

ON GENERALIZED JORDAN TRIPLE (α, β) -DERIVATIONS OF RINGS

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ABSTRACT. In the present paper, we study Jordan triple (α, β) -derivation, generalized Jordan triple (α, β) -derivation on ideal I of a prime ring R and find the necessary and sufficient conditions under which any Jordan triple (α, β) -derivation (resp. generalized Jordan triple (α, β) -derivation) becomes an (α, β) -derivation on ideal I (resp. a generalized (α, β) -derivation on I).

1. INTRODUCTION

Throughout the present paper R will denote an associative ring with center $Z(R)$. A ring R is said to be 2-torsion free, if whenever $2x = 0$, with $x \in R$, implies $x = 0$. Recall that R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that $a = 0$ or $b = 0$, and semiprime if $aRa = \{0\}$ implies $a = 0$. If $I \neq \{0\}$ is an ideal of a prime ring R , then it is straightforward to see that for $a, b \in I$; $aIb = \{0\}$ implies that $a = 0$ or $b = 0$. An additive subgroup L of R is said to be a Lie ideal if $[l, r] \in L$ for all $l \in L$ and $r \in R$. An additive mapping $d : R \rightarrow R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. A famous result due to Herstein [11, Theorem 3.3] state that a Jordan derivation of a prime ring of characteristic not equal to 2 must be a derivation. A brief proof of Herstein's result can also be found in [7]. This result was extended to 2-torsion free semiprime rings by Cusack [9] and subsequently, by Bresar [5].

Before comming to our main results, first we shall review some generalizations of the notion of derivation in rings. An additive mapping $F : R \rightarrow R$ is said to be a generalized derivation (resp. generalized Jordan derivation) on R if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ (resp. $F(x^2) = F(x)x + xd(x)$) holds for all $x, y \in R$. Following [4], an additive mapping $d : R \rightarrow R$ is called a Jordan triple derivation if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in R$. One can easily prove that any Jordan derivation of a 2-torsion free ring is a Jordan triple derivation (see for example [7] where further references can be found). An additive mapping $F : R \rightarrow R$ is said to be a generalized Jordan triple derivation on R if there exists a Jordan triple derivation $d : R \rightarrow R$ such

Date: Received: Dec 23, 2015; Accepted: Apr 2, 2016.

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2010 *Mathematics Subject Classification.* Primary 16W25; Secondary 16N60, 16U80.

Key words and phrases. Prime and semiprime rings, Jordan triple (α, β) -derivations, generalized Jordan triple (α, β) -derivations.

that $F(xyx) = F(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in R$. If $d : R \rightarrow R$ is an additive and if α and β are endomorphisms of R , then d is said to be an (α, β) -derivation of R when for all $x, y \in R$, $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$. Note that for I , the identity map on R , an (I, I) -derivation is just a derivation. An example of (α, β) -derivation when R has a nontrivial central idempotent e is to set $d(x) = ex$, $\alpha(x) = (1 - e)x$, and $\beta = I$ (or d) (formally). Here, d is not a derivation because $d(ee) = eee \neq 2eee = (ee)e + e(ee) = d(e)e + ed(e)$. In any ring with endomorphism α , if we set $d = I - \alpha$, then d is an (α, I) -derivation, but not a derivation when R is semiprime, unless $\alpha = I$. An additive map $F : R \rightarrow R$ is called a generalized (α, β) -derivation, for α and β endomorphisms of R , if there exists an (α, β) -derivation $d : R \rightarrow R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. Clearly, this notion include those of (α, β) -derivation when $F = d$, of derivation when $F = d$ and $\alpha = \beta = I$, and of generalized derivation, which is the case when $\alpha = \beta = I$. Maps of the form $F(x) = ax + xb$ for $a, b \in R$ with $d(x) = xb - bx$ and $\alpha = \beta = I$ are generalized derivations, and more generally, maps $F(x) = a\alpha(x) + \beta(x)b$ are generalized (α, β) -derivation. To see this, observe that $F(xy) = a\alpha(x)\alpha(y) + \beta(x)\beta(y)b = (a\alpha(x) + \beta(x)b)\alpha(y) + \beta(x)(\beta(y)b - b\alpha(y))$, and as we have just seen above, $d(x) = b\alpha(x) - \beta(x)b$ is an (α, β) -derivation of R . As for derivation, a generalized Jordan (α, β) -derivation F assumes $x = y$ in the definition above; that is, we assume only that $F(x^2) = F(x)\alpha(x) + \beta(x)d(x)$, holds for all $x \in R$.

Let α and β be endomorphisms of R . We call an additive mapping $d : R \rightarrow R$ as a Jordan triple (α, β) -derivation if $d(xyx) = d(x)\alpha(yx) + \beta(x)d(y)\alpha(x) + \beta(xy)d(x)$ holds for all $x, y \in R$. An additive mapping $F : R \rightarrow R$ is called a generalized Jordan triple (α, β) -derivation if there exists a Jordan triple (α, β) -derivation $d : R \rightarrow R$ such that $F(xyx) = F(x)\alpha(yx) + \beta(x)d(y)\alpha(x) + \beta(xy)d(x)$, holds for all $x, y \in R$. Clearly, this notion includes those of Jordan triple (α, β) -derivation when $F = d$, of Jordan triple derivation when $F = d$ and $\alpha = \beta = I$, and of generalized Jordan triple derivation which is the case $\alpha = \beta = I$. In [13], Jing and Lu proved that any generalized Jordan triple derivation on a 2-torsion free prime ring is a generalized derivation. Very recently, Vukman [17] extended the above mentioned result for 2-torsion free semiprime ring. However, till now, no attempt seems to have been made to extend Jing and Lu [13, Theorem 2.5] to the case when F is generalized Jordan triple (α, β) -derivation on R . In the present paper, it is shown that on a 2-torsion free prime ring every Jordan triple (α, β) -derivation on ideal I is an (α, β) -derivation on I and every generalized Jordan triple (α, β) -derivation on ideal I is a generalized (α, β) -derivation on I .

2. JORDAN TRIPLE (α, β) -DERIVATIONS

Throughout this section, R will denote a 2-torsion free ring with automorphisms α and β , and d a Jordan triple (α, β) -derivation. It is obvious to see that every derivation is a Jordan triple derivation, but the converse need not to be true in general (see Example 2.10 for $\alpha = \beta = I$, the identity map on R). In [4] Bresar proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a

derivation. Motivated by the result due to Brešar, in the present section it is shown that on a 2-torsion free prime ring R every Jordan triple (α, β) -derivation on ideal I is an (α, β) -derivation on I . More precisely, we prove the following:

Theorem 2.1. *Let R be a prime ring and I be a nonzero ideal of R . A mapping $d : R \rightarrow R$ is a Jordan triple (α, β) -derivation on I if and only if d is an (α, β) -derivation on I .*

To facilitate our discussion, we shall begin with the following lemmas:

Lemma 2.2. *Let R be a semiprime ring and I be a nonzero ideal of R . If $a, b \in R$ such that $a\alpha(xb) + \beta(bx)a = 0$, for all $x \in I$, then $a\alpha(xb) = \beta(bx)a = 0$, for all $x \in I$.*

Proof. We have

$$a\alpha(xb) + \beta(bx)a = 0 \quad \text{for all } x \in I. \quad (2.1)$$

Replacing x by xb in (2.1), we get

$$a\alpha(xb)\alpha(yb) + \beta(b)\beta(xby)a = 0 \quad \text{for all } x, y \in I. \quad (2.2)$$

Now application of (2.2), yields that

$$-\beta(bx)(a\alpha(yb) + \beta(bx)\beta(by)a) = 0. \quad (2.3)$$

Again, by (2.2), we obtain $\beta(bxby)a + \beta(bxby)a = 0$ that is, $2\beta(bxby)a = 0$. Since R is 2-torsion free, we get

$$\beta(bx)\beta(by)a = 0 \quad \text{for all } x, y \in R. \quad (2.4)$$

Taking $x = y\beta^{-1}(a)x$ in (2.4), we obtain $\beta(by)a\beta(x)\beta(by)a = 0$. Since β is onto, we find that $\beta(by)aR\beta(by)a = \{0\}$. Thus, semiprimeness of R yields that $\beta(by)a = 0$ for all $y \in I$. Similarly, one can show that $a\alpha(xb) = 0$ for all $x \in I$.

Lemma 2.3. *Let I be a nonzero ideal of R . Then For arbitrary $a, b, c \in I$, we have*

$$d(abc + cba) = d(a)\alpha(bc) + \beta(a)d(b)\alpha(c) + \beta(ab)d(c) + d(c)\alpha(ba) + \beta(c)d(b)\alpha(a) + \beta(cb)d(a)$$

Proof. Since d is a Jordan triple (α, β) -derivation on I , we have

$$d(aba) = d(a)\alpha(ba) + \beta(a)d(b)\alpha(a) + \beta(ab)d(a) \quad \text{for all } a, b \in I. \quad (2.5)$$

Compute, $W = d((a + c)b(a + c))$ in two different ways. On one hand, we find that $W = d(a + c)\alpha(b(a + c)) + \beta(a + c)d(b)\alpha(a + c) + \beta((a + c)b)d(a + c)$, and on the other hand $W = d(aba) + d(abc + cba) + d(cbc)$. Comparing two expressions we obtain the required result.

Remark 2.4. It is easy to see that every Jordan (α, β) -derivation of a 2-torsion free ring satisfies (2.5) (see [1] for reference).

For the purpose of this section for any $a, b, c \in I$ we shall write; $A(abc) = d(abc) - d(a)\alpha(bc) - \beta(a)d(b)\alpha(c) - \beta(ab)d(c)$, and $B(abc) = abc - cba$. We list few elementary properties of A and B :

- (i) $A(abc) + A(cba) = 0$
- (ii) $A((a + b)cd) = A(acd) + A(bcd)$ and $B((a + b)cd) = B(acd) + B(bcd)$

- (iii) $A(a(b+c)d) = A(abd) + A(acd)$ and $B(a(b+c)d) = B(abd) + B(acd)$
- (iv) $A(ab(c+d)) = A(abc) + A(abd)$ and $B(ab(c+d)) = B(abc) + B(abd)$

Proposition 2.5. *Let R be a semiprime ring and I be a nonzero ideal of R . If $A(abc) = 0$ holds for all $a, b, c \in I$, then d is an (α, β) -derivation of I .*

Proof. We have $A(abc) = 0$ for all $a, b, c \in I$, that is,

$$d(abc) = d(a)\alpha(bc) + \beta(a)d(b)\alpha(c) + \beta(ab)d(c).$$

Let $M = abxab$. We have

$$\begin{aligned} d(M) &= d(a(bxa)b) = d(a)\alpha(bxab) + \beta(a)d(b)\alpha(xab) + \beta(ab)d(x)\alpha(ab) \\ &\quad + \beta(abx)d(a)\alpha(b) + \beta(abxa)d(b) \text{ for all } x, a, b \in I. \end{aligned} \quad (2.6)$$

On the other hand,

$$d(M) = d((ab)x(ab)) = d(ab)\alpha(xab) + \beta(ab)d(x)\alpha(ab) + \beta(abx)d(ab). \quad (2.7)$$

Comparing (2.6) with (2.7) we get

$$\{d(ab) - d(a)\alpha(b) - \beta(a)d(b)\}\alpha(xab) + \beta(abx)\{d(ab) - d(a)\alpha(b) - \beta(a)d(b)\} = 0$$

that is, $a^b\alpha(xab) + \beta(abx)a^b = 0$, where a^b stands for $d(ab) - d(a)\alpha(b) - \beta(a)d(b)$.

Thus by Lemma 2.2 we get

$$a^b\alpha(xab) = 0 \text{ for all } x, a, b \in I. \quad (2.8)$$

Now on linearizing a we find that $a^b\alpha(xcb) + c^b\alpha(xab) = 0$, for all $a, b, c, x \in I$ i.e., ${}^b\alpha(xcb) = -c^b\alpha(xab)$ and hence $a^b\alpha(xcby)a^b\alpha(xcb) = -a^b\alpha(xcby)c^b\alpha(xab)$. This can be rewritten as $(a^b\alpha(xcb)\alpha(y)a^b\alpha(xcb) = -a^b\alpha(xcby\alpha^{-1}(c^b)x)\alpha(ab) = 0$, for all $a, b, c, x, y \in I$. This implies that $a^b\alpha(xcb)Ra^b\alpha(xcb) = \{0\}$ and the semiprimeness of R forces that $a^b\alpha(xcb) = 0$. Similarly, on linearizing b , we obtain $a^b\alpha(xcd) = 0$, for all $a, b, c, d, x \in I$. In particular $a^b\alpha(xc)Ra^b\alpha(xc) = \{0\}$. Again semiprimeness of R forces that $a^b\alpha(xc) = 0$, for all $a, b, c, x \in I$ which leads to $a^b = 0$, for all $a, b \in I$. This proves that $d(ab) = d(a)\alpha(b) + \beta(a)d(b)$, holds for all $a, b \in I$ i.e., d is an (α, β) -derivaton on I .

Lemma 2.6. *Let R be a semiprime ring and I be a nonzero ideal of R . For any $a, b, c, x \in I$, we have*

$$A(abc)\alpha(xB(abc)) = 0.$$

Proof. For any $a, b, c, x \in I$, suppose that $N = abxcba + cbaxabc$. Now we find

$$\begin{aligned} d(N) &= d(a(bxcba)a + c(baxab)c) = d(a(bxcba)a) + d(c(baxab)c) \\ &= d(a)\alpha(bxcba) + \beta(a)d(b)\alpha(cxba) + \beta(ab)d(c)\alpha(xcba) \\ &\quad + \beta(abc)d(x)\alpha(cba) + \beta(abcx)d(c)\alpha(ba) + \beta(abxc)d(b)\alpha(a) \\ &\quad + \beta(abxcba)d(a) + d(c)\alpha(baxabc) + \beta(c)d(b)\alpha(axabc) \\ &\quad + \beta(cb)d(a)\alpha(xabc) + \beta(cba)d(x)\alpha(abc) + \beta(cbax)d(a)\alpha(bc) \\ &\quad + \beta(cbaxa)d(b)\alpha(c) + \beta(cbaxab)d(c). \end{aligned}$$

On the other hand we have

$$\begin{aligned} d(N) &= d((abc)x(cba) + (cba)x(abc)) \\ &= d(abc)\alpha(xcba) + \beta(abc)d(x)\alpha(cba) + \beta(abcx)d(cba) \\ &\quad + d(cba)\alpha(xabc) + \beta(cba)d(x)\alpha(abc) + \beta(cbax)d(abc). \end{aligned}$$

On comparing last two expressions we get

$$-A(cba)\alpha(xcba) + A(cba)\alpha(xabc) + \beta(abcx)A(cba) - \beta(cbax)A(cba) = 0.$$

This implies that $A(abc)\alpha(xB(abc)) + \beta(B(abc)x)A(abc) = 0$. Hence by Lemma 2.2, we get the required result.

Lemma 2.7. *Let R be a semiprime ring and I be a nonzero ideal of R . Then $A(abc)\alpha(xB(rst)) = 0$ holds for all $a, b, c, r, s, t, x \in I$.*

Proof. By Lemma 2.6 we have $\alpha^{-1}(A(a+r)bc)xB((a+r)bc) = 0$ for all $a, b, c, r, x \in I$. This implies that $\alpha^{-1}(A(abc))xB(rbc) + \alpha^{-1}(A(rbc))xB(abc) = 0$, and so

$$\begin{aligned} (A(abc)\alpha(xB(rbc)))\alpha(y)(A(abc)\alpha(xB(rbc))) &= -(A(abc)\alpha(xB(rbc)))\alpha(y)(A(rbc)\alpha(xB(abc))) \\ &= -(A(abc)\alpha(xB(rbc))y\alpha^{-1}(A(rbc)x)\alpha(B(abc))) \\ &= 0 \text{ for all } a, b, c, x, y \in I. \end{aligned}$$

Therefore, by semiprimeness of R , we obtain $A(abc)\alpha(xB(rbc)) = 0$. Similarly, we get $A(abc)\alpha(xB(rsc)) = 0$ and furthermore $A(abc)\alpha(xB(rst)) = 0$, for all $a, b, c, r, s, t, x \in I$.

Lemma 2.8. *Let R be a semiprime ring and I be a nonzero ideal of R . If $B(rst) = 0$ holds for all $r, s, t \in I$, then $I \subseteq Z(R)$.*

Proof. We have $B(rst) = 0$, for all $r, s, t \in I$, then $rst = tsr$. Now replacing t by ty and using $B(rst) = 0$, we get $[r, t]sy + r[s, t]y = 0$ for all $r, s, t, y \in I$. Again relace r by xr , to get $[x, t]rsy = 0$ for all $r, s, t, x, y \in I$ and hence $[x, t]rI[x, t]r = \{0\}$ for all $x, r, t \in I$. Thus, since primeness of R forces that $[x, t]r = 0$ for all $x, t, r \in I$ and hence $[x, t]r[x, t] = \{0\}$. Since R is a semiprime, we find that $[x, t] = 0$ for all $x, t \in I$ i.e. $I \subseteq Z(R)$.

Remark 2.9. It is to note that in the proof of Lemma 2.2 of [1] if we assume that the associated (α, β) -derivation d with F is a Jordan (α, β) -derivation, even then the proof of Lemma 2.2 of [1] follows exactly on the same lines. Hence, we shall use Lemma 2.2 of [1] frequently in the present situation.

Proof of Theorem 2.1. Suppose $d : R \rightarrow R$ is Jordan triple (α, β) -derivation on I . By Lemma 2.7, we have $A(abc)\alpha(xB(rst)) = 0$ for all $a, b, c, x, r, s, t \in I$ that is, $\alpha^{-1}(A(abc))xB(rst) = 0$ and hence $\alpha^{-1}(A(abc))IB(rst) = \{0\}$ for all $a, b, c, r, s, t \in I$. Thus primeness of R forces that either $\alpha^{-1}(A(abc)) = 0$ or $B(rst) = 0$. If $\alpha^{-1}(A(abc)) = 0$ for all $a, b, c \in I$, then $A(abc) = 0$ and hence by Proposition 2.5, we get the required result.

On the other hand, if $B(rst) = 0$, for all $r, s, t \in I$, then by Lemma 2.8, $I \subseteq Z(R)$ and since R is prime, so we obtain R is commutative. Now let $P =$

$a^3bc + cba^3$, then

$$\begin{aligned} d(P) &= d(a^3bc + cba^3) \\ &= d(a^3)\alpha(bc) + \beta(a^3)d(b)\alpha(c) + \beta(a^3b)d(c) + d(c)\alpha(ba^3) \\ &\quad + \beta(c)d(b)\alpha(a^3) + \beta(cb)d(a^3). \end{aligned}$$

On the other hand

$$\begin{aligned} d(P) &= d(abcaa + aaabc) \\ &= d(abc)\alpha(a^2) + \beta(abc)d(a)\alpha(a) + \beta(abca)d(a) + d(a)\alpha(a^2bc) \\ &\quad + \beta(a)d(a)\alpha(abc) + \beta(a^2)d(abc). \end{aligned}$$

These two expressions yield that

$$\{d(abc) - d(c)\alpha(ba) - \beta(c)d(b)\alpha(a) - \beta(cb)d(a)\}\alpha(a^2) +$$

$$\beta(a^2)\{d(abc) - d(a)\alpha(bc) - \beta(a)d(b)\alpha(c) - \beta(ab)d(c)\} = 0,$$

that is, $\beta(a^2)A(abc) - A(abc)\alpha(a^2) = 0$ for all $a, b, c \in I$. Since $I \subseteq Z(R)$, we find that $A(abc)(\beta - \alpha)(a^2) = 0$, and hence $A(abc)(\beta - \alpha)(a)(\beta - \alpha)(a) = 0$, for all $a, b, c \in I$. Now $(A(abc)(\beta - \alpha)(a))x(A(abc)(\beta - \alpha)(a)) = (A(abc)(\beta - \alpha)(a^2))xA(abc) = 0$. Thus primeness of R forces that $A(abc)((\beta - \alpha)(a)) = 0$. A linearization of the above expression with respect to a gives that $A(abc)((\beta - \alpha)(d)) + A dbc)((\beta - \alpha)(a)) = 0$, for all $a, b, c, d \in I$ and hence

$$\begin{aligned} (A(abc)((\beta - \alpha)(d)))x(A(abc)((\beta - \alpha)(d))) &= -(A dbc)((\beta - \alpha)(a)))x(A(abc)((\beta - \alpha)(d))) \\ &= -A dbc)((\beta - \alpha)(a))xA(abc)((\beta - \alpha)(d)) \\ &= -A dbc)((\beta - \alpha)(d))((\beta - \alpha)(a))xA(abc)) \\ &= 0. \end{aligned}$$

Again primeness of R gives that $A(abc)((\beta - \alpha)(d)) = 0$, for all $a, b, c, d \in I$. For any $r \in R$, we get $A(abc)r((\beta - \alpha)(d)) = 0$, that is, $A(abc)R((\beta - \alpha)(d)) = \{0\}$, for all $a, b, c, d \in I$. Thus, primeness of R gives that either $A(abc) = 0$, for all $a, b, c \in I$ or $(\beta - \alpha)(d) = 0$, for all $d \in I$. If $A(abc) = 0$, then by Proposition 2.5, we get the required result. On the other hand if $\beta(d) = \alpha(d)$, for all $d \in I$, then $\alpha = \beta$. Since R is commutative and $\alpha = \beta$, by Lemma 2.3, we find that

$$d(abc) = d(a)\alpha(bc) + \beta(a)d(b)\alpha(c) + \beta(ab)d(c) \quad \text{for all } a, b, c \in I,$$

that is, $A(abc) = 0$ for all $a, b, c \in I$. Thus, by Proposition 2.5, we get the required result.

Conversely, if R admits an (α, β) -derivation d , then d is a Jordan (α, β) -derivation. In view of [1, Lemma 2.2 (ii)] (for $F = d$ and $U = I$), we have $d(xyx) = d(x)\alpha(yx) + \beta(x)d(y)\alpha(x) + \beta(xy)d(x)$ for all $x, y \in I$ i.e., every Jordan (α, β) -derivation on a 2-torsion free ring is a Jordan triple (α, β) -derivation on I . This completes the proof of the theorem.

Now it is natural to question that whether the primeness assumption in Theorem 2.1 is essential. We close this section with the following example which shows that the mentioned condition is necessary:

Example 2.10. Let S be a any ring. Next, let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in S \right\}$.

Define maps $d : R \rightarrow R$ and $\alpha, \beta : R \rightarrow R$ as follows: $d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$, $\beta \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$.

Then, it is straightforward to chek that d is a Jordan triple (α, β) -derivation but not an (α, β) -derivation.

3. GENERALIZED JORDAN TRIPLE (α, β) -DERIVATIONS

Throughout this section, R will represent a 2-torsion free ring and α, β be automorphisms of R . Suppose that R admits a generalized Jordan triple (α, β) -derivation F with associated Jordan triple (α, β) -derivation d . Recently, Jing and Lu [13] introduced a concept of generalized Jordan triple derivation and proved that every generalized Jordan triple derivation on a 2-torsion free prime ring is a generalized derivation. The aim of the present section is to extend the result of Jing and Lu [13] for generalized Jordan triple (α, β) -derivation. In fact, we prove the following:

Theorem 3.1. *Let R be a 2-torsion free prime ring and I be a nonzero ideal of R . Then a mapping $F : R \rightarrow R$ is a generalized Jordan triple (α, β) -derivation on I if and only if F is a generalized (α, β) -derivation on I .*

The following lemmas are required for developing the proof of the above theorem:

Lemma 3.2. *For arbitrary $a, b, c \in I$, we have*

$$F(abc + cba) = F(a)\alpha(bc) + \beta(a)d(b)\alpha(c) + \beta(ab)d(c) + F(c)\alpha(ba) + \beta(c)d(b)\alpha(a) + \beta(cb)d(a).$$

Proof. Using similar arguments as used in the proof of Lemma 2.3, we obtain the assertion of the lemma.

For the purpose of this section we write; $H(abc) = F(abc) - F(a)\alpha(bc) - \beta(a)d(b)\alpha(c) - \beta(ab)d(c)$ and $B(abc) = abc - cba$. The elementary properties of H and B are same as given below Remark 2.4 for A in the previous section.

Lemma 3.3. *Let R be a prime ring and I be a nonzero ideal of R . Then for any $a, b, c, x \in I$,*

$$H(abc)\alpha(xB(abc)) = 0.$$

Proof. Since R is prime, it follows from Theorem 2.1 that d is an (α, β) -derivation on I . Let $W = F(abcxcba + cbaxabc)$ and compute W in two different ways, to get the result.

Proposition 3.4. *Let R be a prime ring and I be a nonzero ideal of R . If $H(abc) = 0$, holds for all $a, b, c \in I$, then F is a generalized (α, β) -derivation on R .*

Proof. We have $H(abc) = 0$, for all $a, b, c \in I$. This can be rewritten as

$$F(abc) = F(a)\alpha(bc) + \beta(a)d(b)\alpha(c) + \beta(ab)d(c).$$

We compute $M = abxab$ in two different ways. On one hand

$$\begin{aligned} F(M) &= F(a)\alpha(bxab) + \beta(a)d(b)\alpha(xab) + \beta(ab)d(x)\alpha(ab) \\ &\quad + \beta(abx)d(a)\alpha(b) + \beta(abxa)d(b). \end{aligned}$$

On the other hand

$$F(M) = F(ab)\alpha(xab) + \beta(ab)d(x)\alpha(ab) + \beta(abx)d(ab).$$

In view of Theorem 2.1, the last two expressions yield that

$$\{F(ab) - F(a)\alpha(b) - \beta(a)d(b)\}\alpha(xab) = 0 \text{ for all } a, b, x \in I.$$

Following notation, $F_a^b = F(ab) - F(a)\alpha(b) - \beta(a)d(b)$, we get $F_a^b\alpha(xab) = 0$, for all $x, a, b \in I$. Thus, using similar arguments as used in the proof of Proposition 2.5, after equation (2.8), we get $F_a^b = 0$, for all $a, b \in I$ that is, F is a generalized (α, β) -derivation on I .

Lemma 3.5. *Let R be a prime ring and I be a nonzero ideal of R . Then for any $a, b, c, x, r, s, t \in I$,*

$$H(abc)\alpha(xB(rst)) = 0.$$

Proof. Use similar techniques as used in the proof of Lemma 2.7.

Now, we are ready to prove our theorem:

Proof of Theorem 3.1. Suppose that $F : R \rightarrow R$ is a generalized Jordan triple (α, β) -derivation with associated Jordan triple (α, β) -derivation d on I . By Lemma 3.5, we find that $H(abc)\alpha(xB(rst)) = 0$, for all $a, b, c, x, r, s, t \in I$ that is, $\alpha^{-1}(H(abc))xB(rst) = 0$ and hence $\alpha^{-1}(H(abc))IB(rst) = \{0\}$. Thus primeness of R gives that either $\alpha^{-1}(H(abc)) = 0$ or $B(rst) = 0$. If $\alpha^{-1}(H(abc)) = 0$, for all $a, b, c \in I$, then $H(abc) = 0$ and hence by Proposition 3.4, we get the required result.

On the other hand if $B(rst) = 0$, for all $r, s, t \in I$, then by Lemma 2.8 R is commutative.

Now let $P = a^3bc + cba^3$, then

$$\begin{aligned} F(P) &= F(a^3bc + cba^3) \\ &= F(a^3)\alpha(bc) + \beta(a^3)d(b)\alpha(c) + \beta(a^3b)d(c) + F(c)\alpha(ba^3) \\ &\quad + \beta(c)d(b)\alpha(a^3) + \beta(cb)d(a^3). \end{aligned}$$

On the other hand

$$\begin{aligned} F(P) &= F(abcaa + aaabc) \\ &= F(abc)\alpha(a^2) + \beta(abc)d(a)\alpha(a) + \beta(abca)d(a) + F(a)\alpha(a^2bc) \\ &\quad + \beta(a)d(a)\alpha(abc) + \beta(a^2)d(abc). \end{aligned}$$

From Theorem 2.1, the last two expressions yield that

$$(F(abc) - F(c)\alpha(ba) - \beta(c)d(b)\alpha(a) - \beta(cb)d(a))\alpha(a^2) = 0,$$

for all $a, b, c \in I$ i.e., $H(abc)\alpha(a^2) = 0$. Now we have

$$(H(abc)\alpha(a))x(H(abc)\alpha(a)) = H(abc)\alpha(a^2)(xH(abc)) = 0, \text{ for all } a, b, c, x \in I.$$

Thus primeness of R forces that $H(abc)\alpha(a) = 0$, and hence replace a by $a + d$, to get $H(abc)\alpha(d) + H(dbc)\alpha(a) = 0$, for all $a, b, c, d \in I$. This implies that

$$\begin{aligned} (H(abc)\alpha(d))x(H(abc)\alpha(d)) &= -H(abc)\alpha(d)x(H(dbc)\alpha(a)) \\ &= -H(abc)\alpha(a)(\alpha(d)xH(dbc)) = 0, \text{ for all } a, b, c, d, x \in I. \end{aligned}$$

Again, by the primeness of R we find that $H(abc)\alpha(d) = 0$. Furthermore $H(abc)\alpha(d)H(abc) = 0$ and so $H(abc) = 0$, for all $a, b, c \in I$. Thus by Proposition 3.4, we get the required result.

Conversely, suppose that F is a generalized (α, β) -derivation on I . Since every generalized (α, β) -derivation is a generalized Jordan (α, β) -derivation on R , in view of Lemma 2.2 (ii) of [1] (for $U = I$), F is a generalized Jordan triple (α, β) -derivation on I . This completes the proof of the theorem.

In view of the above theorem, we get the following corollary:

Corollary 3.6. ([13, Theorem 3.5]). *Let R be a 2-torsion free prime ring. Then every generalized Jordan triple derivation on R is a generalized derivation on R .*

The following example shows that the primeness in the hypotheses of above theorem can not be omitted:

Example 3.7. Consider the rings S , R and automorphisms α, β , as in Example

2.1. Define a map $F : R \longrightarrow R$ such that $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Then, we can

find an associated Jordan triple (α, β) -derivation $d : R \longrightarrow R$ such that

$$d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \text{ It can be easily seen that } F \text{ is a generalized}$$

Jordan triple (α, β) -derivation but not a generalized (α, β) -derivation.

4. GENERALIZED (α, β) -DERIVATIONS IN SEMIPRIME RINGS

Following [21], an additive mapping $T : R \longrightarrow R$ is called a left (resp. right) centralizer if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. An additive mapping $T : R \longrightarrow R$ is called a Jordan left (resp. right) centralizer if $T(x^2) = T(x)x$ (resp. $T(x^2) = xT(x)$) holds for all $x \in R$. If T is both left as well right centralizer, then T is a centralizer. Every left (right) centralizer is a Jordan left(right) centralizer. The converse is not true in general. In [21], Zalar proved that every Jordan left (right) centralizer on a 2-torsion free semiprime ring is a left(right) centralizer. Considerable work has been done on Jordan left(right)

centralizers in prime and semiprime rings during the last couple of decades (see for example; [18, 19, 20] where further references can be found). In [16], Molnar proved that if R is a 2-torsion free prime ring and $T : R \rightarrow R$ is an additive function such that $T(xy) = T(x)y$ for all $x, y \in R$, then T is a left(right) centralizer on R . In the Year 2007, Daif and Tamman [10] showed that if R is a 2-torsion free semiprime ring and $F : R \rightarrow R$ is an additive mapping such that $F(xy) = F(x)y + x d(y)$, holds for all $x, y \in R$, then F is a generalized Jordan derivation on R . The main result of this section is in the spirits of the above result which states as follows:

Theorem 4.1. *Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R . Let α, β be endomorphisms of R such that α is one-one and onto, and $d : R \rightarrow R$ is an (α, β) -derivation of I . Suppose that $F : R \rightarrow R$ is an additive mapping on I . Then $F(x^2) = F(x)\alpha(x) + \beta(x)d(x)$ if and only if $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$, for all $x, y \in I$.*

We begin this section with the following known lemmas. The proofs of both lemmas can be looked into [14].

Lemma 4.2. ([14, Theorem 2]) *Let R be a n -torsion free semiprime ring and let α, β be endomorphisms of R . Let d be a Jordan (α, β) -derivation of R such that either α one-one and onto or β one-one and onto. Then d is an (α, β) -derivation of R .*

Lemma 4.3. ([14, Theorem 3]) *For $n \geq 2$ let R be a 2-torsion free, semiprime with identity 1 and $F, d : R \rightarrow R$ be additive, let β be an endomorphism of R with $\beta(1) = 1$, and let α be an automorphism of R . Assume for all $x \in R$, $F(x^n) = F(x)\alpha(x^{n-1}) + \sum_{j=1}^{n-1} \beta(x^j)d(x)\alpha(x^{n-j-1})$, where $\alpha(x^0) = 1 = \beta(x^0)$. Then d is an (α, β) -derivation of R and F is a generalized (α, β) -derivation of R with respect to d . Further, if d is assumed to be (α, β) -derivation of R , then one needs to assume only that α is unital endomorphism of R .*

Proof of Theorem 4.1. We have

$$F(xy) = F(x)\alpha(y) + \beta(x)d(y) \text{ for all } x, y \in I. \quad (4.1)$$

A straightforward linearization on x yields that, for all $x, y, z \in I$,

$$F(xyz + zyx) = F(x)\alpha(yz) + \beta(z)d(yx) + F(z)\alpha(yx) + \beta(x)d(yz). \quad (4.2)$$

Replacing z by x^2 in (4.1), we get

$$F(xy x^2 + x^2 yx) = F(x)\alpha(yx^2) + \beta(x^2)d(yx) + F(x^2)\alpha(yx) + \beta(x)d(yx^2). \quad (4.3)$$

Now, replace y by $xy + yx$ in the relation (4.1), to get

$$F(x^2 yx + xy x^2) = F(x)\alpha(xy x) + F(x)\alpha(yx^2) + \beta(x)d(xy x) + \beta(x)d(yx^2). \quad (4.4)$$

From (4.3) and (4.4), we obtain

$$\{F(x^2) - F(x)\alpha(x) - \beta(x)d(x)\}\alpha(yx) = 0, \text{ for all } x, y \in I$$

that is,

$$H(x)\alpha(yx) = 0 \text{ for all } x, y \in I, \quad (4.5)$$

where $H(x)$ stands for $F(x^2) - F(x)\alpha(x) - \beta(x)d(x)$. Our intention is to prove that F is a Jordan generalized (α, β) -derivation that is,

$$H(x) = 0 \text{ for all } x \in I. \quad (4.6)$$

Again replace y by $xy\alpha^{-1}(H(x))$ in (4.5), to get $H(x)\alpha(xy)H(x)\alpha(x) = 0$ for all $x, y \in I$ that is, $H(x)\alpha(x)RH(x)\alpha(x) = \{0\}$ for all $x \in I$, and the semiprimeness of R yields that

$$H(x)\alpha(x) = 0 \text{ for all } x \in I. \quad (4.7)$$

Now, multiplying the relation (4.5) from the left side by $\alpha(x)$ and from the right by $H(x)$, we obtain $\alpha(x)H(x)\alpha(y)\alpha(x)H(x) = 0$ for all $x, y \in I$ that is, $\alpha(x)H(x)R\alpha(x)H(x) = \{0\}$, for all $x \in I$. Thus, again by the semiprimeness of R , it follows that

$$\alpha(x)H(x) = 0 \text{ for all } x \in I. \quad (4.8)$$

The linearization of (4.7) gives that

$$\{F(x+y)^2 - F(x+y)\alpha(x+y) - \beta(x+y)d(x+y)\}\alpha(x+y) = 0 \text{ for all } x, y \in I.$$

This can be rewritten as

$$H(x)\alpha(y) + \delta(x, y)\alpha(x) + H(y)\alpha(x) + \delta(x, y)\alpha(y) = 0 \text{ for all } x, y \in I, \quad (4.9)$$

where $\delta(x, y)$ denotes $F(xy + yx) - F(x)\alpha(y) - F(y)\alpha(x) - \beta(x)d(y) - \beta(y)d(x)$. Now, replacing x by $-x$ in (4.9), we get

$$H(x)\alpha(y) + \delta(x, y)\alpha(x) - H(y)\alpha(x) - \delta(x, y)\alpha(y) = 0. \quad (4.10)$$

Adding (4.9) and (4.10) and using the fact that R is 2-torsion free, we find that

$$H(x)\alpha(y) + \delta(x, y)\alpha(x) = 0 \text{ for all } x, y \in I.$$

Now, multiply the above equation by $H(x)$ from right and use (??) to get $H(x)\alpha(y)H(x) = 0$ for all $x, y \in I$ that is, $H(x)RH(x) = \{0\}$ for all $x \in I$. Since R is semiprime, we find that $H(x) = 0$ for all $x \in I$. This proves that $F(x^2) = F(x)\alpha(x) + \beta(x)d(x)$ holds for all $x \in I$. In other words, F is a generalized Jordan (α, β) -derivation.

Conversely, suppose that F is a generalized Jordan (α, β) -derivation on R . Since R is 2-torsion free ring and F is a generalized Jordan (α, β) -derivation, by Lemma 2.2(ii) of [1] for $U = R$, we get the required result. This completes the proof of the theorem.

In view of Theorem 4.11 and Lemma 4.3, we get the following result:

Corollary 4.4. *Let R be a 2-torsion free semiprime with identity 1, β be an endomorphism of R with $\beta(1) = 1$, and let α be an automorphism of R . Suppose that $d : R \rightarrow R$ is an (α, β) -derivation of R . If an additive mapping $F : R \rightarrow R$ satisfies $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$, for all $x, y \in R$, then $F(xy) = F(x)\alpha(x) + \beta(x)d(y)$, for all $x, y \in R$.*

Setting $F = d$ in Theorem 4.1 and using Lemma 4.2, we obtain the following corollary:

Corollary 4.5. *Let R be a 2-torsion free semiprime ring. Suppose that α, β be endomorphisms of R such that α is one-one and onto. If an additive mapping $d : R \longrightarrow R$ satisfies $d(xyx) = d(x)\alpha(yx) + \beta(x)d(yx)$, for all $x, y \in R$, then $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$, for all $x, y \in R$.*

It is clear that if we use the derivation d to be zero derivation and α to be an identity map on R in Theorem 4.1, then we get the following corollary which generalizes the result of Molnar [16].

Corollary 4.6. *Let R be a 2-torsion free semiprime ring and let $T : R \longrightarrow R$ be an additive mapping. If $T(xyx) = T(x)yx$, holds for all $x, y \in R$, then T is a left centralizer.*

The following example demonstrates that Theorem 4.1 is not true in the case of arbitrary rings with $\alpha = \beta = I$ and $d = 0$.

Example 4.7. Let S be a ring such that the square of each element in S is zero, but the product of some elements in S is non zero. Next, let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$.

Define a map $F : R \longrightarrow R$ such that $F \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. Then F is generalized Jordan derivation but $F(xyx) \neq F(x)yx$ for some $x, y \in R$.

In conclusion, it is tempting to conjecture as follows:

Conjecture 4.8. Let R be a semiprime ring and L be a Lie ideal of R with suitable torsion restrictions. Suppose that α, β are endomorphisms of R and d is a Jordan triple (α, β) -derivation on L . If $F : R \longrightarrow R$ is a generalized Jordan triple (α, β) -derivation on L , then F is a generalized (α, β) -derivation on L .

Acknowledgement. This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Under grant No. (363- 157-D1435). The authors , therefore acknowledge technical and financial support of KAU.

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