Gulf Journal of Mathematics
Vol 4, Issue 2 (2016) 13-17

REMARKS ON RELATIVE PROJECTIVE OF MACKEY FUNCTORS

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Abstract. The aim of this paper is to study and discuss the concept of relative projectivity in the case of Mackey functors. Then we use the definitions of a vertex and of a source of an indecomposable Mackey functor to recall some results of Green's theory. Higman Criterion is the main tool.

1. Introduction and preliminaries

The notion of Mackey functor is due to several authors. There are many approaches to tackle this topic. We shall use the definition of a Mackey functor as in [3]. Our aims are to parallel some well known results in module theory and G-algebras to the mackey functors and Green functors. Some of these results may appear in the literature, but our contribution is rather different. Our notation are the same as in [11, Chapter 8, Section 53]. The coefficients are from a ring $\mathcal{O}$ which can be obtained from some construction of a suitable p-modular system for prime number p.

The idea of Mackey functors is to get some sort of operations which are similar to conjugation, restriction and induction of group characters. One can see such methods in several places such as K-theory, G-algebras, algebraic number theory, Burnside algebras and representation rings.

As a survey of this concept, the reader can consult also beginning of this approach by A. Dress [2]. However, the most useful and very clear for our task is the paper [3] as we have already mentioned above. The structure of Mackey functors can be seen in [10]. In fact, J. Thévenaz made a remarkable contribution in this direction and the reader can see that in [8], [9] and the book [11]. For extension groups between simple Mackey functors as well as the socle of a projective Mackey functor for a p-group, the reader can see that in the excellent work by M. Nicollerat [6] and [7]. Induced representation and Mackey theory with good exposition and with functional flavor can be seen in [4]. Similar treatments for classical results such as Clifford theory, inflations and imprimitivity of Mackey functors can be seen in [12], [13] and [14].

Date: Received: 2 Aug, 2015; Accepted: 2 Feb, 2016.

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2010 *Mathematics Subject Classification.* Primary 20C20; 20J05 Secondary 19A22.

Key words and phrases. Algebra, Representation of Finite groups, Relative Projective, Green Theory, Higman Criterion.

Now let G be a finite group and $H \leq G$. Then we write $M(H)$ for a module over H which is the image under a Mackey functor M of G over $\mathcal{O}$. For $K \leq H \leq G$ we use for the trace and restriction maps the standard notations $t_K^H : A(K) \longrightarrow A(H)$ and $r_K^H : A(H) \longrightarrow A(K)$ respectively. We write $Ind_H^G(M)$ to denote the induced Mackey functor of M from subgroup H to G and $Res_H^G(N)$ to the restriction of Mackey functor N from G to its subgroup H respectively. For all basic facts and elementary properties of Mackey functors, the reader can see that in [10, 11]. The paper contains the following section which is devoted to the definitions as well as the main results.

2. Main results

The first definition we shall use is the relative projectivety in the case of Mackey functors.

Definition 2.1. Let G be a finite group and H be a subgroup of G. An indecomposable Mackey functor M for G is said to be projective relative to the subgroup H (H-projective) if there is a Mackey functor N for H such that M is a direct summand of the induced Mackey functor; $Ind_H^G(N)$. We can write that terminology as $M|Ind_H^G(N) = \sum_{g\in[G/H]} n_g \otimes g, \quad n_g \in N(H)$.

Since Mackey functor is a generalization of Green theory of G-modules, the concept of source module can be seen in this setting as follows.

Definition 2.2. Let $H \leq G$, we say that a Mackey functor N for H is a source of a Mackey functor M for G if $M \mid Ind_H^G(N)$.

Definition 2.3. Let G be a finite group and H be a subgroup of G. Let M be an indecomposable Mackey functor for G over $\mathcal{O}$. We define $S = \{H \leq G : M \ is \ H - projective\}$.

Then the definition gives us that the minimal subgroup H of the set S is called a vertex of M, and we write $vertex(M) = H$.

The following lemma is the transitivity of the relative projectively in Mackey functor.

Lemma 2.4. Let M be a Mackey functor for G over $\mathcal{O}$. If M is an H-projective then M is K-projective for all subgroups K such that $H \leq K \leq G$.

Proof: Since M is an H-projective then there is a Mackey functor N for H such that $M|Ind_H^G(N) = Ind_K^G(Ind_H^K(N))$. □

The following lemma says that a Mackey functor respects the conjugation action of subgroups.

Lemma 2.5. Let M be a Mackey functor for G over $\mathcal{O}$. If M is an H-projective then M is H^g-projective for all $g \in G$.

Proof: Since M is an H-projective then there is a Mackey functor N for H such that $M|Ind_H^G(N)$. Then $M^g|(Ind_H^G(N))^g$. But M is a Mackey functor for G. So, $M|Ind_{H^g}^{G^g}(N^g) = Ind_{H^g}^G(N')$; where $N' = N^g$ is a Mackey functor for H^g. Therefore, M is H^g-projective for all $g \in G$. □

The following lemma deals with the concept of internal tensor product of Mackey functors.

Lemma 2.6. Let M be a Mackey functor for G over $\mathcal{O}$. If M is an H-projective then $N \otimes M$ is H-projective for all Mackey functors N for G.

Proof: Since M is an H-projective then there is a Mackey functor W for H such that $M|Ind_H^G(W)$. Now

$$N \otimes M|N \otimes Ind_H^G(W) \quad \cong \quad Ind_H^G((r_H^G(N)) \otimes W).$$

Therefore, $N \otimes M$ is H-projective for all Mackey functors N for G. □

Here we recall without proof a Higman-type result. It is very important in the sense of getting a characterization of relative projective of Mackey functor. The proof is similar to the case of G-modules. See [5, Theorem 2.2].

Lemma 2.7. Let $H \leq G$ and M be a Mackey functor for G over $\mathcal{O}$. Then the following are equivalent:

(1) M is H-projective.
(2) (Higman's Criterion) The identity map on M is in the image of t_H^G. This means that there exists $\phi \in End_H(M)$ such that $t_H^G(\phi) = id_M$. Here, $End_H(M) = [End(M)]^H$ is the fixed points under H-action. $t_H^G : End_H(M) \to End_G(M)$; $id_M \in Im(t_H^G(End(M))) = [End(M)]_H^G$.
(3) $End_G(M) = [End(M)]_H^G$. This means that the trace map is surjective.
(4) For the following diagram of the Mackey functors N, W for G and morphisms of Mackey functor f, μ over G.

$$\begin{array}{ccccc} & & M & & \\ & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle f} & & \\ N & \xrightarrow[\mu]{} & W & \longrightarrow & 0 \end{array}$$

There exists a morphism of Mackey functor over G; $h : M \longrightarrow N$ such that $\mu \circ h = f$, provided there exists a morphism of Mackey functor over H; $g : M \longrightarrow N$ such that $f = \mu \circ g$.
(5) Every surjective morphism over G; $\mu : N \longrightarrow M$ of Mackey functor for G splits if it splits as a morphism over H.
(6) The surjective morphism over G; $\lambda : Ind_H^G(Res_H^G(M)) \longrightarrow M$ $(\lambda(\sum_{i=1}^{[G:H]} m_i \otimes x_i) = \sum_{i=1}^{n} m_i x_i)$ splits.
(7) M is a direct summand of $Ind_H^G(Res_H^G(M))$.

Our construction of a Makey functor of a finite group G is over p-modular system. Here, the notation $p \nmid [G : H]$ means that the prime number p does not divide the index $[G : H]$ for a subgroup H of G.

Theorem 2.8. Let $H \leq G$, M be a Mackey functor for G over $\mathcal{O}$. If $p \nmid [G : H]$ then every Mackey functor is H-projective, where p is prime number.

Proof: Since $p \nmid [G : H]$ then $[G : H]$ is a unit in $\mathcal{O}$. Then we have

$$id_M = t_H^G([G : H]^{-1} id_M) \in Im(t_H^G(End(M))) = [End(M)]_H^G.$$

Then by Lemma 2.7 (2), every Mackey functor such that $p \nmid [G : H]$ is H-projective. □

Corollary 2.9. All Mackey functors for G are P-projective for all $P \in Syl_p(G)$.

Proof: Since if $P \in Syl_p(G)$ we have $p \nmid [G : P]$. Then by Theorem 2.8, M is P-projective for all $P \in Syl_p(G)$. □
The following result can be seen as Maschke-type theorem in the case of Mackey functors.

Corollary 2.10. If $gcd(|G|, p) = 1$ then every Mackey functors is projective (1_G-projective).

Lemma 2.11. Let M be a Mackey functor for G over $\mathcal{O}$. If M is an H-projective then M is Q-projective for all $Q \in Syl_p(H)$.

Proof: Since M is H-projective then there exists a Mackey functor N for H such that $M|Ind_H^G(N)$. Now if $Q \in Syl_p(H)$ we have $p \nmid [H : Q]$ then $[H : Q]$ is a unit in $\mathcal{O}$. Then by Corollary 2.9, every Mackey functor for H is Q-projective. Then there exists a Mackey functor W for Q such that $N|Ind_Q^H(W)$. Therefore,

$$M|Ind_H^G(N)|Ind_H^G(Ind_Q^H(W)) = Ind_Q^G(W).$$

Then M is Q-projective for all $Q \in Syl_p(H)$. □

Lemma 2.12. Let M be a Mackey functor for G over $\mathcal{O}$. Then M is H-projective if and only if every direct summand of M is H-projective.

Proof: Suppose that M_1 is a Mackey functor for G such that $M_1|M$ and M is H-projective (i.e $M|Ind_H^G(N)$ for some Mackey functor N for H). Then $M_1|M|Ind_H^G(N)$. Therefore, M_1 is H-projective. Now suppose that every direct summand of M is H-projective. (For instance if $M = M_1 \oplus M_2$ such that M_1 and M_2 are H-projective). We will prove that M is H-projective. Now M_i is H-projective meaning that there exists a Mackey functor N_i for H such that $M_i|Ind_H^G(N_i)$ for $i = 1, 2$. Then $Ind_H^G(N_i) = M_i \oplus X_i$, where X_i is a Mackey functor for G. Then

$$\begin{aligned} Ind_H^G(N_1 \oplus N_2) &\cong Ind_H^G(N_1) \oplus Ind_H^G(N_2). \\ &= M_1 \oplus X_1 \oplus M_2 \oplus X_2 \\ &= \underbrace{M_1 \oplus M_2}_{M} \oplus X_1 \oplus X_2. \end{aligned}$$

Therefore, $M|Ind_H^G(N_1 \oplus N_2)$ and M is H-projective. □

Acknowledgement. The results in this paper is part of the master dissertation [1] of the second author under supervision of the first author.

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