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CONTACT CR-SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLDS WITH RESPECT TO QUARTER SYMMETRIC NON-METRIC CONNECTION

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Abstract. The present paper deals with the study of contact CR-submanifolds of trans-Sasakian manifolds with respect to quarter symmetric non-metric connection. We investigate totally geodesic leaves and integrability of the distributions and also study the totally umbilical contact CR-submanifolds of trans-Sasakian manifolds. At last we give an example to verify a relation.

1. Introduction and preliminaries

Trans-Sasakian manifold was introduced by J.A. Oubina in 1985 [15] and the local structure of trans-Sasakian manifolds of dimension $n \geq 5$ was completely characterized by J.C. Marrero [12]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu. Trans-Sasakian structures of type (0,0), (α,0) and (0,β) are cosymplectic, α-Sasakian and β-Kenmotsu respectively.

An n-dimensional (n is odd and $n > 1$) manifold $\tilde{M}$ with the almost contact metric structure (ϕ, ξ, η, g) (where ϕ, ξ, η are tensor fields on $\tilde{M}$ of type (1,1), (1,0), (0,1) respectively and g is a compatible metric with the almost structure) is called a *trans-Sasakian manifold of type* (α, β) if the following equation holds

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{1.1}$$

for smooth functions α and β on $\tilde{M}$ and $\forall X, Y \in \Gamma(T\tilde{M})$. Hence a trans-Sasakian manifold of type (0,0), (1,0) and (0,1) is called *cosymplectic manifold* [18], *Sasakian manifold* [13] and *Kenmotsu manifold* [2] respectively.

On a trans-Sasakian manifold $\tilde{M}$, authors in [12] and [15] have obtained

$$\tilde{\nabla}_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi], \tag{1.2}$$

also due to the almost contact metric structure we have $\forall X, Y \in \Gamma(T\tilde{M})$,

$$\eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0,\ \phi^2(X) = -X + \eta(X)\xi, \tag{1.3}$$

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$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ g(\phi X, Y) = -g(X, \phi Y). \tag{1.4}$$

Let M be an m-dimensional submanifold of an n-dimensional trans-Sasakian manifold $\tilde{M}$ $(m < n)$ with induced metric g and induced connections ∇ and $\nabla^{\perp}$ on TM and $T^{\perp}M$ respectively. Then for $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1.5}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{1.6}$$

respectively, where h and A_V are second fundamental form and shape operator respectively for the immersion of M satisfying the relation [21]

$$g(h(X, Y), V) = g(A_V X, Y). \tag{1.7}$$

M is *totally umbilical* if $\forall X, Y \in \Gamma(TM)$,

$$h(X, Y) = g(X, Y)H, \tag{1.8}$$

where H is the mean curvature vector on M and M becomes *minimal* if $H = 0$ and *totally geodesic* if $h = 0$.

CR-submanifold was introduced by A. Bejancu [3, 4, 5]. There are several research papers on geometry of CR-submanifolds [9, 10, 11, 13]. Some research papers on CR-submanifolds of trans-Sasakian manifolds are given by [8, 14, 19, 20].

A submanifold M of a trans-Sasakian manifold $\tilde{M}$ is called a *contact CR-submanifold* if ξ is tangent to M and there is a differential distribution D and its orthogonal complementary distribution $D^{\perp}$ such that [5]
(i) $\phi(D) \subseteq D$ and
(ii) $\phi(D^{\perp}) \subseteq T^{\perp}M$,
where D (respectively $D^{\perp}$) is called horizontal (respectively vertical) distribution.

M is called ξ-*horizontal* (respectively ξ-*vertical*) if $\xi \in D$ (respectively $\xi \in D^{\perp}$).

Now,

$$TM = D \oplus D^{\perp} \ \textit{and} \ T^{\perp}M = \phi(D^{\perp}) \oplus \mu, \tag{1.9}$$

where μ is a normal sub-bundle invariant to ϕ. For $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$ we write

$$X = PX + QX \tag{1.10}$$

and

$$\phi V = BV + CV, \tag{1.11}$$

where $PX \in D, \ QX \in D^{\perp}, \ BV = tan(\phi V)$ and $CV = nor(\phi V)$.

A contact CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ is called a *contact CR-product* [13] if M is locally a Riemannian product of M^T and $M^{\perp}$, where M^T and $M^{\perp}$ denote the leaves of D and $D^{\perp}$ respectively.

Quarter symmetric linear connection on a smooth manifold $\tilde{M}$ introduced by S. Golab [7], is a linear connection $\bar{\nabla}$ such that its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y.$$

If $\phi X = X$ in particular, then it reduces to *semi-symmetric connection* introduced by A. Friedmann and J.A. Shouten [6]. Further, if $(\bar{\nabla}_X g)(Y,Z) \neq 0\ \forall X,Y,Z \in \chi(\tilde{M})$, then $\bar{\nabla}$ is called a *quarter symmetric non-metric connection.*

M. Ahmad studied CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with quarter symmetric metric connection [1] and T. Pal et al. discussed CR-submanifolds of $(LCS)_n$-manifolds with respect to quarter symmetric non-metric connection [16] in detail.

Motivated from all the research papers mentioned above and specially from the work done by T. Pal et al. [16], in this paper we have studied contact CR-submanifolds of trans-Sasakian manifolds with respect to quarter symmetric non-metric connection. We have investigated totally geodesic leaves and integrability of the distributions and also have studied the totally umbilical contact CR-submanifolds of trans-Sasakian manifolds. At last we have given an example to verify a relation.

Let us consider a linear connection $\bar{\bar{\tilde{\nabla}}}$ on a trans-Sasakian manifold $\tilde{M}$ by [16]

$$\bar{\bar{\tilde{\nabla}}}_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi X + a(X)\phi Y, \tag{1.12}$$

where a is a 1-form associated to a vector field A on $\tilde{M}$ by

$$g(X,A) = a(X) \tag{1.13}$$

$\forall X,Y \in \chi(\tilde{M})$. If $\bar{\bar{\tilde{T}}}$ be the torsion tensor of $\tilde{M}$ with respect to $\bar{\bar{\tilde{\nabla}}}$, then from (1.12) we find

$$\bar{\bar{\tilde{T}}}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y + a(X)\phi Y - a(Y)\phi X. \tag{1.14}$$

Furthermore $\forall X,Y,Z \in \chi(\tilde{M})$,

$$(\bar{\bar{\tilde{\nabla}}}_X g)(Y,Z) = -\eta(Y)g(\phi X,Z) - \eta(Z)g(\phi X,Y) - 2a(X)g(\phi Y,Z). \tag{1.15}$$

Thus $\bar{\bar{\tilde{\nabla}}}$ given in (1.12) satisfying (1.14) and (1.15) is a quarter symmetric non-metric connection.

Now, for $\tilde{M}$ with respect to $\bar{\bar{\tilde{\nabla}}}$ we get,

$$(\bar{\bar{\tilde{\nabla}}}_X \phi)Y = \alpha g(X,Y)\xi + \beta g(\phi X,Y)\xi + (1-\alpha)\eta(Y)X - \beta\eta(Y)\phi X - \eta(X)\eta(Y)\xi \tag{1.16}$$

and

$$\bar{\bar{\tilde{\nabla}}}_X \xi = (1-\alpha)\phi X + \beta[X - \eta(X)\xi]. \tag{1.17}$$

We have used all the above equations in the next chapter of the paper.

2. Trans-Sasakian manifolds with respect to quarter symmetric non-metric connection

This chapter consists of three sections but before proving the results of those sections we now first compute the curvature tensor, Ricci tensor and scalar curvature of a trans-Sasakian manifold with respect to the quarter symmetric non-metric connection given in (1.12).

Let with respect to $\bar{\tilde{\nabla}}$ and $\tilde{\nabla}$, the curvature tensors of an n-dimensional trans-Sasakian manifold $\tilde{M}$ of type (α, β) be $\bar{\tilde{R}}$ and $\tilde{R}$ respectively, the Ricci tensors of $\tilde{M}$ be $\bar{\tilde{S}}$ and $\tilde{S}$ respectively and the scalar curvatures of $\tilde{M}$ be $\bar{\tilde{r}}$ and $\tilde{r}$ respectively. Then we obtain $\forall X, Y, Z \in \chi(\tilde{M})$,

$$\begin{aligned}\bar{\tilde{R}}(X,Y)Z &= \tilde{R}(X,Y)Z + da(X,Y)\phi Z + (1-\alpha)[a(Y)\eta(Z)X - a(X)\eta(Z)Y] \\ &+ \alpha[-\eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y - a(X)g(Y,Z)\xi + a(Y)g(X,Z)\xi + g(\phi Y,Z)\phi X \\ &- g(\phi X,Z)\phi Y] + \beta[2\eta(Z)g(\phi X,Y)\xi - a(X)g(\phi Y,Z)\xi + a(Y)g(\phi X,Z)\xi \\ &+ a(X)\eta(Z)\phi Y - a(Y)\eta(Z)\phi X - g(Y,Z)\phi X + g(X,Z)\phi Y] + a(X)\eta(Y)\eta(Z)\xi \\ &- a(Y)\eta(X)\eta(Z)\xi, \end{aligned} \tag{2.1}$$

after contraction we obtain,

$$\begin{aligned}\bar{\tilde{S}}(Y,Z) &= \tilde{S}(Y,Z) + da(Y,\phi Z) + [(n-1)(1-\alpha) + \alpha - \lambda\beta - 1]a(Y)\eta(Z) \\ &+ [-\alpha n + a(\xi)]\eta(Y)\eta(Z) + [\alpha\{1-a(\xi)\} - \lambda\beta]g(Y,Z) \\ &+ [-\alpha\lambda + \beta\{1-a(\xi)\}]g(\phi Y,Z) + \beta a(\phi Y)\eta(Z) \end{aligned} \tag{2.2}$$

and

$$\bar{\tilde{r}} = \tilde{r} + \mu + [(n-1)(1-2\alpha) - 2\lambda\beta]a(\xi) - \lambda\beta(n-1) - \alpha\lambda^2, \tag{2.3}$$

where $\lambda = trace(\phi)$ and $\mu = trace(da)$. Thus we have the following:

Theorem 2.1. *$\bar{\tilde{R}}$, $\bar{\tilde{S}}$ and $\bar{\tilde{r}}$ of an n-dimensional trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)) are given in (2.1), (2.2) and (2.3) respectively.*

2.1. Contact CR-submanifolds of a trans-Sasakian manifold with respect to quarter symmetric non-metric connection.

In this section we state and prove some results regarding a contact CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ with respect to the quarter symmetric non-metric connection given in (1.12).

Let ∇ be the induced connection on M from the connection $\tilde{\nabla}$ and $\bar{\nabla}$ be the induced connection on M from the connection $\bar{\tilde{\nabla}}$. Let h and $\bar{h}$ be second fundamental forms with respect to ∇ and $\bar{\nabla}$ respectively. Then we have for $X, Y \in \Gamma(TM)$,

$$\bar{\tilde{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{h}(X,Y). \tag{2.4}$$

From (1.5), (1.12) and (2.4) we get

$$\bar{\nabla}_X Y + \bar{h}(X,Y) = \nabla_X Y + h(X,Y) + \eta(Y)\phi X + a(X)\phi Y. \tag{2.5}$$

Using (1.10) in (2.5) we get

$$P\bar{\nabla}_X Y+Q\bar{\nabla}_X Y+\bar{h}(X,Y) = P\nabla_X Y+Q\nabla_X Y+h(X,Y)+\eta(Y)\phi PX+\eta(Y)\phi QX + a(X)\phi PY + a(X)\phi QY. \tag{2.6}$$

Comparing horizontal, vertical and normal parts from both sides of (2.6) we get

$$P\bar{\nabla}_X Y = P\nabla_X Y + \eta(Y)\phi PX + a(X)\phi PY, \tag{2.7}$$

$$Q\bar{\nabla}_X Y = Q\nabla_X Y, \tag{2.8}$$

$$\bar{h}(X,Y) = h(X,Y) + \eta(Y)\phi QX + a(X)\phi QY. \tag{2.9}$$

Now, for $X, Y \in D$ from (2.5) we get

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X + a(X)\phi Y \tag{2.10}$$

and

$$\bar{h}(X,Y) = h(X,Y). \tag{2.11}$$

Hence we have the following:

Theorem 2.2. *If M is a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ admitting $\bar{\tilde{\nabla}}$ (given in (1.12)), then*
(i) the induced connection $\bar{\nabla}$ on M is also a quarter symmetric non-metric connection,
(ii) the second fundamental forms h and $\bar{h}$ are equal.

Again for $X \in TM$ and $V \in T^{\perp}M$ from Weingarten formula for quarter symmetric non-metric connection $\bar{\tilde{\nabla}}$ we have

$$\bar{\tilde{\nabla}}_X V = -\bar{A}_V X + \bar{\nabla}^{\perp}_X V. \tag{2.12}$$

Also from (1.6) and (1.12) we get

$$\bar{\tilde{\nabla}}_X V = -A_V X + \nabla^{\perp}_X V + a(X)\phi V. \tag{2.13}$$

From (2.12) and (2.13) we obtain

$$-\bar{A}_V X + \bar{\nabla}^{\perp}_X V = -A_V X + \nabla^{\perp}_X V + a(X)\phi V. \tag{2.14}$$

Now, for $Z \in D^{\perp}$ we have $\phi Z \in T^{\perp}M$ and hence from (2.14) we obtain

$$-\bar{A}_{\phi Z} X + \bar{\nabla}^{\perp}_X \phi Z = -A_{\phi Z} X + \nabla^{\perp}_X \phi Z + a(X)[-Z + \eta(Z)\xi]$$

from which we get

$$\bar{A}_{\phi Z} X = A_{\phi Z} X - a(X)[-Z + \eta(Z)\xi] \tag{2.15}$$

and

$$\bar{\nabla}^{\perp}_X \phi Z = \nabla^{\perp}_X \phi Z. \tag{2.16}$$

Theorem 2.3. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$ (given in (1.12)), then* $\forall X, Y \in TM$,

$$P\bar{\nabla}_X \phi PY - P\bar{A}_{\phi QY}X = \phi P(\bar{\nabla}_X Y) + \alpha g(X,Y)P\xi + \beta g(\phi X, Y)P\xi + (1-\alpha)\eta(Y)PX - \beta\eta(Y)\phi PX - \eta(X)\eta(Y)P\xi, \tag{2.17}$$

$$Q\bar{\nabla}_X \phi PY - Q\bar{A}_{\phi QY}X = B\bar{h}(X,Y) + \alpha g(X,Y)Q\xi + (1-\alpha)\eta(Y)QX + \beta g(\phi X, Y)Q\xi - \eta(X)\eta(Y)Q\xi, \tag{2.18}$$

$$\bar{h}(X, \phi PY) + \bar{\nabla}^{\perp}_X \phi QY = \phi Q(\bar{\nabla}_X Y) + C\bar{h}(X,Y) - \beta\eta(Y)\phi QX. \tag{2.19}$$

Proof. From (1.16) we have $\forall X, Y \in TM$,

$$\bar{\bar{\nabla}}_X \phi Y - \phi(\bar{\bar{\nabla}}_X Y) = \alpha g(X,Y)\xi + \beta g(\phi X, Y)\xi + (1-\alpha)\eta(Y)X - \beta\eta(Y)\phi X - \eta(X)\eta(Y)\xi. \tag{2.20}$$

Applying (1.10), (1.11), (2.4) and (2.12) on (2.20) we get

$$P\bar{\nabla}_X \phi PY + Q\bar{\nabla}_X \phi PY + \bar{h}(X, \phi PY) - P\bar{A}_{\phi QY}X - Q\bar{A}_{\phi QY}X + \bar{\nabla}^{\perp}_X \phi QY - \phi P(\bar{\nabla}_X Y) - \phi Q(\bar{\nabla}_X Y) - B\bar{h}(X,Y) - C\bar{h}(X,Y) = \alpha g(X,Y)P\xi + \alpha g(X,Y)Q\xi + \beta g(\phi X, Y)P\xi + \beta g(\phi X, Y)Q\xi + (1-\alpha)\eta(Y)PX + (1-\alpha)\eta(Y)QX - \beta\eta(Y)\phi PX - \beta\eta(Y)\phi QX - \eta(X)\eta(Y)P\xi - \eta(X)\eta(Y)Q\xi. \tag{2.21}$$

Equating horizontal, vertical and normal parts from both sides of (2.21) we get (2.17), (2.18) and (2.19) respectively.

□

2.2. **Totally geodesic leaves and integrability of the distributions.** In this section we obtain the necessary and sufficient conditions of integrability of the distributions D and $D^{\perp}$ of a contact CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ in different cases. We also discuss the cases where the leaves of D and $D^{\perp}$ are totally geodesic.

Lemma 2.4. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\bar{\bar{\nabla}}$ (given in (1.12)). Then* $\forall Z, W \in D^{\perp}$,

$$\phi P[W, Z] = A_{\phi W}Z - A_{\phi Z}W - a(Z)[-W + \eta(W)\xi] + a(W)[-Z + \eta(Z)\xi] + (\alpha - 1)[\eta(Z)W - \eta(W)Z].$$

Proof. $\forall Z, W \in D^{\perp}$, $\bar{\bar{\nabla}}_Z \phi W = (\bar{\bar{\nabla}}_Z \phi)W + \phi(\bar{\bar{\nabla}}_Z W)$.

Using (1.10), (1.11), (1.16), (2.4) and (2.12) in the above equation we get

$$\bar{\bar{\nabla}}^{\perp}_Z \phi W = \bar{A}_{\phi W}Z + \phi P(\bar{\nabla}_Z W) + \phi Q(\bar{\nabla}_Z W) + B\bar{h}(W,Z) + C\bar{h}(W,Z) + \alpha g(W,Z)\xi + (1-\alpha)\eta(W)Z - \beta\eta(W)\phi Z - \eta(Z)\eta(W)\xi. \tag{2.22}$$

Also from (2.19) we get

$$\bar{\bar{\nabla}}^{\perp}_Z \phi W = \phi Q(\bar{\nabla}_Z W) + C\bar{h}(W,Z) - \beta\eta(W)\phi Z. \tag{2.23}$$

Using (2.23) in (2.22) we obtain
$$\phi P(\bar{\nabla}_Z W) = -\bar{A}_{\phi W} Z - B\bar{h}(W,Z) - \alpha g(W,Z)\xi + (\alpha-1)\eta(W)Z + \eta(Z)\eta(W)\xi. \quad (2.24)$$

Interchanging Z and W in (2.24) and subtracting (2.24) from the resultant equation we get
$$\phi P[W,Z] = \bar{A}_{\phi W} Z - \bar{A}_{\phi Z} W + (\alpha-1)[\eta(Z)W - \eta(W)Z].$$

Using (2.15) in the above equation we obtain
$$\phi P[W,Z] = A_{\phi W} Z - A_{\phi Z} W - a(Z)[-W + \eta(W)\xi] + a(W)[-Z + \eta(Z)\xi]$$
$$+ (\alpha-1)[\eta(Z)W - \eta(W)Z].$$

□

Theorem 2.5. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\bar{\nabla}}$ (given in (1.12)). Then the distribution $D^{\perp}$ is integrable if and only if $\forall Z, W \in D^{\perp}$,*
$A_{\phi W} Z - A_{\phi Z} W = a(Z)[-W + \eta(W)\xi] - a(W)[-Z + \eta(Z)\xi] + (\alpha-1)[\eta(W)Z - \eta(Z)W]$.

Proof. It is obvious from Lemma 2.2.1.

□

Corollary 2.6. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\bar{\nabla}}$ (given in (1.12)). Then the distribution $D^{\perp}$ is integrable if and only if*
$$\forall Z, W \in D^{\perp},\ A_{\phi W} Z - A_{\phi Z} W = a(W)Z - a(Z)W.$$

Remark 2.7. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\tilde{\nabla}$. Then the distribution $D^{\perp}$ is integrable if and only if $\forall Z, W \in D^{\perp}$,*
$$A_{\phi W} Z - A_{\phi Z} W = \alpha[\eta(W)Z - \eta(Z)W].$$

Remark 2.8. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\tilde{\nabla}$. Then the distribution $D^{\perp}$ is integrable if and only if $\forall Z, W \in D^{\perp}$,*
$$A_{\phi W} Z = A_{\phi Z} W.$$

Theorem 2.9. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\bar{\nabla}}$ (given in (1.12)). Then the distribution D is integrable if and only if $\forall X, Y \in D$,*
$$h(X,\phi Y) = h(\phi X, Y).$$

Proof. From (2.11) and (2.19) we have $\forall X, Y \in D$,
$$\phi Q(\bar{\nabla}_X Y) = h(X,\phi Y) - Ch(X,Y). \quad (2.25)$$

Interchanging X and Y in (2.25) and subtracting the resultant equation from (2.25) we get
$$\phi Q[X,Y] = h(X,\phi Y) - h(Y,\phi X).$$

Hence the distribution D is integrable if and only if $\forall X, Y \in D$, $h(X, \phi Y) = h(\phi X, Y)$.

$\square$

Remark 2.10. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\tilde{\nabla}$. Then the distribution D is integrable if and only if $\forall X, Y \in D$, $h(X, \phi Y) = h(\phi X, Y)$.*

Theorem 2.11. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)). If the leaf of D is totally geodesic in M, then $\forall X, Y \in D, Z \in D^{\perp}$,*
$-g(h(X,Y), \phi Z) + (\alpha - 1)\eta(Z)g(X,Y) + \beta\eta(Z)g(\phi X, Y) + \eta(X)\eta(Y)\eta(Z) = 0.$

Proof. As the leaf of D is totally geodesic in M, $\bar{\nabla}_X \phi Y \in D$ $\forall X, Y \in D$(since $\phi Y \in D$). Now, $\forall Z \in D^{\perp}$ from (2.21) we have

$$\phi P(\bar{\nabla}_X Z) = -\bar{A}_{\phi Z}X + \nabla^{\perp}_X \phi Z - \phi Q(\bar{\nabla}_X Z) - \phi\bar{h}(X,Z) + (\alpha-1)\eta(Z)X + \beta\eta(Z)\phi X + \eta(X)\eta(Z)\xi. \tag{2.26}$$

Using (1.4), (1.7), (1.10), (2.11) and (2.26) we get

$$0 = g(\bar{\nabla}_X \phi Y, Z) = -g(\phi Y, \bar{\nabla}_X Z) = -g(\phi Y, P(\bar{\nabla}_X Z)) = g(Y, \phi P(\bar{\nabla}_X Z))$$
$$= -g(\bar{A}_{\phi Z}X, Y) + (\alpha - 1)\eta(Z)g(X,Y) + \beta\eta(Z)g(\phi X, Y) + \eta(X)\eta(Z)\eta(Y)$$
$$= -g(h(X,Y), \phi Z) + (\alpha - 1)\eta(Z)g(X,Y) + \beta\eta(Z)g(\phi X, Y) + \eta(X)\eta(Y)\eta(Z).$$

$\square$

Corollary 2.12. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)), then the leaf of D is totally geodesic in M if and only if $\forall X, Y \in D, Z \in D^{\perp}$,*

$$g(h(X,Y), \phi Z) = 0.$$

Proof. The direct part follows from Theorem 2.2.3.

Conversely, $\forall X, Y \in D, Z \in D^{\perp}$ (since $\phi Y \in D$),

$$0 = g(h(X, \phi Y), \phi Z) = g(\bar{\tilde{\nabla}}_X \phi Y, \phi Z) = g(\phi \bar{\tilde{\nabla}}_X Y, \phi Z) = g(\bar{\tilde{\nabla}}_X Y, Z) = g(\bar{\nabla}_X Y, Z)$$

which implies that $\forall X, Y \in D$, $\bar{\nabla}_X Y \in D$. Hence the leaf of D is totally geodesic in M.

$\square$

Theorem 2.13. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)). If the leaf of $D^{\perp}$ is totally geodesic in M, then $\forall Z, W \in D^{\perp}, X \in D$,*
$g(h(X,Z), \phi W) + a(X)g(Z,W) + \alpha\eta(X)g(Z,W) - a(X)\eta(Z)\eta(W) - \eta(X)\eta(Z)\eta(W) = 0.$

Proof. As the leaf of $D^{\perp}$ is totally geodesic in M, $\forall Z, W \in D^{\perp}, \bar{\nabla}_Z W \in D^{\perp}$. Now from (2.21) we have

$$\phi P(\bar{\nabla}_Z W) = -\bar{A}_{\phi W}Z + \bar{\nabla}^{\perp}_Z \phi W - \phi Q(\bar{\nabla}_Z W) - \phi\bar{h}(Z,W) + (\alpha - 1)\eta(W)Z + \beta\eta(W)\phi Z + \eta(Z)\eta(W)\xi - \alpha g(Z,W)\xi. \tag{2.27}$$

Taking inner product of (2.27) with $X \in D$ we get (since $\bar{\nabla}_Z W \in D^\perp$)
$0 = g(\phi P(\bar{\nabla}_Z W), X) = -g(\bar{A}_{\phi W} Z, X) - \alpha g(Z, W)\eta(X) + \eta(Z)\eta(W)\eta(X).$(2.28)

Using (1.4), (1.7) and $\bar{h}(X, Z) = h(X, Z) + a(X)\phi Z$ (obtained from (2.9)) in (2.28) we get
$0 = -[g(h(X, Z), \phi W) + a(X)g(Z, W) - a(X)\eta(Z)\eta(W)] - \alpha\eta(X)g(Z, W)$
$+ \eta(X)\eta(Z)\eta(W).$

□

Corollary 2.14. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)). If the leaf of $D^\perp$ is totally geodesic in M, then $\forall Z, W \in D^\perp, X \in D$,*

$$g(h(X, Z), \phi W) + a(X)g(Z, W) + \alpha\eta(X)g(Z, W) = 0.$$

Corollary 2.15. *Let M be a ξ-vertical contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)). If the leaf of $D^\perp$ is totally geodesic in M, then $\forall Z, W \in D^\perp, X \in D$,*

$$g(h(X, Z), \phi W) + a(X)g(Z, W) - a(X)\eta(Z)\eta(W) = 0.$$

Theorem 2.16. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)). If M is a contact CR-product, then $\forall X \in D, W \in D^\perp$,*

$$A_{\phi W} X + a(X)W + \alpha\eta(X)W = 0.$$

Proof. As the leaf of $D^\perp$ is totally geodesic in M, from Corollary 2.2.3 we have $\forall Z, W \in D^\perp, X \in D$,

$$g(A_{\phi W} X + a(X)W + \alpha\eta(X)W, Z) = 0$$

which implies that

$$A_{\phi W} X + a(X)W + \alpha\eta(X)W \in D. \tag{2.29}$$

Again as the leaf of D is totally geodesic in M and $\forall Y \in D$, $\phi Y \in D$, we have $\bar{\nabla}_X \phi Y \in D$. Hence we get
$g(A_{\phi W} X + a(X)W + \alpha\eta(X)W, Y) = g(A_{\phi W} X, Y) = g(h(X, Y), \phi W)$
$= -g(\phi h(X, Y), W) = -g(\phi(\bar{\tilde{\nabla}}_X Y - \bar{\nabla}_X Y), W) = -g(\phi(\bar{\tilde{\nabla}}_X Y), W)$
$= -g(\bar{\tilde{\nabla}}_X \phi Y, W) = -g(\bar{\nabla}_X \phi Y, W) = 0$ (using (1.4), (1.7), (1.16), (2.4), (2.11))
which implies that $A_{\phi W} X + a(X)W + \alpha\eta(X)W \in D^\perp.$ (2.30)

From (2.29) and (2.30) we obtain $A_{\phi W} X + a(X)W + \alpha\eta(X)W = 0.$

□

2.3. **Totally umbilical contact CR-submanifolds.** In this section we study totally umbilical contact CR-submanifolds of a trans-Sasakian manifold with respect to $\tilde{\nabla}$.

Let M be a totally umbilical contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\tilde{\nabla}$, then from (1.1) we have $\forall Z, W \in D^\perp$,

$$\tilde{\nabla}_Z \phi W - \phi(\tilde{\nabla}_Z W) = \alpha[g(Z, W)\xi - \eta(W)Z] - \beta\eta(W)\phi Z. \tag{2.31}$$

Using (1.5), (1.6) and (1.10) in (2.31) we get
$-A_{\phi W}Z+\nabla^{\perp}_{Z}\phi W=\phi P(\nabla_Z W)+\phi Q(\nabla_Z W)+\phi h(Z,W)+\alpha[g(Z,W)\xi-\eta(W)Z]$
$-\beta\eta(W)\phi Z.$ (2.32)

Taking inner product of (2.32) with Z and using (1.7) we get

$$-g(h(Z,Z),\phi W)=g(\phi h(Z,W),Z)+\alpha[g(Z,W)\eta(Z)-\eta(W)\|Z\|^2]. \quad (2.33)$$

Using (1.8) in (2.33) we get

$$g(H,\phi W)=-\frac{1}{\|Z\|^2}[g(Z,W)g(\phi H,Z)+\alpha\{g(Z,W)\eta(Z)-\eta(W)\|Z\|^2\}]. \quad (2.34)$$

Interchanging Z and W in (2.34) and using (1.4) we get

$$g(\phi H,Z)=\frac{1}{\|W\|^2}[-g(Z,W)g(H,\phi W)+\alpha\{g(Z,W)\eta(W)-\eta(Z)\|W\|^2\}]. \quad (2.35)$$

Substituting (2.35) in (2.34) we obtain
$g(H,\phi W)=-\frac{g(Z,W)}{\|Z\|^2}\Big[-\frac{1}{\|W\|^2}g(Z,W)g(H,\phi W)+\frac{\alpha}{\|W\|^2}\{g(Z,W)\eta(W)-\eta(Z)\|W\|^2\}\Big]$
$-\frac{\alpha}{\|Z\|^2}\{g(Z,W)\eta(Z)-\eta(W)\|Z\|^2\}$
$\Rightarrow\Big[1-\frac{g(Z,W)^2}{\|Z\|^2\|W\|^2}\Big]g(H,\phi W)+\alpha\frac{g(Z,W)}{\|Z\|^2}\Big[\frac{\eta(W)g(Z,W)}{\|W\|^2}-\eta(Z)\Big]+\alpha\Big[\frac{\eta(Z)g(Z,W)}{\|Z\|^2}-\eta(W)\Big]$
$=0.$ (2.36)

Hence from (2.36) we get the following theorems:

Theorem 2.17. *Let M be a ξ-vertical totally umbilical contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect $\tilde{\nabla}$, then dim$D^{\perp}$=1.*

Theorem 2.18. *Let M be a ξ-horizontal totally umbilical contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\tilde{\nabla}$, then*
(i) M is minimal in $\tilde{M}$
or (ii) dim$D^{\perp}$=1
or (iii) $H\in\Gamma(\mu)$.

Remark 2.19. *Theorem 2.3.2 also holds good in case of considering $\tilde{M}$ with respect to the quarter symmetric non-metric connection $\bar{\tilde{\nabla}}$ (given in (1.12)).*

3. Example

Here we give an example of a 3-dimensional trans-Sasakian manifold from [17] and then verify the relation (2.1).

Let us consider a 3-dimensional manifold $\tilde{M}=\{(x,y,z)\in\mathbb{R}^3 : z\neq 0\}$, where (x,y,z) are the standard co-ordinates of $\mathbb{R}^3$. Let the vector fields

$$E_1=e^{-2z}\Big(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big),\ E_2=-e^{-2z}\Big(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\Big),\ E_3=\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 0, & for\ i \neq j \\ 1, & for\ i = j \end{cases}$$

and η be the 1-form defined by $\eta(X) = g(X, E_3)\ \forall X \in \chi(\tilde{M})$. Let ϕ be the (1,1) tensor field defined by $\phi E_1 = E_2,\ \phi E_2 = -E_1,\ \phi E_3 = 0$. Then we have $\eta(E_3) = 1,\ \phi^2(X) = -X + \eta(X)E_3,\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\ \forall X, Y \in \chi(\tilde{M})$. Let $\tilde{\nabla}$ be the Riemannian connection on $\tilde{M}$ with respect to the metric g. Then we obtain $[E_1, E_2] = 0,\ [E_1, E_3] = 2E_1,\ [E_2, E_3] = 2E_2$. Now, using Koszul's formula for g, it can be calculated that

$$\begin{gathered} \tilde{\nabla}_{E_1}E_1 = -2E_3,\ \tilde{\nabla}_{E_1}E_2 = 0,\ \tilde{\nabla}_{E_1}E_3 = 2E_1, \\ \tilde{\nabla}_{E_2}E_1 = 0,\ \tilde{\nabla}_{E_2}E_2 = -2E_3,\ \tilde{\nabla}_{E_2}E_3 = 2E_2, \\ \tilde{\nabla}_{E_3}E_1 = 0,\ \tilde{\nabla}_{E_3}E_2 = 0,\ \tilde{\nabla}_{E_3}E_3 = 0. \end{gathered}$$

Since $\{E_1, E_2, E_3\}$ forms a basis for $\tilde{M}$, then any vector field $X, Y \in \chi(\tilde{M})$ can be written as

$$X = x_1E_1 + x_2E_2 + x_3E_3,\ Y = y_1E_1 + y_2E_2 + y_3E_3,$$

where $x_i, y_i \in \mathbb{R},\ i = 1, 2, 3$. Hence $g(X, Y) = x_1y_1 + x_2y_2 + x_3y_3$.

Thus $\tilde{\nabla}_X Y = 2x_1y_3E_1 + 2x_2y_3E_2 - 2(x_1y_1 + x_2y_2)E_3$. (3.1)

Therefore $\tilde{\nabla}_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi]\ \forall X \in \chi(\tilde{M})$ holds for $\alpha = 0,\ \beta = 2$ and $\xi = E_3$. Thus $(\tilde{M}, g)$ is a 3-dimensional trans-Sasakian manifold of type (0,2).

We set $A = E_1$. Then $a(X) = g(X, E_1) = x_1\ \forall X = x_1E_1 + x_2E_2 + x_3E_3 \in \chi(\tilde{M})$. Hence using (3.1) in (1.12) we get

$$\bar{\tilde{\nabla}}_X Y = (2x_1y_3 - x_2y_3 - x_1y_2)E_1 + (2x_2y_3 + x_1y_3 + x_1y_1)E_2 - 2(x_1y_1 + x_2y_2)E_3. \tag{3.2}$$

Also for $Z = z_1E_1 + z_2E_2 + z_3E_3 \in \chi(\tilde{M})$ we have

$$(\bar{\tilde{\nabla}}_X g)(Y, Z) = x_2y_3z_1 - x_1y_3z_2 + (x_2y_1 - x_1y_2)z_3 \neq 0.$$

Hence $\bar{\tilde{\nabla}}$(given by (3.2)) is a quarter symmetric non-metric connection on $\tilde{M}$.

Now, we will verify the relation (2.1) for $X = E_1, Y = E_2, Z = E_2$.

Using the values of $\tilde{\nabla}_{E_i}E_j$ $(i, j = 1, 2, 3)$ given above, we obtain

$$\tilde{R}(E_1, E_2)E_2 = -4E_1, \tag{3.3}$$

and using (3.2) we obtain $\bar{\tilde{R}}(E_1, E_2)E_2 = -4E_1 - 2E_2$. (3.4)

Now, from (2.1) we get $\bar{\tilde{R}}(E_1, E_2)E_2 = \tilde{R}(E_1, E_2)E_2 - 2E_2$ which is satisfied by (3.3) and (3.4). Hence the relation (2.1) holds for $X = E_1, Y = E_2, Z = E_2$. Similarly we can prove that the relation (2.1) holds for other values of $X, Y, Z \in \chi(\tilde{M})$.

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