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THE RESTRICTED HILL FOUR-BODY PROBLEM WITH VARIABLE MASS

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ABSTRACT. The presentation of the paper consists the extended version of the restricted Hill three-body problem i.e. the restricted Hill four-body problem. Here the mass of the fourth smallest body is supposed to variable with time and also suppose that the other three massive bodies are remain fixed at the apices of an equilateral triangle. After shifting the origin and using the various transformations, we determine the equations of motion and quasi-Jacobi integral for this model. The properties like the equilibrium points and stability are performed analytically.

1. INTRODUCTION

The extended version of restricted three-body problem i.e. the restricted four-body problem is used to study in this manuscript where we have shifted the origin to one of the primaries and applied the transformation with limit as $\mu \to 0$, to get the restricted Hill four-body problem with constant mass. And again used the Jeans law as well as Meshcherskii space time transformations to evaluate the equation of motion for the restricted Hill four-body problem with variable mass.

The Hill problem in three-body configuration with constant mass has studied by many researchers with various types of perturbations. The Hill problem under the effects of oblateness and solar radiation pressure of the massive bodies are studied by [22, 23, 24]. They have shown the location of equilibrium points and zero-velocity curves analytically as well as numerically. They also illustrated about the applications of the problem. The three-dimensional version of Hill's model with oblateness of the primary where they revealed analytically as well as numerically the in-plane and out-of-plane equilibrium points and also their stability are performed by [28]. They also shown some interesting results for this model. This study is different from the classical Hill model. The new version of Hill's problem which contains the effects of radiation as well as the oblateness of the primaries are introduced by [21]. They used iterative methods for obtaining the equilibrium points and their stability where all the equilibrium points are unstable. For determining the expression for the Lyapunov families emanating

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from the equilibrium points, they apply singular perturbations method and numerical techniques in coplanar and spatial case. The existence and location of the collinear equilibrium points and their stability in the generalized Hill model are investigated by [17] and also illustrated their fractal basins of attraction.

The effect of the oblateness of the bigger primary on the location of five equilibrium points in the restricted problem is investigated by [31] numerically. The effects of various perturbations on the equilibrium points and their stability in the restricted problem are investigated by [1] and [7].

The effect of variation of mass is studied by many researchers [6], [8], [9], [11], [33].

The restricted four-body problem with constant mass is investigated by various scientists as [2], [10] ,[12], [13], [19], [20], [26], [27], [29], [30], [32].

Some more new dynamical models are investigated in the dynamical system and studied by [4] and [16].

This work is organized in various sections as: The overview literature is given in section 1. The equations of motion and quasi-Jacobian integral are performed in section 2 . Section 3 presents the analytical studies with subsections 3.1 and 3.2. Finally, conclusion is drawn in section 4.

2. Equations of motion

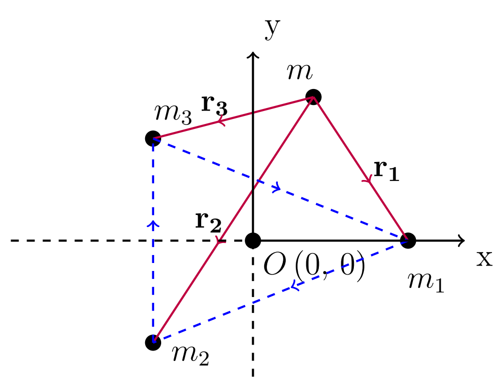

Figure 1. Equilateral triangular configuration in the restricted four-body problem

Assuming the four-body configuration where three bodies with spherical shapes of masses m_i (i = 1, 2, 3) are placed on the vertices of an equilateral triangle with side length ℓ and moving around their common center of mass which is taken as origin and the fourth body of mass m is moving under the influence of the gravitational forces of the other three bodies in the space while this fourth body is not affecting them. We also assume that sidereal and synodic coordinates system coincide on each other. Hence the geometric center coincide to the center of mass. Therefore the equations of motion of fourth body of constant mass in the restricted four-body configuration in the non-dimensional units can perform as [14]:

$$
\begin{aligned}
\ddot{x} - 2\,\dot{y} &= \Omega_x, \\
\ddot{y} + 2\,\dot{x} &= \Omega_y,
\end{aligned}
\tag{2.1}
$$

where,

$$
\begin{aligned}
\Omega &= \frac{1}{2}(x^2+y^2) + \frac{1-2\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu}{r_3}, \\
r_1^2 &= (x-\sqrt{3}\mu)^2 + y^2, \\
r_2^2 &= (x+\frac{\sqrt{3}}{2}(1-2\,\mu))^2 + (y+\frac{1}{2})^2, \\
r_3^2 &= (x+\frac{\sqrt{3}}{2}(1-2\,\mu))^2 + (y-\frac{1}{2})^2 \\
m_1 &= 1-2\,\mu,\ \ m_2 = m_3 = \mu\ \ \text{(say)}.
\end{aligned}
\tag{2.2}
$$

To get the Hill restricted four-body problem with constant mass, we will shift the origin to m_3 with scale $x \to x - \dfrac{\sqrt{3}}{2}(1-2\,\mu)$, $y \to y + \dfrac{1}{2}$ and using the transformation $x \to x\,\mu^{1/3}$, $y \to y\,\mu^{1/3}$, then take limit as $\mu \to 0$. Utilizing the above said expressions, we obtain the system of equations as

$$
\begin{aligned}
\ddot{x} - 2\,\dot{y} &= -\frac{3\,\sqrt{3}}{4}\,y + \frac{9}{4}\,x - \frac{x}{(x^2+y^2)^{3/2}}, \\
\ddot{y} + 2\,\dot{x} &= -\frac{3\,\sqrt{3}}{4}\,x + \frac{3}{4}\,y - \frac{y}{(x^2+y^2)^{3/2}}.
\end{aligned}
\tag{2.3}
$$

Following the procedure given by [3] and [5], we can write the equations of motion of the fourth variable mass body for Hill four-body system where the variation of mass originates from one point and have zero momenta as:

$$
\begin{aligned}
\frac{\dot{m}}{m}\,(\dot{x} - y) + (\ddot{x} - 2\,\dot{y}) &= -\frac{3\,\sqrt{3}}{4}\,y + \frac{9}{4}\,x - \frac{x}{(x^2+y^2)^{3/2}}, \\
\frac{\dot{m}}{m}\,(\dot{y} + x) + (\ddot{y} + 2\,\dot{x}) &= -\frac{3\,\sqrt{3}}{4}\,x + \frac{3}{4}\,y - \frac{y}{(x^2+y^2)^{3/2}}.
\end{aligned}
\tag{2.4}
$$

Due to variable mass of the fourth body, we will use Jean's law [18] and Meshcherskii space-time transformations [25]. These are as follow:

$$\begin{aligned} m &= m_0\, e^{-\psi_1 t},\ (x, y) = \psi_2^{-1/2}\,(\alpha, \beta), \\ (\dot{x}, \dot{y}) &= \psi_2^{-1/2}\left[\left(\dot{\alpha} + \frac{\psi_1}{2}\,\alpha\right), \left(\dot{\beta} + \frac{\psi_1}{2}\,\beta\right)\right], \\ (\ddot{x}, \ddot{y}) &= \psi_2^{-1/2}\left[\left(\ddot{\alpha} + \psi_1\,\dot{\alpha} + \frac{\psi_1^2}{4}\,\alpha\right), \left(\ddot{\beta} + \psi_1\,\dot{\beta} + \frac{\psi_1^2}{4}\,\beta\right)\right], \end{aligned} \tag{2.5}$$

where ψ_1 is variation constant and $\psi_2 = \dfrac{m}{m_0}$, where m_0 is the starting mass of the fourth body.
Utilizing Eqs. (2.4) and (2.5), we obtain

$$\begin{aligned} \ddot{\alpha} - 2\,\dot{\beta} &= \frac{\partial V}{\partial \alpha}, \\ \ddot{\beta} + 2\,\dot{\alpha} &= \frac{\partial V}{\partial \beta}, \end{aligned} \tag{2.6}$$

where,

$$V = \frac{\psi_1^2}{8}\,(\alpha^2 + \beta^2) + \frac{3}{8}\,(3\,\alpha^2 + \beta^2) - \frac{3\sqrt{3}}{4}\,\alpha\,\beta + \frac{\psi_2^{3/2}}{\sqrt{\alpha^2 + \beta^2}}. \tag{2.7}$$

The Quasi-Jacobi integral will be admitted from the Eq. (2.6) as

$$\dot{\alpha}^2 + \dot{\beta}^2 = 2\,V - C - 2\int_{t_0}^{t}\left(\frac{\partial V}{\partial t}\right) dt, \tag{2.8}$$

where C is the quasi-Jacobi constant for this model.

3. Analytical Studies

In this section, we have analytically determined the locations of the equilibrium points and their stability states for this model in two different subsections.

3.1. **Evaluation of equilibrium points.** In this subsection, we evaluate the locations of equilibrium points by solving the right hand side of the Eqs. (2.6) with putting zero as:

$$V_\alpha = \frac{\psi_1^2}{4}\,\alpha + \frac{9}{4}\,\alpha - \frac{3\sqrt{3}}{4}\,\beta - \frac{\alpha\,\psi_2^{3/2}}{(\alpha^2 + \beta^2)^{3/2}} = 0, \tag{3.1}$$

$$V_\beta = \frac{\psi_1^2}{4}\,\beta + \frac{3}{4}\,\beta - \frac{3\sqrt{3}}{4}\,\alpha - \frac{\beta\,\psi_2^{3/2}}{(\alpha^2 + \beta^2)^{3/2}} = 0, \tag{3.2}$$

Eqs. (3.1) and (3.2) will give us the equilibrium points in $\alpha-\beta$-plane. By putting $\beta=0$, in Eq. (3.1), we get $|\alpha| = \dfrac{\sqrt{\psi_2}}{\left(\frac{\psi_1^2}{4}+\frac{9}{4}\right)^{1/3}}$, i.e. $\left(\pm\dfrac{\sqrt{\psi_2}}{\left(\frac{\psi_1^2}{4}+\frac{9}{4}\right)^{1/3}},\ 0\right)$. Here the value of positive and negative sign represents L_1 and L_2 respectively. Same-way if we put $\alpha=0$, in Eq. (3.2), we get $|\beta| = \dfrac{\sqrt{\psi_2}}{\left(\frac{\psi_1^2}{4}+\frac{3}{4}\right)^{1/3}}$,

i.e. $\left(0,\ \pm\dfrac{\sqrt{\psi_2}}{\left(\frac{\psi_1^2}{4}+\frac{3}{4}\right)^{1/3}}\right)$. Here the value of positive and negative sign represents for L_3 and L_4 respectively. In this way, we got four equilibrium points L_1, L_2, L_3 and L_4 out of which two L_1 and L_2 are on the α-axis as well as the other two L_3 and L_4 are on the β-axis.

3.2. **Stability states.** In this subsection, we shall examine the stability states for each equilibrium points separately. Since L_1 and L_2 are symmetrical about β-axis as well as L_3 and L_4 are symmetrical about α-axis. Therefore it is sufficient to show the stability of L_1 and L_3. Following the procedure given by [15], we can write the characteristic polynomial of the Eq. (2.6) in the planar case as:

$$\lambda^4 + f_3\,\lambda^3 + f_2\,\lambda^2 + f_1\,\lambda + f_0 = 0, \tag{3.3}$$

where

$$\begin{aligned}
f_3 &= -2\,\psi_1,\\
f_2 &= 4 + \frac{3\,\psi_1^2}{2} - (V_{\alpha\,\alpha})^0 - (V_{\beta\,\beta})^0,\\
f_1 &= -\psi_1(4 + \frac{\psi_1^2}{2} - (V_{\alpha\,\alpha})^0 - (V_{\beta\,\beta})^0),\\
f_0 &= \frac{\psi_1^4}{16} + \frac{\psi_1^2}{4}(\,4 - (V_{\alpha\,\alpha})^0 - (V_{\beta\,\beta})^0) + (V_{\alpha\,\alpha})^0\,(V_{\beta\,\beta})^0\ -\ ((V_{\alpha\,\beta})^0)^2,
\end{aligned} \tag{3.4}$$

the superscript zero represents the values of the second derivatives of the potential function V corresponding to the equilibrium point. And also the second derivatives of the potential function are defined from Eq. (2.7) as:

$$\begin{aligned}
V_{\alpha\,\alpha} &= \frac{\psi_1^2}{4}+\frac{9}{4}+\frac{3\,\alpha^2\,\psi_2^{3/2}}{(\alpha^2+\beta^2)^{5/2}}-\frac{\psi_2^{3/2}}{(\alpha^2+\beta^2)^{3/2}},\\
V_{\beta\,\beta} &= \frac{\psi_1^2}{4}+\frac{3}{4}+\frac{3\,\beta^2\,\psi_2^{3/2}}{(\alpha^2+\beta^2)^{5/2}}-\frac{\psi_2^{3/2}}{(\alpha^2+\beta^2)^{3/2}},\\
V_{\alpha\,\beta} &= V_{\beta\,\alpha}=-\frac{3\sqrt{3}}{4}+\frac{3\,\alpha\,\beta\,\psi_2^{3/2}}{(\alpha^2+\beta^2)^{5/2}}.
\end{aligned} \tag{3.5}$$

3.2.1. *Stability states corresponding to L_1.* The second derivatives of the potential function corresponding to L_1 i.e. $\left(\frac{\sqrt{\psi_2}}{\left(\frac{\psi_1^2}{4}+\frac{9}{4}\right)^{1/3}},\ 0\right)$ can be written as

$$V_{\alpha\,\alpha} = \frac{3\,\psi_1^2}{4}+\frac{27}{4},\quad V_{\beta\,\beta} = -\frac{3}{2},\quad V_{\alpha\,\beta} = V_{\beta\,\alpha}=-\frac{3\sqrt{3}}{4}. \tag{3.6}$$

Therefore,the values of the coefficients of the characteristic equation are

$$\begin{aligned}
f_3 &= -2\,\psi_1,\ <0,\\
f_2 &= -\frac{11}{4}+\frac{3\,\psi_1^2}{4},\ <0,\\
f_1 &= \psi_1\left(\frac{\psi_1^2}{4}+\frac{5}{2}\right),\ >0,\\
f_0 &= -\frac{\psi_1^4}{8}-\frac{19\,\psi_1^2}{8}-\frac{189}{16},\ <0.
\end{aligned} \tag{3.7}$$

From here we got that there are three changes in sign of the coefficients of the characteristic equation. According to the Descart's rule of signs, at least one positive real root exists. And hence the equilibrium point L_1 is unstable.

3.2.2. *Stability states corresponding to L_3.* The second derivatives of the potential function corresponding to L_3 i.e. $\left(0\,,\ \frac{\sqrt{\psi_2}}{\left(\frac{\psi_1^2}{4}+\frac{3}{4}\right)^{1/3}}\right)$ can be written as

$$V_{\alpha\,\alpha} = \frac{3}{2},\quad V_{\beta\,\beta} = \frac{3\,\psi_1^2}{4}+\frac{9}{4},\quad V_{\alpha\,\beta} = V_{\beta\,\alpha}=-\frac{3\sqrt{3}}{4}. \tag{3.8}$$

Therefore, the values of the coefficients of the characteristic equation are

$$\begin{aligned} f_3 &= -2\,\psi_1,\ < 0, \\ f_2 &= \frac{3\,\psi_1^2}{4} + \frac{1}{4},\ > 0, \\ f_1 &= \psi_1(\frac{\psi_1^2}{4} - \frac{1}{4}),\ < 0, \\ f_0 &= -\frac{\psi_1^4}{8} + \frac{19\,\psi_1^2}{16} + \frac{27}{16},\ > 0. \end{aligned} \tag{3.9}$$

From here we got that there are four changes in sign of the coefficients of the characteristic equation. According to the Descart's rule of signs, there must exist positive real roots. And hence the equilibrium point L_3 is unstable.

4. Conclusion

The restricted Hill four-body problem is investigated with the variation effect of mass of the fourth body by shifting the origin to one of the primaries and by many transformations with making use of limit $\mu \to 0$. The obtained equations of motion are admitted the quasi-Jacobi integral. With the use of equations of motion, we have determined the four equilibrium points out of which two lie on the abscissa and other two lie on the ordinate. The equilibrium points lie on the abscissa are symmetrical about the ordinate while the equilibrium points lie on the ordinate are symmetrical about the abscissa. In the next subsection, we have examined the stability of these equilibrium points. Utilizing the Descart's rule of sign, we obtained that all four equilibrium points are unstable because there exists at least one positive real root.

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