Gulf Journal of Mathematics
Vol 4, Issue 2 (2016) 18-24

ASYMPTOTIC ORTHOGONALITY IN NORMED SPACES

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Abstract. In this paper we want to generalize the concept of best approximation and orthogonality in normed space and consider relation between them. Also we give important results for asymptotic best approximation in dual space.

1. Introduction

Let X be a normed linear space and M be a subset of X. A point $y_0 \in M$ is said to be a best approximation for $x \in X$, if for every $y \in W$,

$$\|x - y_0\| \leq \|x - y\|.$$

The set of all the best approximation of x in M is denoted by $P_M(x)$.

The problem of best approximation has a long history and continues to generate much interest and such problems have been extended extensively, see, for example [3-5]. Now, we introduce a generalization of the best approximation concept with sequences that is important in different aspect of mathematics.

Definition 1.1. *Let X be a normed space, M a closed subset of X and $\{x_n\}_{n\geq 1} \subseteq X \backslash M$. Put*

$$d(\{x_n\}_{n\geq 1}, M) = \liminf_{n\to\infty} \inf_{y\in M} \|x_n - y\|.$$

A point $y_0 \in M$ is said to be a asymptotic best approximation for $\{x_n\}_{n\geq 1} \subseteq X \backslash M$, if

$$\liminf_{n\to\infty} \| x_n - y_0\| = d(\{x_n\}_{n\geq 1}, M).$$

If every sequence $\{x_n\}_{n\geq 1} \subseteq X$ has at least one asymptotic best approximation in M, then M is called a asymptotic proximinal subset of X. If each sequence $\{x_n\}_{n\geq 1} \subseteq X$ has an unique asymptotic best approximation in M, then M is called a asymptotic Chebyshev subset of X. For $x \in X$ put,

$$P^a_M(\{x_n\}_{n\geq 1}) = \{y_0 \in M : \liminf_{n\to\infty} \| x_n - y_0\| = d(\{x_n\}_{n\geq 1}, M)\}.$$

This set may be empty, a singleton, or certain infinitely many points. It is clear that

$$P^a_M(\{x_n\}_{n\geq 1}) = \{\cap_{n=1}^{\infty} \cup_{i=n}^{\infty} B_{d(\{x_n\}_{n\geq 1}, M)}(x_i)\} \cap M.$$

Date: Received: Dec 2, 2015; Accepted: Mar 7, 2016.
2010 *Mathematics Subject Classification.* 46A32,46M05,41A17.
Key words and phrases. Best approximation, dual space, orthogonal basis.

Also, if $\{x_n\}_{n\geq 1}$ converges strongly to $x \in C$, then $P^a_M(\{x_n\}_{n\geq 1}) = \{x\}$.

2. Existence and Uniqueness

In the following we give some conditions that guaranties the existence, uniqueness or compactness of the set of asymptotic best approximations.

Theorem 2.1. *Let M be a nonempty subset of Banach space X and $\{x_n\}_{n\geq 1} \subseteq X\backslash M$. If M is weakly compact and convex, then $P^a_M(\{x_n\}_{n\geq 1})$ is a nonempty weakly compact set.*

Proof. Because M is weakly compact convex set and the function $x \to \liminf \|x_n - x\|$ is continuous on M. Thus, $P^a_M(\{x_n\}_{n\geq 1})$ is nonempty and also, since $P^a_M(\{x_n\}_{n\geq 1})$ is closed, weakly compact.

□

Proposition 2.2. *Let M be a nonempty subset of Banach space X and $\{x_n\}_{n\geq 1} \subseteq X$. Then*

$$diam(P^a_M(\{x_n\}_{n\geq 1})) \leq \epsilon_0(X) d(\{x_n\}_{n\geq 1}, M).$$

Proof. Set $d = diam(P^a_M(\{x_n\}_{n\geq 1}))$ that $d > 0$. Let $0 < r < d$ and $x, y \in P^a_M(\{x_n\}_{n\geq 1})$ with $\|x - y\| \geq d - r$. By the convexity of $P^a_M(\{x_n\}_{n\geq 1})$, $\frac{x+y}{2} \in P^a_M(\{x_n\}_{n\geq 1})$. Also from the property of modulus of convexity for every $s \in C$ we have

$$d(\{x_n\}_{n\geq 1}, M) = \liminf \|x_n - \frac{x-y}{2}\| \leq d(\{x_n\}_{n\geq 1}, M)[1 - \delta_X(\frac{\|x-y\|}{d(\{x_n\}_{n\geq 1}, M)})].$$

Therefore

$$\delta_X(\frac{\|x-y\|}{d(\{x_n\}_{n\geq 1}, M)}) \leq 0.$$

By definition of $\epsilon_0(X)$, $d - r \leq \|x - y\| \leq \epsilon_0(X) d(\{x_n\}_{n\geq 1}, M)$. Because $r > 0$ is arbitrary, we have

$$diam(P^a_M(\{x_n\}_{n\geq 1})) \leq \epsilon_0(X) d(\{x_n\}_{n\geq 1}, M).$$

□

Theorem 2.3. *Let M be a nonempty weakly compact convex subset of a uniformly convex Banach space X and $\{x_n\}_{n\geq 1} \subseteq X$. Then $P^a_M(\{x_n\}_{n\geq 1})$ is a singleton set.*

Proof. By Proposition 2.2, we obtain that $P^a_M(\{x_n\}_{n\geq 1}) \neq \emptyset$. Also by the uniform convexity of X, $\epsilon_0(X) = 0$, it follows that from last proposition $diam(P^a_M(\{x_n\}_{n\geq 1})) = 0$, i.e., $P^a_M(\{x_n\}_{n\geq 1})$ is a singleton set. □

3. Orthogonality and Basis

Many authors have introduced the concept of orthogonality in different ways[1,2,5]. One of the important definition is the Brushoff orthogonality [2]. Let X be a normed linear space, if $x, y \in X$, x is said to be Brushoff orthogonal to y and is denoted by $x \perp y$ if and only if for ever scaler α, $\|x\| \leq \|x + \alpha y\|$. Now we generalize the concept of Brushoff orthogonality with sequences.

Definition 3.1. Let $\{x_n\}_{n\geq 1} \subseteq X$ and $y \in X$. We say that $\{x_n\}_{n\geq 1}$ is asymptotic orthogonal to y and is denoted by $\{x_n\}_{n\geq 1} \perp^a y$ if for ever scaler α,

$$\liminf_{n\to\infty} \|x_n\| \leq \liminf_{n\to\infty} \|x_n + \alpha y\|.$$

A sequence $\{x_n\}_{n\geq 1} \subseteq X$ is said to be asymptotic orthogonal to a set $M \subseteq X$ and we write $\{x_n\}_{n\geq 1} \perp^a M$ if for every $y \in M$, $\{x_n\}_{n\geq 1} \perp^a y$.

Proposition 3.2. *Let X be a normed linear space, M a linear subspace of X, $\{x_n\}_{n\geq 1} \subseteq X \setminus M$ and $g_0 \in M$. Then $g_0 \in P^a_M(\{x_n\}_{n\geq 1})$ if and only if*

$$\{x_n - g_0\} \perp^a M \ (*).$$

Proof. By definition of asymptotic orthogonality, condition $(*)$ means that for every scaler α and $g \in M$

$$\liminf_{n\to\infty} \| x_n - g_0\| \leq \liminf_{n\to\infty} \| x_n - y_0 + \alpha g\|$$

and this is obviously equivalent to $g_0 \in P^a_M(\{x_n\}_{n\geq 1})$. □

In continue, we will give new result about asymptotic orthogonality in inner product space. we start by following lemma.

Lemma 3.3. *Let X be an inner product space, $\{x_n\}_{n\geq 1} \subseteq X$ and $y \in X$. Then $\{x_n\}_{n\geq 1} \perp^a y$ if and only if $\liminf_{n\to\infty}\langle x_n, y\rangle = 0$.*

Proof. Suppose $\{x_n\}_{n\geq 1} \perp^a y$ and $\liminf_{n\to\infty} |\langle x_n, y\rangle| \neq 0$. Put $\alpha = -\liminf_{n\to\infty} \frac{|\langle x_n, y\rangle|}{\langle y, y\rangle}$. We have

$$\begin{aligned}
\liminf_{n\to\infty} \|x_n + \alpha y\| &= \liminf_{n\to\infty}\langle x_n - \liminf_{n\to\infty} \frac{\langle x_n, y\rangle}{\langle y, y\rangle} y, x_n - \liminf_{n\to\infty} \frac{\langle x_n, y\rangle}{\langle y, y\rangle} y\rangle \\
&= \liminf_{n\to\infty}(\langle x_n, x_n\rangle - 2\liminf_{n\to\infty} \frac{\langle x_n, y\rangle}{\langle y, y\rangle}\langle x_n, y\rangle \\
&\quad + \liminf_{n\to\infty} \frac{\langle x_n, y\rangle^2}{\langle y, y\rangle^2}\langle y, y\rangle) \\
&\leq \liminf_{n\to\infty}\langle x_n, x_n\rangle - \liminf_{n\to\infty} \frac{\langle x_n, y\rangle^2}{\langle y, y\rangle} \\
&< \liminf_{n\to\infty}\langle x_n, x_n\rangle = \liminf_{n\to\infty} \|x_n\|^2,
\end{aligned}$$

whence $\{x_n\}_{n\geq 1}$ is not asymptotic orthogonal to y, while if $\liminf_{n\to\infty}\langle x_n, y\rangle = 0$, then for every scaler α we have

$$\begin{aligned}\liminf_{n\to\infty} \|x_n + \alpha y\|^2 &= \liminf_{n\to\infty} \langle x_n + \alpha y, x + \alpha y\rangle \\ &\geq \liminf_{n\to\infty} \|x_n\|^2 + |\alpha|^2\|y\|^2 \geq \liminf_{n\to\infty} \|x_n\|^2. \blacksquare\end{aligned}$$

□

Theorem 3.4. *Let X be a inner product space, M a subset of X, $\{x_n\}_{n\geq 1} \subseteq X \setminus M$ and $g_0 \in M$. Then $g_0 = P^a_M(\{x_n\}_{n\geq 1})$ if and only if*

$$\limsup_{n\to\infty} \langle x_n - g_0, g - g_0\rangle \leq 0, \ \forall g \in M \ (*).$$

Proof. If $(*)$ holds and $y \in M$, then

$$\begin{aligned}\|x_n - g_0\|^2 &= \langle x_n - g_0, x_n - g_0\rangle \\ &= \langle x_n - g_0, x_n - g\rangle + \langle x_n - g_0, g - g_0\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\liminf_{n\to\infty} \|x_n - g_0\|^2 &\leq \liminf_{n\to\infty} \langle x_n - g_0, x_n - g\rangle + \limsup_{n\to\infty} \langle x_n - g_0, g - g_0\rangle \\ &\leq \liminf_{n\to\infty} \langle x_n - g_0, x_n - g\rangle \leq \liminf_{n\to\infty} \|x_n - g_0\|\|x_n - g\|\end{aligned}$$

by Schwarz's inequality. Therefore

$$\liminf_{n\to\infty} \|x_n - g_0\| \leq \liminf_{n\to\infty} \|x_n - g\|,$$

and so $g_0 = P^a_M(\{x_n\}_{n\geq 1})$.

Conversely, suppose $(*)$ fails. Then for some $y \in M$ have $\limsup_{n\to\infty}\langle x_n - g_0, x - g\rangle > 0$. For every $0 < \lambda < 1$, the element $y_\lambda := \lambda g + (1-\lambda)g_0 \in M$ by convexity and

$$\begin{aligned}\|x_n - g_\lambda\|^2 &= \langle x_n - g_\lambda, x_n - g_\lambda\rangle \\ &= \|x_n - g_\lambda\|^2 - 2\lambda\langle x_n - g_0, g - g_0\rangle + \lambda^2\|g - g_0\|^2.\end{aligned}$$

Then

$$\begin{aligned}\liminf_{n\to\infty} \|x_n - g_\lambda\|^2 &< \liminf_{n\to\infty} \|x_n - g_\lambda\|^2 - 2\lambda \limsup_{n\to\infty} \langle x_n - g_0, g - g_0\rangle \\ &< \liminf_{n\to\infty} \|x_n - g_0\|^2\end{aligned}$$

and so $g_0 \neq P^a_M(\{x_n\}_{n\geq 1})$. □

Theorem 3.5. *Let X be a inner product space, M a subspace of X, $\{x_n\}_{n\geq 1} \subseteq X \setminus M$ and $g_0 \in M$. Then $g_0 \in P^a_M(\{x_n\}_{n\geq 1})$ if and only if*

$$\limsup_{n\to\infty} \langle x_n - g_0, g\rangle = 0, \ \forall g \in M \ (**).$$

Proof. Since M is subspace, therefore by the last theorem the proof is complete.

□

Corollary 3.6. *Let $\{y_1, ..., y_n\}$ be a basis for the n-dimensional subspace M of X and $g_0 \in P_M^a(\{x_n\}_{n\geq 1})$. Then*

$$g_0 = \sum_{i=1}^{n} \alpha_i x_i,$$

where the scalers α_i are the unique solution to the normal equations

$$\limsup_{n\to\infty} \langle x_n, y_j\rangle = \sum_{i=1}^{n} \alpha_i \langle y_i, y_j\rangle, \ j = 1, ..., n.$$

Corollary 3.7. *Let $\{y_1, ..., y_n\}$ be a orthogonal basis for the n-dimensional subspace M of X and $g_0 \in P_M^a(\{x_n\}_{n\geq 1})$. Then*

$$g_0 = \sum_{i=1}^{n} \alpha_i x_i,$$

where $\alpha_i = \limsup_{n\to\infty} \langle x_n, y_i\rangle, \ i = 1, ..., n$.

Theorem 3.8. *Let X be a Banach space, M a subset of X and $\{x_n\}_{n\geq 1} \subseteq X \setminus M$. Then there is subsequence $\{y_n\}_{n\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that for every subsequence $\{y_{n_k}\}_{k\geq 1}$ of $\{y_n\}_{n\geq 1}$ we have*

$$d(\{y_n\}_{n\geq 1}, M) = d(\{y_{n_k}\}_{k\geq 1}, M).$$

Proof. Define

$$r_0 = \sup \big\{ d(\{x_n\}_{n\geq 1}, M) : \ \{x_{n_k}\}_{k\geq 1} \textit{is a subsequence of } \{x_n\}_{n\geq 1} \big\}.$$

Then there is a subsequence $\{x_{n_{k(1)}}\}$ of $\{x_n\}$ such that

$$r_0 \leq d(\{x_{k(1)}\}, M) + 1.$$

Let $\{x_{k(i)}\}$ be a subsequence of $\{x_{k(i-1)}\}$ and define

$$r_i = \sup \big\{ d(\{x_{k(i)_j}\}, M) : \ \{x_{k(i)_j}\} \textit{ is a subsequence of } \{x_{k(i)}\} \big\}.$$

Then there is a subsequence $\{x_{n_{k(i+1)}}\}$ of $\{x_{n_{k(i)}}\}$ such that

$$r_i \leq d(\{x_{n_{k(i+1)}}\}, M) + \frac{1}{i+1}. \ (*)$$

Because $0 \leq ... \leq r_3 \leq r_2 \leq r_1$, it follows that there exists $r \geq 0$, $r = \lim_{i\to\infty} r_i$. Then from $(*)$,

$$r = \lim_{i\to\infty} d(\{x_{n_{k(i+1)}}\}, M).$$

Now, consider the diagonal sequence $\{x_{n_{k(k)}}\}$ and $\bar{r} = d(\{x_{n_{k(k)}}\}, M)$. Because $\{x_{n_{k(k)}}\}$ is a subsequence $\{x_{n_{k(i)}}\}$, it follows that $\bar{r} \leq r_i$ and so $\bar{r} \leq r$. Also, from $(*)$ we have $r \leq \bar{r}$. Hence $r = \bar{r}$ and the proof is completed. □

Corollary 3.9. *Let X be a Banach space, M a separable subset of X and $\{x_n\}_{n\geq 1} \subseteq X \setminus M$. Then there is subsequence $\{y_n\}_{n\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that for every subsequence $\{y_{n_k}\}_{k\geq 1}$ of $\{y_n\}_{n\geq 1}$ we have*

$$P_M^a(\{y_n\}_{n\geq 1}) = P_M^a(\{y_{n_k}\}_{k\geq 1}).$$

Proof. By Theorem 3.8, $\{x_n\}$ has a subsequence $\{y_n\}$ that

$$d(\{y_n\}_{n\geq 1}, M) = d(\{y_{n_k}\}_{k\geq 1}, M).$$

Because M is separable can be used to obtain a subsequence of $\{y_n\}$, which we again denoted by $\{y_n\}$ such that $\lim_{n\to\infty} \|y_n - y\|$ exists for all $y \in M$ and so the proof is completed. □

It is notable that, since every sequence in $\mathbb{R}^n$ has subsequence convergence, $\mathbb{R}^n$ has this property.

4. Functionals and Asymptotic Proximinal

Reader can find some important results of the proximinal subset in dual space in [4]. In this section, we give new results of the relation between functionals of dual space X and asymptotic proximinal of X.

Theorem 4.1. *Let M be a closed subspace of X. Then for every sequence $\{x_n\}_{n\geq 1} \subseteq X\backslash M$, $g_0 \in P^a_M(\{x_n\}_{n\geq 1})$ if and only if there exists a sequence $\{x^*_n\} \subseteq X^*$ such that*

$$\|x^*_n\| = 1,\ x^*_n|_M = 0,\ x^*_n(x_n - g_0) \geq \liminf_{n\to\infty} \|x_n - g_0\|,\ n \in \mathbb{N}.$$

Proof. Suppose $\{x_n\}_{n\geq 1} \subseteq X\backslash M$. Define $m^*_n(y + \alpha x_n) = \alpha d(\{x_n\}_{n\geq 1}, M)$ for each scalar α and $y \in Y$. Then for every $n \in N$, m^*_n is a linear functional on $M + \langle x_n\rangle$ such that $m^*_n(x) = d(\{x_n\}_{n\geq 1}, M)$ and $m^*_n(y) = 0$ for each $y \in M$. Whenever for each scalar $\alpha \neq 0$ and $y \in M$,

$$|m^*_n(y + \alpha x)| \leq |\alpha|\ \|x_n + \alpha^{-1} y\| = \|y + \alpha x_n\|.$$

Hance $\|m^*_n\| \leq 1$. Also, for each $y \in M$

$$d(\{x_n\}_{n\geq 1}, M) = m^*_n(x_n - y) \leq \|m^*_n\|\|x_n - y\| \leq d(\{x_n\}_{n\geq 1}, M).$$

Then $m^*_n \geq 1$ and so $m^*_n = 1$. Finally, suppose x^*_n be any Hahn-Banach extension of m^*_n to X for every $n \in N$.

Conversely, suppose that there exists a sequence $\{x^*_n\} \subseteq X^*$ such that

$$\|x^*_n\| = 1,\ x^*_n|_M = 0,\ x^*_n(x_n - g_0) \geq \liminf_{n\to\infty} \|x_n - g_0\|,\ n \in \mathbb{N}.$$

Therefore

$$\begin{aligned}
\liminf_{n\to\infty} \|x_n - g_0\| &\leq x^*_n(x_n - g_0)\\
&= x^*_n(x_n - g)\\
&\leq |x^*_n(x_n - g)|\\
&\leq \|x^*_n\|\|x_n - g\|\\
&\leq \liminf_{n\to\infty} \|x_n - g\|,\ \forall g \in M.
\end{aligned}$$

Therefore $g_0 \in P^a_M(\{x_n\}_{n\geq 1})$. □

Corollary 4.2. *Let X be a normed space, M a closed subspace of X, $C \subseteq M$ and $x \in X \backslash M$. Then the following statements are equivalence:*

i) $C \subseteq P_M^a(\{x_n\}_{n\geq 1})$.

ii) There exists a sequence $\{x_n^*\} \subseteq X^*$ *such that*

$$\|x_n^*\| = 1,\ x_n^*|_M = 0,\ x_n^*(x_n - g) \geq d(\{x_n\}_{n\geq 1}, M),\ n \in \mathbb{N},\ g \in C.$$

Proof. $i \Rightarrow ii$. Suppose $C \subseteq P_M^a(\{x_n\}_{n\geq 1})$. We consider a $g_0 \in M$. Then a sequence $\{x_n^{*g_0}\} \subseteq X^*$ such that

$$\|x_n^{*g_0}\| = 1,\ x_n^{*g_0}|_M = 0,\ x_n^{*g_0}(x_n - g_0) \geq d(\{x_n\}_{n\geq 1}, M),\ n \in \mathbb{N}.$$

New, for every $g \in M$, we have

$$x_n^{*g_0}(x_n - g) = x_n^{*g_0}(x_n - g_0) \geq d(\{x_n\}_{n\geq 1}, M)$$

and so we have ii.
$ii \Rightarrow i$ It is trivial. □

Corollary 4.3. *Let X be a normed space such that X^* boundedly compact. Also assume M a closed subspace of X and $\{x_n\}_{n\geq 1} \subseteq X \backslash M$. If $g_0 \in P_M^a(\{x_n\}_{n\geq 1})$, then there are $x^* \in X^*$ and $x \in X$ such that*

$$\|x^*\| = 1,\ x^*_{|_M} = 0,\ x^*(x - g_0) \geq \liminf_{n\to\infty} \|x_n - g_0\|.$$

Proof. Suppose $g_0 \in P_M^a(\{x_n\}_{n\geq 1})$. By Corollary 4.2 there exists a sequence $\{x_n^*\} \subseteq X^*$ such that

$$\|x_n^*\| = 1,\ x_n^*|_M = 0,\ x_n^*(x_n - g_0) \geq d(\{x_n\}_{n\geq 1}, M),\ n \in \mathbb{N}.$$

Therefore there are $x^* \in X^*$ and subsequence $\{x_{n_k}\}$ such that w^*-converge to x^*. Also, since $\{x_n\}_{n\geq 1} \subseteq X \backslash M$ is a boundedly compact sequence, has subsequence $\{x_{n_k}\}$ converge to $x \in X$. Therefore $x_{n_k}^*(x_{n_k}) \to x^*(x)$. Hence $x^*(x - g_0) \geq \liminf \|x_n - g_0\|$. □

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