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# UNFILTRABLE VECTOR BUNDLES AND SPANNED VECTOR BUNDLES

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ABSTRACT. Let $X$ be an integral $n$-dimensional projective variety. We study the existence of rank $r \geq 2$ vector bundles on $X$ which are not extensions of two vector bundles (e.g. they exist if $r = 2$ and $X$ is a smooth surface with Kodaira dimension $\geq 0$). Their existence implies the existence of many spanned rank $n$ vector bundles $E$ on $X$ and $(n+r)$-dimensional linear subspaces $W \subseteq H^0(E)$ spanning $E$ such that the evaluation map $W \otimes \mathcal{O}_X \to E$ does not factor through $f : W \otimes \mathcal{O}_X \to F$ and $g : F \to E$ with $F$ locally free, $f$, $g$ surjective and $n + r > \mathrm{rank}(F) > \mathrm{rank}(E)$.

## 1. Introduction

Let $X$ be an integral projective variety and $E$ a rank $r \geq 2$ vector bundle on $X$. We say that $F$ is *partially filtrable* if there is a subsheaf $A$ of $F$ with $0 < \mathrm{rank}(A) < r$ and both $A$ and $F/A$ locally free, i.e. if $F$ is an extension of two vector bundles (this is not the notion of filtrable used in [3] for the case $r = 2$). If $\mathrm{rank}(E) > \dim(X)$, then $E$ is partially filtrable ([2, Theorem 2]). If $\dim(X) = 1$, then every vector bundle of rank $\geq 2$ on $X$ is partially filtrable ([2, §4]). We say that $E$ is *universally not partially filtrable* if for each integral projective variety $Y$ with $\dim(Y) = \dim(X)$ and every finite map $f : Y \to X$ the bundle $f^*(E)$ is not partially filtrable. It is essential to require that $f$ is finite (Remark 2.1). If $X$ is smooth, one may define an a priori weaker notion restricting to the pairs $(Y, f)$ with $Y$ smooth. Since $\mathbb{P}^n$ is the only smooth variety that it is the target of all smooth varieties of the same dimension ([6]), it would be interesting to know the existence or not of universally not partially filtrable bundles on $\mathbb{P}^n$. By [4] if $n$ is even, then $T\mathbb{P}^n$ is not partially filtrable, while if $n \geq 3$ is odd the rank $n-1$ null-correlation bundles, i.e. the kernels of a surjection $T\mathbb{P}^n \to \mathcal{O}_X(2)$, are not partially filtrable.

**Question 1.1.** Let $X$ be an integral projective variety with $\dim(X) \geq n$. Is there a vector bundle $E$ on $X$ with $\mathrm{rank}(E) = \dim(X)$ and $E$ not partially filtrable? And universally not partially filtrable?

We prove the following result.

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**Theorem 1.2.** *Let $X$ be a smooth and connected projective surface with Kodaira dimension $\ge 0$. Then there is a polarization $H$ on $X$ and a rank 2 $H$-stable vector bundle $E$ on $X$ which is not partially filtrable.*

We collect from the references the existence of not partially filtrable vector bundles on $\mathbb{P}^n$ when $n = 3$ and $r = 3$ (Proposition 2.10), when $n = 4$ and $r = 3$ (Proposition 2.11) and when $n = 5$ and $r = 3$ (Proposition 2.12).

We explain now how we came to this topic, while studying spanned vector bundles with some extremal properties. For any vector bundle $E$ on $X$ and any linear subspace $W \subseteq H^0(E)$ let $e_{E,W} : W \otimes \mathcal{O}_X \to E$ denote the evaluation map. Set $e_E := e_{E,H^0(E)}$. $E$ is spanned if and only if $e_E$ is surjective. Now assume that $E$ is spanned, but not trivial. We say that $E$ is *extremely generated* if $e_E$ does not factor through a surjection $F \to E$ with $F$ a vector bundle of rank $> \mathrm{rank}(E)$ and not a trivial bundle. We say that $E$ is *totally extremely generated* if for every linear subspace $W \subseteq H^0(E)$ with $e_{E,W}$ surjective, the map $e_{E,W}$ does not factor through a surjection $F \to E$ with $F$ a vector bundle of rank $> \mathrm{rank}(E)$ and not a trivial bundle, i.e. there are no $(F, f, g)$, where $F$ is a vector bundle, $f : W \otimes \mathcal{O}_X \to F$ and $g : F \to E$ are surjections, $e_{E,W} = g \circ f$ and neither $f$ nor $g$ are isomorphisms (if $F, f, g$ exists, then we call it a *decomposition* of $e_{E,W}$). Assume $\dim(W) \ge \mathrm{rank}(E) + 2$ and that $W$ spans $E$. The bundle $\ker(e_{E,W})$ is partially filtrable if and only if $e_{E,W}$ has a factorization (Lemma 2.6). If $h^0(E) = \mathrm{rank}(E) + 1$, then $E$ is totally extremely generated (Lemma 2.4). If $\dim(X) = 1$, then $E$ is extremely generated if and only if $h^0(E) = \mathrm{rank}(E) + 1$ (Lemma 2.5). We prove the following result.

**Proposition 1.3.** *Set $n := \dim(X)$. Fix an integer $x$ such that $2 \le x \le n$. There is a pair $(E, W)$, where $E$ is a rank $n$ vector bundle on $X$, $W$ is an $(n+x)$-dimensional linear subspace of $H^0(E)$ spanning $W$ and $e_{E,W}$ is not decomposable if and only if there is a rank $x$ vector bundle on $X$ which is not partially filtrable.*

## 2. The proofs

*Remark* 2.1. Let $X$ be an integral projective variety $X$ and $E$ a rank $r \ge 2$ vector bundle on $X$. There is a birational morphism $f : Y \to X$ such that $f^*(E)$ has a rank 1 subbundle (take a an ample line bundle $R$ on $X$, and blow up the zero-scheme of a non-zero section of $E \otimes R^{\otimes t}$, $t \gg 0$). In characteristic zero we may take $Y$ smooth by the resolution of singularities.

*Remark* 2.2. A vector bundle $E$ is totally extremely generated if it is extremely generated and no proper linear subspace of $H^0(E)$ spans $E$.

*Remark* 2.3. Let $E$ be a rank $r$ spanned vector bundle on $X$ which is not totally extremely generated and $E \ne \mathcal{O}_X^{\oplus r}$. Let $W \subseteq H^0(E)$ be a linear subspace such that $e_W$ is surjective and $e_W = f \circ g$ with $g : W \otimes \mathcal{O}_X \to F$, $F$ a vector bundle, $f : F \to E$, $g, f$ surjective and neither $g$ nor $f$ an isomorphism. A surjection of vector bundles is an isomorphism if and only if the two vector bundles have the same rank. Notice that the surjection $g$ (resp. $f$) is not an isomorphism if and only if $\dim(W) > \mathrm{rank}(F)$ (resp. $\mathrm{rank}(F) > r$).

**Lemma 2.4.** *Let $E$ be a rank $r$ spanned vector bundle on $X$ with $h^0(E) = r+1$. Then $E$ is totally extremely generated.*

*Proof.* Since $h^0(E) = r+1$, $E$ is not trivial. Apply Remark 2.3. □

*Remark* 2.5. Let $G$ be a vector bundle on $X$ with $\mathrm{rank}(G) > \dim(X)$. Then $G$ has a rank 1 subbundle (apply [2, Theorem 2] to a twist of $G$ by a power of an ample line bundle). Hence $G$ is partially filtrable.

**Lemma 2.6.** *Let $E$ a spanned vector bundle and take a linear subspace $W \subseteq H^0(E)$ such that $W$ spans $E$. There are a vector bundle $F$ on $X$ and surjections $g : W \otimes \mathcal{O}_X \to F$ and $f : F \to E$ such that $\dim(W) > \mathrm{rank}(F) > \mathrm{rank}(E)$ if and only if the bundle $\ker(e_{E,W})$ is partially filtrable.*

*Proof.* If $F$, $f$, and $g$ exist, then $\ker(g)$ is a proper subheaf of $\ker(e_{E,W})$ with $\ker(e_{E,W})/\ker(g) \cong F/E$ and hence both $\ker(g)$ and $\ker(e_{E,W})/\ker(g)$ are non-zero and locally free. Hence $\ker(e_{E,W})$ is partially filtrable. Now assume that $\ker(e_{E,W})$ is partially filtrable, say it is an extension of the vector bundle $B$ by the vector bundle $A \subset \ker(e_{E,W})$. Let $i : \ker(e_{E,W}) \to W \otimes \mathcal{O}_X$ be the inclusion. Set $j := i|A$ and write $F := W \otimes \mathcal{O}_X/j(A)$. There is a surjection $g : F \to E$ with $\ker(h) \cong B$ and $e_{E,W}$ factors through $g$. □

**Lemma 2.7.** *Let $E$ be a spanned vector bundle on $X$. If $h^0(E) > \mathrm{rank}(E) + \dim(X)$, then $E$ is not extremely generated.*

*Proof.* Use Lemma 2.6 and Remark 2.5. □

**Corollary 2.8.** *Let $R$ be an ample line bundle on $X$. Let $F$ be a vector bundle on $X$. Then there is an integer $t_0$ such that $F \otimes R^{\otimes t}$ is spanned, but not extremely spanned for all $t \geq t_0$.*

*Proof.* Set $n := \dim(X)$ and $r := \mathrm{rank}(F)$. There is an integer $t_0$ such that $F \otimes R^{\otimes t}$ is spanned and $h^0(F \otimes R^{\otimes t}) \geq r + n + 1$ for all $t \geq t_0$. Fix any integer $t \geq t_0$ and set $E := F \otimes R^{\otimes t}$. Set $G := \ker(e_E)$. By construction $G$ has rank $\geq n+1$. Hence it is partially filtrable (Remark 2.5). Apply Proposition 2.7. □

*Remark* 2.9. Let $X$ be an integral projective variety such that any extension of two line bundles splits; by Lefschetz-Noether theorem this is is the case if $X$ is a either a smooth complete intersection and $\dim(X) \geq 3$ or $X$ is a general complete intersection surface not of multidegree $(2,2)$, 3 or 2. Then no rank 2 indecomposable vector bundle on $X$ is partially filtrable. If $\dim(X) = 2$, then there are many indecomposable rank two vector bundles (e.g. the stable ones with respect to any polarization). Now assume $n := \dim(X) = 3, 4$. Take a stable rank 2 vector bundle $F$ on $\mathbb{P}^n$ (if $n = 4$ take the Horrocks-Mumford 's bundle) and take the pull-back by a suitable finite map $X \to \mathbb{P}^n$.

*Proof of Proposition 1.3:* Assume the existence of a rank $x$ vector bundle $G$ on $X$ which is not partially filtrable. By tensoring $G$ with a large power of the dual of an ample line bundle we may assume that $G^\vee$ is spanned and that $h^0(G^\vee) \geq x+n$. Hence there is an $(n+x)$-dimensional linear subspace $V$ of $H^0(G^\vee)$ such that $e_{G^\vee,V}$ is surjective. Dualizing $e_{G^\vee,V}$ we get an inclusion $j : G \to \mathcal{O}_X^{\oplus(n+x)}$ with a locally free cokernel. Set $E$ : coker$(j)$ and let $W \subseteq H^0(E)$ be the image of the map $\alpha : H^0(\mathcal{O}_X^{\oplus(n+x)} \to H^0(E)$ induced by the surjection associated to the cokernel of $j$. Since $V \subseteq H^0(G^\vee)$, $\alpha$ is injective and hence $\dim(W) = n + x$. Apply Lemma 2.6. Conversely, gives $E$ and $W$ with $e_{E,W}$ not decomposable, Lemma 2.6 gives that coker$(e_{E,W})$ is not partially filtrable. □

*Proof of Theorem 1.2:* For any ample line bundle $H$ on $X$ and any integer $c_2$ let $\overline{M}(X; c_2; H)$ be the coarse moduli scheme of rank 2 $H$-semistable sheaves $E$ on $X$ with $\det(E) \cong \mathcal{O}_X$ and $c_2(E) = c_2$.

First assume that $X$ is a minimal surface and that $X$ has Kodaira dimension $\geq 1$. By [9] there is $c_2 \gg 0$ and $H$ such that canonical divisor of $\overline{M}(X; c_2; H)$ is nef. For large $c_2$ we may also assume that a general point of a (component of) $\overline{M}(X; c_2; H)$ is an $H$-stable vector bundle $E$. We claim that general $E \in \overline{M}(X; c_2; H)$ in not partially filtrable. $E$ is locally free. Assume that $E$ is partially filtrable. Since $\det(E) \cong \mathcal{O}_X$, there is $L \in \mathrm{Pic}(X)$ such that $E$ is an extension of $L$ by $L^\vee$. Since $E$ is $H$-stable, it is indecomposable and hence $h^1((L^{\otimes 2})^\vee) > 0$. Since stability is an open condition, a general extension of $L$ by $L^\vee$ is $H$-stable. Since $\overline{M}(X; c_2; H)$ has non-negative Kodaira dimension and $E$ is general, $E$ is not contained in a rational curve. Hence $h^1((L^{\otimes 2})^\vee) = 1$. We get a contradiction, because for $c_2 \gg 0$ we have $\dim(\overline{M}(X; c_2; H)) \sim 4c_2 > q(X)$.

Now assume that $X$ has Kodaira dimension 0 and it is minimal. We take an odd large integer $c_2$ and get that $\overline{M}(X; c_2; H)$ has Kodaira dimension 0 ([10, Corollary 1.2]). Hence we may again use the proof of the claim.

Now assume that $X$ is not minimal. We use induction on the number of blowing-downs needed to pass from $X$ to its minimal model. Let $u : X \to X'$ be the blowing down of a point. Assume to have found an ample line bundle $H$ on $X'$ and a large integer $c_2$ such that $\overline{M}(X'; c_2; H)$ has non-negative Kodaira dimension. Apply part 1 of [7, Theorem 1] to get a corresponding polarization for $X$ and then use the proof of the claim. □

**Proposition 2.10.** *There is rank 3 vector bundle on $\mathbb{P}^3$, which is not partially filtrable.*

*Proof.* Fix the union $C \subset \mathbb{P}^3$ of two disjoint conics. Since $\omega_C(2) \cong \mathcal{O}_C(1)$, $\omega_C(2)$ is spanned and $h^0(\omega_C(2)) > 1$. By the Hartshorne-Serre correspondence ([1]) there is a rank 3 vector bundle $E$ on $\mathbb{P}^3$ fitting in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}^2 \to E \to \mathcal{I}_C(2) \to 0$$

Since $c_2(E) = \deg(C)$ and $c_3(E) = \deg(\omega_C(2))$, $E$ has $1 + 2h + 4h^2 + 4h^3$ as its Chern polynomial. Assume that $E$ has a partial filtration and call $1 + ah$ and $1 + bh + ch^2$ the Chern polynomials of the bundles appearing in a partial filtration of $E$. We get $a + b = 2$, $4 = c + ab$ and $4 = ac$. Hence $a \in \{-4, -2, -1, 1, 2, 4\}$ and $c = 4/a$. If $a = -4$, then, $b = 6$, $c = 28$, a contradiction. If $a = -2$, then $b = 4$ and $c = 12$, a contradiction. If $a = -1$, then $b = 3$, $c = 7$, a contradiction. If $a = 1$, then $b = 1$ and $c = 3$, a contradiction. If $a = 2$, then $b = 0$ and $c = 4$, a contradiction. If $a = 4$, then $b = -2$ and $c = 12$, a contradiction. □

**Proposition 2.11.** *The rank 3 Tango bundle on $\mathbb{P}^4$ is not partially filtrable.*

*Proof.* Since the Tango bundle $E$ fits in an exact sequence ([8])

$$0 \to T\mathbb{P}^4(-2) \to \mathcal{O}_{\mathbb{P}^4}^7 \to F \to 0$$

and $T\mathbb{P}^4(-2)$ has $1 - 3h + 4h^2 - 2h^3 + h^4$ as its Chern polynomial (use a twist of the Euler's sequence of $T\mathbb{P}^4$), then $E$ has $1 + 3h + 5h^2 + 5h^3$ as its Chern polynomial. Since 5 is prime, we easily see that $1 + 3h + 5h^2 + 5h^3$ is irreducible over $\mathbb{Z}$ and hence $E$ is not partially filtrable. □

**Proposition 2.12.** *There is rank* 3 *vector bundle on* $\mathbb{P}^5$*, which is not partially filtrable (in characteristic* $\neq 2$*).*

*Proof.* We use one of the Horrocks bundles constructed in [5]. Let $E$ be the parent bundle of [5]. Use the last four lines of [5, §1]. □

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## References

1. E. Arrondo, *A home-made Hartshorne-Serre correspondence*. Rev. Mat. Complut. **20** (2007), no. 2, 423–443.
2. M. F. Atiyah, *Vector bundles over an elliptic curve*. Proc. London Math. Soc. (3) **7** (1957), 414–472; reprinted in: Michael Atiyah Collected Works, Vol. 1, pp. 105–143, Oxford Science Publications, 1988.
3. G. Elencwajg and O. Forster, *Vector bundles on manifolds without divisors and a theorem on deformations*. Ann. Inst. Fourier (Grenoble) **32** (1982), no. 4, 25–51.
4. C, De Concini and Z. Reichstein, *Nesting maps of Grassmannians*. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **15** (2004), no. 2, 109–118.
5. G. Horrocks, *Examples of rank three vector bundles on five-dimensional projective space*. J. London Math. Soc. **18** (1978), 15–27.
6. R. Lazarsfeld, *Some applications of the theory of positive vector bundles*. Complete intersections (Acireale, 1983), 29–61, Lecture Notes in Math., 1092, Springer, Berlin, 1984.
7. T. Nakashima, *Moduli of stable bundles on blown up surfaces*. J. Math. Kyoto Univ. **33** (1993), no. 3, 571–581
8. H. Tango, *An example of indecomposable vector bundle of rank* $n-1$ *on* $\mathbb{P}^n$. J. Math. Kyoto Univ. **16** (1976), no. 1, 137–141.
9. K. Yamada, *Flips and variation of moduli scheme of sheaves on a surface*. J. Math. Kyoto Univ. **49** (2009), no. 2, 419–425.
10. K. Yamada, *Singularities and Kodaira dimension of moduli scheme of stable sheaves on Enriques surfaces*. Kyoto J. Math. **53** (2013), no. 1, 145–153.

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