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SOME RESULTS ON UNIQUENESS OF ENTIRE FUNCTIONS SHARING A POLYNOMIAL

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Abstract. In this paper, we study the uniqueness problem on entire functions that share a nonconstant polynomial in which multiplicities are not taken into consideration. The results of the paper improve and generalize the recent results due to Liu - Yang [9].

1. Introduction and preliminaries

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We assume that the reader is familiar with standard notations and fundamental results of Nevanlinna value distribution theory as described in [7, 13]. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic function of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ outside of a possible exceptional set E of finite linear measure. The meromorphic function a is called a small function of f if $T(r, a) = S(r, f)$.

We say that two nonconstant meromorphic functions f and g share a small function a IM (ignoring multiplicities), when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities). A finite value z_0 is called a fixed point of $f(z)$ if $f(z_0) = z_0$.

To answer a familiar question posed by Hayman [6] in 1959, the following result was proved by Fang - Hua [4] in 1996.

Theorem A. *Let f and g be two nonconstant entire functions, $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

In 2002 Fang [3] proved the following results considering kth derivative of f^n and g^n which may be taken as an extension of Theorem A.

Theorem B. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either*

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$f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.

Theorem C. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f = g$.*

Natural question arises: Is it possible to replace the value sharing with fixed point sharing in Theorems A-C ? Afterwards research works concerning the above question have been done by many mathematicians such as Fang - Qiu [5], Lin - Yi [8], Qi - Yang [10], Sahoo [11], Zhang [14]. Here we recall the following results proved by Qi - Yang [10] in 2010.

Theorem D. *Let f and g be two transcendental entire functions, n, m and k be three positive integers with $n > 2k + m^* + 4$, λ and μ be constants that satisfy $|\lambda| + |\mu| \neq 0$. If $[f^n(\lambda f^m + \mu)]^{(k)}$ and $[g^n(\lambda g^m + \mu)]^{(k)}$ share z CM, then the following conclusions hold:*
(i) If $\lambda\mu \neq 0$, then $f^d(z) = g^d(z)$, where $d = \gcd(n, m)$; especially when $d = 1$, $f = g$.
(ii) If $\lambda\mu = 0$, then either $f = tg$ for a constant t that satisfies $t^{n+m^*} = 1$ or $k = 1$ and $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$ for three constants c_1, c_2 and c that satisfy

$$4(\lambda + \mu)^2(c_1c_2)^{n+m^*}[(n + m^*)c]^2 = -1,$$

where

$$m^* = \begin{cases} m & \text{if } \lambda \neq 0; \\ 0 & \text{if } \lambda = 0. \end{cases}$$

In 2011 Dou - Qi - Yang [2] studied the uniqueness problem on entire functions sharing fixed point concerning some general differential polynomials and proved the following results which may be treated as a generalization of Theorem D.

Theorem E. *Let $P(z) = a_mz^m + a_{m-1}z^{m-1} + ... + a_1z + a_0$ or $P(z) = C$, where a_0, a_1,...,a_{m-1}, $a_m(\neq 0)$, $C(\neq 0)$ are complex constants. Suppose that f and g be two transcendental entire functions, and let n, k and m be three positive integers with $n > 2k + m^{**} + 4$. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share z CM, then the following conclusions hold:*
(i) If $P(z) = a_mz^m + a_{m-1}z^{m-1} + ... + a_1z + a_0$ is not a monomial, then either $f = tg$ for a constant t that satisfies $t^d = 1$, where $d = (n+m, ..., n+m-i, ..., n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, ..., m$; or f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(w_1, w_2) = w_1^n(a_mw_1^m + ... + a_1w_1 + a_0) - w_2^n(a_mw_2^m + ... + a_1w_2 + a_0).$$

(ii) When $P(z) = C$ or $P(z) = a_mz^m$, then either $f = tg$ for a constant t that satisfies $t^{n+m^{**}} = 1$, or $f(z) = b_1e^{bz^2}$, $g(z) = b_2e^{-bz^2}$ for three constants b_1, b_2 and b that satisfies $4a_m^2(b_1b_2)^{n+m}((n + m)b)^2 = -1$, or $4C^2(b_1b_2)^n(nb)^2 = -1$, where m^{**} is defined by

$$m^{**} = \begin{cases} m & \text{if } P(z) \neq C; \\ 0 & \text{if } P(z) = C. \end{cases}$$

One may now ask the following question :

Question 1. *What are the IM analogous of Theorems D and E ?*

Recently Liu - Yang [9] answered the above question by proving the following results.

Theorem F. *Let f and g be two transcendental entire functions, and let n, m and k be three positive integers with $n > 5k + 4m^* + 7$, λ and μ be constants that satisfy $|\lambda| + |\mu| \neq 0$. If $[f^n(\lambda f^m + \mu)]^{(k)}$ and $[g^n(\lambda g^m + \mu)]^{(k)}$ share z IM, then the conclusions (i) and (ii) of Theorem D hold.*

Theorem G. *Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$ or $P(z) = C$, where a_0, a_1,...,a_{m-1}, $a_m(\neq 0)$, $C(\neq 0)$ are complex constants. Suppose that f and g be two transcendental entire functions, and let n, k and m be three positive integers with $n > 5k + 4m^{**} + 7$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share z IM, then the conclusions (i) and (ii) of Theorem E hold.*

Regarding Theorems F and G the following question is inevitable which is the motive of the authors.

Question 2. *What can be said if the fixed point sharing in Theorems F and G is replaced with sharing a nonconstant polynomial ?*

In this paper we will concentrate our attention on the above question. We will prove the following two theorems first one of which improves and generalizes Theorem F and second one improves and generalizes Theorem G. The following are the main results of the paper.

2. Main results

Theorem 1. *Let f and g be two transcendental entire functions, $P_1(z)$ be a nonconstant polynomial of degree p, and let n, k and m be three positive integers with $n > 5k + 2p + 4m^* + 5$. Suppose that either k, p are co-prime or $k > p$ when $p \geq 2$. If $[f^n(\lambda f^m + \mu)]^{(k)} - P_1$ and $[g^n(\lambda g^m + \mu)]^{(k)} - P_1$ share the value 0 IM, where λ, μ are constants satisfying $|\lambda| + |\mu| \neq 0$, then the following conclusions hold:*
(i) If $\lambda\mu \neq 0$, then $f^d(z) = g^d(z)$, where $d = \gcd(n, m)$; especially when $d = 1$, $f = g$.
(ii) If $\lambda\mu = 0$, then either $f = tg$ for a constant t that satisfies $t^{n+m^*} = 1$ or $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where $Q(z)$ is a polynomial without constant such that $Q'(z) = P_1(z)$, b_1, b_2 and b are three constants satisfying $\mu^2(nb)^2(b_1b_2)^n = -1$ or $\lambda^2((n+m)b)^2(b_1b_2)^{n+m} = -1$.

Theorem 2. *Let f and g be two transcendental entire functions, $P_1(z)$ be a nonconstant polynomial of degree p, and let n, k and m be three positive integers with $n > 5k+2p+4m^{**}+5$. Let $P(z)$ be defined as in Theorem G. If $[f^n P(f)]^{(k)} - P_1$ and $[g^n P(g)]^{(k)} - P_1$ share 0 IM, then the following conclusions hold:*
(i) If $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$ is not a monomial, then either $f = tg$ for a constant t that satisfies $t^d = 1$, where $d = (n+m, ..., n+m-i, ..., n)$,

$a_{m-i} \neq 0$ *for some* $i = 0, 1, 2, ..., m$*; or* f *and* g *satisfy the algebraic equation* $R(f, g) = 0$*, where*

$$R(w_1, w_2) = w_1^n(a_m w_1^m + ... + a_1 w_1 + a_0) - w_2^n(a_m w_2^m + ... + a_1 w_2 + a_0).$$

(ii) When $P(z) = C$ *or* $P(z) = a_m z^m$*, then either* $f = tg$ *for a constant* t *that satisfies* $t^{n+m^{**}} = 1$*, or* $f(z) = b_1 e^{bQ(z)}$*,* $g(z) = b_2 e^{-bQ(z)}$*, where* $Q(z)$ *is a polynomial without constant such that* $Q'(z) = P_1(z)$*,* b_1*,* b_2 *and* b *are three constants satisfying* $C^2(nb)^2(b_1b_2)^n = -1$ *or* $a_m^2((n+m)b)^2(b_1b_2)^{n+m} = -1$*.*

We now give the following definitions and notations which are used in the paper.

Definition 1. [7] *Let* $a \in \mathbb{C} \cup \{\infty\}$*. For a positive integer* p *we denote by* $N(r, a; f \mid\leq p)$ *the counting function of those* a*-points of* f *(counted with proper multiplicities) whose multiplicities are not greater than* p*. By* $\overline{N}(r, a; f \mid\leq p)$ *we denote the corresponding reduced counting function.*

Analogously we can define $N(r, a; f \mid\geq p)$ *and* $\overline{N}(r, a; f \mid\geq p)$*.*

Definition 2. *Let* a *be any value in the extended complex plane, and let* k *be an arbitrary nonnegative integer. We denote by* $N_k(r, a; f)$ *the counting function of* a*-points of* f*, where an* a*-point of multiplicity* p *is counted* p *times if* $p \leq k$ *and* k *times if* $p > k$*. Then*

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + ... + \overline{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$*.*

Here we present some lemmas which will be needed in the sequel.

Lemma 1. [15] *Let* f *be a nonconstant meromorphic function, and* p*,* k *be two positive integers. Then*

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2.1}$$

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.2}$$

Lemma 2. [1] *Let* f *and* g *be two nonconstant meromorphic functions that share the value 1 IM and*

$$\frac{f''}{f'} - \frac{2f'}{f-1} \not\equiv \frac{g''}{g'} - \frac{2g'}{g-1}.$$

Then
$T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + 2\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + 2\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) + S(r, g)$*.*
Similar inequation holds for g *also.*

Lemma 3. [12] *Let* f *be a nonconstant meromorphic function and let* $a_n(z)(\not\equiv 0)$*,* $a_{n-1}(z)$*, ... ,* $a_0(z)$ *be small functions of* f*. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 4. [7] *Suppose that f is a nonconstant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r, \infty; f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S\left(r, \frac{f'}{f}\right),$$

then $f = e^{az+b}$, where $a(\neq 0)$, b are constants.

Lemma 5. [11] *Let f and g be two nonconstant entire functions and let n, k be two positive integers. Suppose that $F_1 = (f^n P(f))^{(k)}$ and $G_1 = (g^n P(g))^{(k)}$ where $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$, $a_0(\neq 0)$, a_1, ... ,a_{m-1}, $a_m(\neq 0)$ are complex constants. If there exist two nonzero constants c_1 and c_2 such that $\overline{N}(r, c_1; F_1) = \overline{N}(r, 0; G_1)$ and $\overline{N}(r, c_2; G_1) = \overline{N}(r, 0; F_1)$, then $n \leq 2k + m + 2$.*

Lemma 6. *Let f and g be two nonconstant entire functions, n, m and k be three positive integers. Suppose that $F_1 G_1 = P_1^2$, where F_1, G_1 are defined as in Lemma 5 and $P_1(z)$ is defined as in Theorem 1. Then $n \leq k + 2p$.*

Proof. If possible, we assume that $n > k + 2p$. From $F_1 G_1 = P_1^2$, we have

$$(f^n P(f))^{(k)} (g^n P(g))^{(k)} = P_1^2.$$

Let z_0 be a zero of f with multiplicity l. Then z_0 is a zero of $(f^n P(f))^{(k)}$ with multiplicity $nl - k$. Since g is an entire function and $n > k + 2p$, it follows that z_0 is a zero of P_1^2 with multiplicity at least $2p + 1$, which is absurd. Thus f has no zeros. We put $f = e^\alpha$, where α is a nonconstant entire function. Now

$$(a_m f^{n+m})^{(k)} = t_m(\alpha', \alpha'', ..., \alpha^{(k)}) e^{(n+m)\alpha}, \tag{2.3}$$

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$$(a_0 f^n)^{(k)} = t_0(\alpha', \alpha'', ..., \alpha^{(k)}) e^{n\alpha}, \tag{2.4}$$

where $t_i(\alpha', \alpha'', ..., \alpha^{(k)})$ $(i = 0, 1, ..., m)$ are differential polynomials in α', α'', . . . , $\alpha^{(k)}$. Obviously

$$t_i(\alpha', \alpha'', ..., \alpha^{(k)}) \not\equiv 0$$

for $i = 0, 1, 2, ..., m$, and

$$(f^n P(f))^{(k)} \not\equiv 0.$$

Therefore from (2.3) and (2.4) we obtain

$$t_m(\alpha', \alpha'', ..., \alpha^{(k)}) e^{m\alpha} + ... + t_0(\alpha', \alpha'', ..., \alpha^{(k)}) \not\equiv 0. \tag{2.5}$$

Since α is an entire function, we have $T(r, \alpha^{(j)}) = S(r, f)$ for $j = 1, 2, ..., k$, and hence $T(r, t_i) = S(r, f)$ for $i = 0, 1, 2, ..., m$. Therefore using (2.5), Lemma 3 and the second fundamental theorem for small functions we deduce that

$$\begin{aligned} mT(r,f) &= T(r, t_m e^{m\alpha} + ... + t_1 e^{\alpha}) + S(r,f) \\ &\leq \overline{N}(r, 0; t_m e^{m\alpha} + ... + t_1 e^{\alpha}) + \overline{N}(r, 0; t_m e^{m\alpha} + ... + t_1 e^{\alpha} + t_0) \\ &\quad + S(r,f) \\ &\leq \overline{N}(r, 0; t_m e^{(m-1)\alpha} + ... + t_1) + S(r,f) \\ &\leq (m-1)T(r,f) + S(r,f), \end{aligned}$$

a contradiction. Hence $n \leq k + 2p$. This proves the lemma. □

Lemma 7. [10] *Let f and g be two nonconstant entire functions, n, m and k be three positive integers, and let $F_2 = [f^n(\lambda f^m + \mu)]^{(k)}$ and $G_2 = [g^n(\lambda g^m + \mu)]^{(k)}$ where $|\lambda| + |\mu| \neq 0$, and $\lambda\mu = 0$. If there exist two nonzero constants c_1 and c_2 such that $\overline{N}(r, c_1; F_2) = \overline{N}(r, 0; G_2)$ and $\overline{N}(r, c_2; G_2) = \overline{N}(r, 0; F_2)$, then $n \leq 2k + m^* + 2$.*

Lemma 8. *Let f and g be two nonconstant entire functions, n, m and k be three positive integers with $n > 5k + 2p + 4m^* + 5$. Further assume that either k, p are co-prime or $k > p$ when $p \geq 2$. Suppose that $F_2G_2 = P_1^2$, where F_2, G_2 are defined as in Lemma 7, $|\lambda| + |\mu| \neq 0$ and $P_1(z)$ is same as in Theorem 1. Then $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $\lambda^2((n+m)b)^2(b_1b_2)^{n+m} = -1$ or $\mu^2(nb)^2(b_1b_2)^n = -1$ and $Q(z)$ is same as in Theorem 1.*

Proof. We discuss the following two cases separately.
Case I. Let $\lambda\mu = 0$. Noting that $|\lambda| + |\mu| \neq 0$, we take $\mu = 0$, $\lambda \neq 0$ and therefore $m^* = m$. The case $\mu \neq 0$, $\lambda = 0$ can be proved similarly. First we assume that $k = 1$. Then $F_2G_2 = P_1^2$ gives

$$(\lambda f^{n+m})'(\lambda g^{n+m})' = P_1^2. \tag{2.6}$$

Since f and g are entire functions and $n > 5k + 2p + 4m + 5$, it follows from (2.6) that f and g have no zeros. We put

$$f = e^{\alpha}, \qquad g = e^{\beta}, \tag{2.7}$$

where α and β are two nonconstant entire functions. Therefore from (2.6) we get

$$\lambda^2(n+m)^2\alpha'\beta' e^{(n+m)(\alpha+\beta)} = P_1^2. \tag{2.8}$$

From (2.8) it follows that α, β must be polynomials and $\alpha + \beta \equiv C$, where C is a constant. Thus $deg(\alpha) = deg(\beta)$. Therefore $\alpha' + \beta' \equiv 0$ and

$$\lambda^2(n+m)^2\alpha'\beta' e^{(n+m)C} = P_1^2.$$

Simplifying we obtain $\alpha' = bP_1(z)$ and $\beta' = -bP_1(z)$, where $b(\neq 0)$ is a constant. This gives $\alpha = bQ(z) + d_1$ and $\beta = -bQ(z) + d_2$, where $Q(z)$ is a polynomial without constant such that $Q'(z) = P_1(z)$ and d_1, d_2 are constants. Therefore

$$f = b_1 e^{bQ(z)}, \qquad g = b_2 e^{-bQ(z)},$$

where b_1, b_2 and b are three constants satisfying

$$\lambda^2((n+m)b)^2(b_1b_2)^{n+m} = -1.$$

Next we assume that $k \geq 2$. Then from $F_2G_2 = P_1^2$ we obtain

$$(\lambda f^{n+m})^{(k)}(\lambda g^{n+m})^{(k)} = P_1^2. \tag{2.9}$$

Since f and g are transcendental entire function, from (2.9) we obtain

$$N(r, 0; (\lambda f^{n+m})^{(k)}) = O\{\log r\}.$$

From this and (2.7) we see that

$$N(r, \infty; \lambda f^{n+m}) + N(r, 0; \lambda f^{n+m}) + N(r, 0; (\lambda f^{n+m})^{(k)}) = O\{\log r\}.$$

Suppose that α is a transcendental entire function. Then by Lemma 4 we deduce that α is a polynomial, a contradiction. Next we assume that α, β are polynomials of degree p_1 and p_2 respectively. If $p_1 = p_2 = 1$, then

$$f = e^{Az+B}, \qquad g = e^{Cz+D},$$

where $A(\neq 0)$, B, $C(\neq 0)$ and D are constants. So from (2.9) we obtain

$$\lambda^2(AC)^k(n+m)^{2k}e^{(n+m)\{(A+C)z+(B+D)\}} = P_1^2,$$

which is not possible. Hence $\max\{p_1, p_2\} > 1$. We assume that $p_1 > 1$. Then $(\lambda f^{n+m})^{(k)} = Q_1e^{(n+m)\alpha}$ and $(\lambda g^{n+m})^{(k)} = Q_2e^{(n+m)\beta}$, where Q_1, Q_2 are polynomials of degree $k(p_1-1)$ and $k(p_2-1)$ respectively. Therefore from (2.9) we obtain $\alpha + \beta \equiv k_1$, a constant, and therefore $p_1 = p_2$ and $k(p_1 - 1) = p$. This shows that $p \geq k \geq 2$, contradicting with the assumption that k, p are prime to each other.

Case II. Let $\lambda\mu \neq 0$. Since $n > 5k + 2p + 4m + 5 > k + 2p$, using the argument similar as in Lemma 6 we reach at a contradiction. This proves the lemma. □

Lemma 9. [10] *Suppose that F_2 and G_2 are given as in Lemma 7 where $\lambda\mu = 0$. If $n > 2k + m^*$ and $F_2 = G_2$, then $f = tg$ for a constant t satisfying $t^{n+m^*} = 1$.*

Lemma 10. [10] *Suppose that F_2 and G_2 are given as in Lemma 7 where $\lambda\mu \neq 0$. If $n > 2k + m$ and $F_2 = G_2$, then $f^d(z) = g^d(z)$ where $d = \gcd(n, m)$.*

Proof of Theorem 2. We discuss the following three cases separately.

Case (i) Let $P(z) = a_mz^m + a_{m-1}z^{m-1} + ... + a_1z + a_0$, where $a_0(\neq 0)$, a_1, . . . ,a_{m-1}, $a_m(\neq 0)$ are complex constants. We consider $F = \frac{(f^nP(f))^{(k)}}{P_1(z)}$ and $G = \frac{(g^nP(g))^{(k)}}{P_1(z)}$. Then F and G are transcendental meromorphic functions that share the value 1 ignoring multiplicities. Let

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.10}$$

If possible, we suppose that $H \not\equiv 0$. Then using Lemma 3 and (2.1) we obtain

$$\begin{aligned} N_2(r,0;F) &\leq N_2\left(r,0;(f^nP(f))^{(k)}\right) + S(r,f) \\ &\leq T\left(r,(f^nP(f))^{(k)}\right) - (n+m)T(r,f) + N_{k+2}(r,0;f^nP(f)) \\ &\quad +S(r,f) \\ &\leq T(r,F) - (n+m)T(r,f) + N_{k+2}(r,0;f^nP(f)) \\ &\quad +S(r,f). \end{aligned} \tag{2.11}$$

In a similar manner we obtain

$$N_2(r,0;G) \leq T(r,G) - (n+m)T(r,g) + N_{k+2}(r,0;g^nP(g)) + S(r,g). \quad (2.12)$$

From (2.11) and (2.12) we get

$$\begin{aligned}(n+m)\{T(r,f)+T(r,g)\} \leq\; & T(r,F)+T(r,G)+N_{k+2}(r,0;f^nP(f))\\ & +N_{k+2}(r,0;g^nP(g)) - N_2(r,0;F)\\ & -N_2(r,0;G)+S(r,f)+S(r,g). \quad (2.13)\end{aligned}$$

Again from (2.2) we deduce that

$$N_2(r,0;F) \leq N_{k+2}(r,0;f^nP(f)) + S(r,f), \quad (2.14)$$

and

$$N_2(r,0;G) \leq N_{k+2}(r,0;g^nP(g)) + S(r,g). \quad (2.15)$$

Using Lemma 2, Lemma 3, (2.2), (2.14) and (2.15) we obtain from (2.13)

$$\begin{aligned}(n+m)\{T(r,f)+T(r,g)\} \leq\; & N_2(r,0;F)+N_2(r,0;G)+2N_2(r,\infty;F)\\ & +2N_2(r,\infty;G)+3\overline{N}(r,0;F)+3\overline{N}(r,0;G)\\ & +3\overline{N}(r,\infty;F)+3\overline{N}(r,\infty;G)\\ & +N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))\\ & +S(r,f)+S(r,g)\\ \leq\; & 2N_{k+2}(r,0;f^nP(f))+2N_{k+2}(r,0;g^nP(g))\\ & +3N_{k+1}(r,0;f^nP(f))+3N_{k+1}(r,0;g^nP(g))\\ & +S(r,f)+S(r,g)\\ \leq\; & (5k+5m+7)\{T(r,f)+T(r,g)\}\\ & +S(r,f)+S(r,g).\end{aligned}$$

From this we get

$$(n-5k-4m-7)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

contradicting with the fact that $n > 5k+2p+4m+5$.
We now assume that $H = 0$. That is

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \quad (2.16)$$

where $A(\neq 0)$ and B are constants. Now we consider the following three subcases.

Subcase (i) Let $B \neq 0$ and $A = B$. Then from (2.16) we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \quad (2.17)$$

If $B=-1$, then from (2.17) we obtain
$$FG=1,$$
i.e.,
$$(f^nP(f))^{(k)}(g^nP(g))^{(k)}=P_1^2,$$
a contradiction by Lemma 6. If $B\neq -1$, from (2.17), we have $G=\frac{-1}{BF-(B+1)}$ and so $\overline{N}(r,\frac{B+1}{B};F)=\overline{N}(r,G)$. Now using (2.1) we deduce from the second fundamental theorem of Nevanlinna
$$\begin{aligned}T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}\left(r,\frac{B+1}{B};F\right)+S(r,F)\\ &\leq \overline{N}(r,0;F)+S(r,F)\\ &\leq T(r,F)-(n+m)T(r,f)+N_{k+1}(r,0;f^nP(f))+S(r,f).\end{aligned}$$
This gives
$$(n+m)T(r,f) \leq (k+m+1)T(r,f)+S(r,f),$$
a contradiction as $n>5k+2p+4m+5$.
Subcase (ii) Let $B\neq 0$ and $A\neq B$. Then from (2.16) we get $G=\frac{(B-A)F+(A-B-1)}{BF-(B+1)}$ and so $\overline{N}(r,\frac{B+1}{B};F)=\overline{N}(r,G)$. Arguing similarly as in **Subcase (i)** we reach at a contradiction.
Subcase (iii) Let $B=0$ and $A\neq 0$. Then from (2.16) we can write $F=\frac{G+A-1}{A}$ and $G=AF-(A-1)$. Assuming $A\neq 1$, we have $\overline{N}(r,\frac{A-1}{A};F)=\overline{N}(r,0;G)$ and $\overline{N}(r,1-A;G)=\overline{N}(r,0;F)$. So by Lemma 5 we obtain $n\leq 2k+m+2$, a contradiction. Therefore $A=1$ and thus $F=G$. That is
$$[f^nP(f)]^{(k)}=[g^nP(g)]^{(k)}.$$
Integrating we get
$$[f^nP(f)]^{(k-1)}=[g^nP(g)]^{(k-1)}+c_{k-1},$$
where c_{k-1} is a constant. If $c_{k-1}\neq 0$, from Lemma 5 we obtain $n\leq 2k+m$, a contradiction. Hence $c_{k-1}=0$. Repeating k-times, we obtain
$$f^nP(f)=g^nP(g).$$
That is
$$\begin{aligned}&f^n(a_mf^m+a_{m-1}f^{m-1}+...+a_1f+a_0)\\ &=g^n(a_mg^m+a_{m-1}g^{m-1}+...+a_1g+a_0).\end{aligned}\tag{2.18}$$
Let $h=\frac{f}{g}$. If h is a constant, by putting $f=gh$ in (2.18) we get
$$a_mg^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+...+a_0g^n(h^n-1)=0,$$
which implies $h^d=1$, where $d=(n+m,...,n+m-i,...,n+1,n)$, $a_{m-i}\neq 0$ for some $i=0,1,...,m$. Thus $f=tg$ for a constant t such that $t^d=1$, $d=(n+m,...,n+m-i,...,n+1,n)$, $a_{m-i}\neq 0$ for some $i=0,1,...,m$.

If h is not a constant, then from (2.18) we see that f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(w_1,w_2)=w_1^n(a_mw_1^m+...+a_1w_1+a_0)-w_2^n(a_mw_2^m+...+a_1w_2+a_0).$$

Case (ii) Now we assume that $P(z)=a_mz^m$, where a_m $(\neq 0)$ is a complex constant. Let $F=\frac{(a_mf^{n+m})^{(k)}}{P_1(z)}$ and $G=\frac{(a_mg^{n+m})^{(k)}}{P_1(z)}$. Then F and G are transcendental meromorphic functions that share the value 1 ignoring multiplicities. Proceeding in a similar manner as Case (i) above we obtain either $FG=1$ or $F=G$.
If $FG=1$, then

$$(a_mf^{n+m})^{(k)}(a_mg^{n+m})^{(k)}=P_1^2.$$

So by Lemma 8 we obtain $f(z)=b_1e^{bQ(z)}$, $g(z)=b_2e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $a_m^2((n+m)b)^2(b_1b_2)^{n+m}=-1$ and $Q(z)$ is same as in Theorem 1. If $F=G$, then using Lemmas 7 and 9 we obtain $f=tg$ for a constant t such that $t^{n+m}=1$.
Case (iii) Let $P(z)=C$. Taking $F=\frac{(Cf^n)^{(k)}}{P_1(z)}$, $G=\frac{(Cg^n)^{(k)}}{P_1(z)}$ and arguing similarly as in Case (ii) we obtain either $f(z)=b_1e^{bQ(z)}$, $g(z)=b_2e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $C^2(nb)^2(b_1b_2)^n=-1$, $Q(z)$ is same as in Theorem 1 or $f=tg$ for a constant t satisfying $t^n=1$. This completes the proof of the theorem. □

Proof of Theorem 1. Let $F=\frac{[f^n(\lambda f^m+\mu)]^{(k)}}{P_1(z)}$ and $G=\frac{[g^n(\lambda g^m+\mu)]^{(k)}}{P_1(z)}$. Then F and G are transcendental meromorphic functions that share the value 1 ignoring multiplicities. Then proceeding similarly as in Theorem 2 we obtain either $FG=1$ or $F=G$. First we assume that $\lambda\mu\neq 0$. Then $FG\not\equiv 1$, by Lemma 6. Hence $F=G$ and so by Lemmas 5 and 10 we obtain $f^d(z)=g^d(z)$ where $d=\gcd(n,m)$. Next we assume that $\lambda\mu=0$. Let $\lambda\neq 0$ and $\mu=0$. Then if $FG=1$, by Lemma 8 we have $f(z)=b_1e^{bQ(z)}$, $g(z)=b_2e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $\lambda^2((n+m)b)^2(b_1b_2)^{n+m}=-1$ and $Q(z)$ is same as in Theorem 1. Similar result holds when $\mu\neq 0$ and $\lambda=0$. If $F=G$, by Lemmas 7 and 9 we conclude that $f=tg$ for a constant t that satisfies $t^{n+m^*}=1$. This proves the theorem. □

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