now, since $C_{-\infty}(X) < \frac{(1+\frac{1}{R(1,X)})^2}{2}$, it follows that X satisfies the (DL)-condition by Theorem 3.6. Therefore, T has a fixed point by Theorem 2.6. □

By virtue of Theorems 2.7 and 3.6, we immediately get the following result.

Corollary 3.8. *Let E be a nonempty bounded closed convex subset of a Banach space X such that*

$$C_{-\infty}(X) < \frac{\left(1 + \frac{1}{R(1,X)}\right)^2}{2}$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 3.9. *Let X be a Banach space such that*

$$C_{-\infty}(X) < \frac{\left(1 + \frac{1}{R(1,X)}\right)^2}{2}.$$

Then X has normal structure.

Proof. Because X is reflexive, it then implies that normal structure and weak normal structure are the same. Now, since $C_{-\infty}(X) < \frac{(1+\frac{1}{R(1,X)})^2}{2}$, it follows that X satisfies the (DL)-condition by Theorem 3.6. Hence, X has normal structure by Theorem 2.5. $\square$

Remark 3.10. Corollaries 3.3 and 3.8 not only improve [25, Corollary 3.2], but also improve Corollary 3.6 of [25]; a Banach space X with

$$C_Z(X) < \frac{\left(1 + \frac{1}{R(1,X)}\right)^2}{2}$$

satisfies the (DL)-condition.

Remark 3.11. It is well known that

$$\frac{1}{2}(J(X))^2 \leq C_p'(X) \leq C_p(X).$$

Thus, it is easy to see that Theorems 3.1 and 3.6 include [11, Theorem 3] and Corollaries 3.3 and 3.8 include [11, Corollary 2]. Meanwhile, Corollaries 3.4 and 3.9 include [20, Corollary 24].

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