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GENERALIZATION OF TITCHMARSH'S THEOREM IN THE SPACE $L^2(\mathbb{R}, A_{\alpha,\beta}(x)dx)$

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ABSTRACT. In this paper, using a generalized Jacobi-Dunkl translation operator, we prove an analog of Titchmarsh's theorem for functions satisfying the ϕ-Jacobi-Dunkl Lipschitz condition in $L^2(\mathbb{R}, A_{\alpha,\beta}(x)dx), \alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$.

1. INTRODUCTION

Titchmarsh's ([10],Theorem 85) characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz Condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

Theorem 1.1. [10] *Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents*

(a) $\|f(x+h) - f(x)\| = 0(h^\alpha), \quad as \quad h \to 0,$

(b) $\displaystyle\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha}), \quad as \quad r \to \infty,$

where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we prove in analog of Theorem 1.1 for the Jacobi-Dunkl transform for functions satisfying the ϕ-Jacobi-Dunkl Lipschitz condition in the space $L^2(\mathbb{R}, A_{\alpha,\beta}(x)dx)$. For this purpose, we use the generalized translation operator. Similar results have been established in the context of non compact rank one Riemannian symetric spaces [9].
In section 2 below, we recapitulate from [1, 2, 3, 5] some results related to the harmonic analysis associated with Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$.
Section 3 is devoted to the main result after defining the class $Lip(\phi, 2, \alpha, \beta)$ of functions in $L^2_{\alpha,\beta}(\mathbb{R})$ satisfying the ϕ-Jacobi-Dunkl Lipschitz condition correspondent to the generalized Jacobi-Dunkl translation.

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2. Notation and Preliminaries

The Jacobi-Dunkl function with parameters $(\alpha,\beta), \alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$, defined by the formula

$$\forall x \in \mathbb{R}, \psi_\lambda^{\alpha,\beta}(x) = \begin{cases} \varphi_\mu^{\alpha,\beta}(x) - \frac{i}{\lambda}\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x), & \text{if } \lambda \in \mathbb{C}\backslash\{0\} \\ 1, & \text{if } \lambda = 0 \end{cases}$$

with $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha + \beta + 1$ and $\varphi_\mu^{\alpha,\beta}$ is the Jacobi function given by

$$\varphi_\mu^{\alpha,\beta}(x) = F\left(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}, \alpha + 1, -(\sinh(x))^2\right),$$

F is the Gausse hypergeometric function (see [1, 6, 7]).
$\psi_\lambda^{\alpha,\beta}$ is the unique C^∞-solution on $\mathbb{R}$ of the differentiel-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta}\mathcal{U} = i\lambda\mathcal{U} \quad, \lambda \in \mathbb{C} \\ \mathcal{U}(0) = 1 \end{cases}$$

where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by

$$\Lambda_{\alpha,\beta}\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + [(2\alpha+1)\coth x + (2\beta+1)\tanh x] \times \frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}].$$

The operator $\Lambda_{\alpha,\beta}$ is a particular case of the operator D given by

$$D\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + \frac{A'(x)}{A(x)} \times \left(\frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}\right),$$

where $A(x) = |x|^{2\alpha+1}B(x)$, and B a function of class C^∞ on $\mathbb{R}$, even and positive. The operator $\Lambda_{\alpha,\beta}$ corresponds to the function

$$A(x) = A_{\alpha,\beta}(x) = 2^\rho(\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

Using the relation

$$\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) = -\frac{\mu^2+\rho^2}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x),$$

the function $\psi_\lambda^{\alpha,\beta}$ can be written in the form above (see [2])

$$\forall x \in \mathbb{R}, \psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + i\frac{\lambda}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x).$$

Denote $L^2_{\alpha,\beta}(\mathbb{R}) = L^2_{\alpha,\beta}(\mathbb{R}, A_{\alpha,\beta}(x)dx)$ the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{L^2_{\alpha,\beta}(\mathbb{R})} = \left(\int_\mathbb{R} |f(x)|^2 A_{\alpha,\beta}(x)dx\right)^{1/2} < +\infty.$$

Using the eigenfunctions $\psi_\lambda^{\alpha,\beta}$ of the operator $\Lambda_{\alpha,\beta}$ called the Jacobi-Dunkl kernels, we define the Jacobi-Dunkl transform of a function $f \in L^2_{\alpha,\beta}(\mathbb{R})$ by

$$\mathcal{F}_{\alpha,\beta}f(\lambda) = \int_\mathbb{R} f(x)\psi_\lambda^{\alpha,\beta}(x)A_{\alpha,\beta}(x)dx, \quad \lambda \in \mathbb{R},$$

and the inversion formula

$$f(x)=\int_{\mathbb{R}}\mathcal{F}_{\alpha,\beta}f(\lambda)\psi_{-\lambda}^{\alpha,\beta}(x)d\sigma(\lambda),$$

where

$$d\sigma(\lambda)=\frac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}|C_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2})|}1_{\mathbb{R}\backslash]-\rho,\rho[}(\lambda)d\lambda.$$

Here,

$$C_{\alpha,\beta}(\mu)=\frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))},\quad \mu\in\mathbb{C}\backslash(i\mathbb{N}),$$

and $1_{\mathbb{R}\backslash]-\rho,\rho[}$ is the characteristic function of $\mathbb{R}\backslash]-\rho,\rho[$.
The Jacobi-Dunkl transform is a unitary isomorphism from $L^2_{\alpha,\beta}(\mathbb{R})$ onto $L^2(\mathbb{R},d\sigma(\lambda))$, i.e.,

$$\|f\|:=\|f\|_{L^2_{\alpha,\beta}(\mathbb{R})}=\|\mathcal{F}_{\alpha,\beta}(f)\|_{L^2(\mathbb{R},d\sigma(\lambda))}. \tag{2.1}$$

The operator of Jacobi-Dunkl translation is defined by

$$T_xf(y)=\int_{\mathbb{R}}f(z)d\nu_{x,y}^{\alpha,\beta}(z),\quad \forall x,y\in\mathbb{R},$$

where $\nu_{x,y}^{\alpha,\beta}(z),\quad x,y\in\mathbb{R}$ are the signed measures given by

$$d\nu_{x,y}^{\alpha,\beta}(z)=\begin{cases}K_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz, & \text{if } x,y\in\mathbb{R}^*\\ \\ \delta_x, & \text{if } y=0\\ \delta_y, & \text{if } x=0\end{cases}$$

Here, δ_x is the Dirac measure at x. And,

$$\begin{aligned}K_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}(\sinh(|x|)\sinh(|y|)\sinh(|z|))^{-2\alpha}\mathbb{I}_{I_{x,y}}\times\int_0^\pi\rho_\theta(x,y,z)\\ &\times\ (g_\theta(x,y,z))_+^{\alpha-\beta-1}sin^{2\beta}\theta d\theta\\ I_{x,y} &= [-|x|-|y|,-||x|-|y||]\cup[||x|-|y||,|x|+|y|]\\ \rho_\theta(x,y,z) &= 1-\sigma_{x,y,z}^\theta+\sigma_{z,x,y}^\theta+\sigma_{z,y,x}^\theta,\end{aligned}$$

$$\forall z\in\mathbb{R},\theta\in[0,\pi],\sigma_{x,y,z}^\theta=\begin{cases}\frac{\cosh(x)+\cosh(y)-\cosh(z)\cos(\theta)}{\sinh(x)\sinh(y)}, & \text{if } xy\neq0\\ \\ 0, & \text{if } xy=0\end{cases}$$

$$g_\theta(x,y,z)=1-\cosh^2(x)-\cosh^2(y)-\cosh^2(z)+2\cosh(x)\cosh(y)\cosh(z)\cos\theta$$

$$t_+=\begin{cases}t, & \text{if } t>0\\ \\ 0, & \text{if } t\leq0\end{cases}$$

and,

$$M_{\alpha,\beta}=\begin{cases}\frac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}, & \text{if } \alpha>\beta\\ \\ 0, & \text{if } \alpha=\beta\end{cases}$$

In [2], we have

$$\mathcal{F}_{\alpha,\beta}(T_h f)(\lambda) = \psi_\lambda^{\alpha,\beta}(h)\mathcal{F}_{\alpha,\beta}(f)(\lambda), \tag{2.2}$$
$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta} f)(\lambda) = i\lambda\mathcal{F}_{\alpha,\beta}(f)(\lambda). \tag{2.3}$$

For $\alpha \geq \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind defined by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n}}{n!\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}.$$

Moreover, we see that

$$\lim_{x\to 0}\frac{j_\alpha(x)-1}{x^2} \neq 0,$$

by consequence, there exists $C_1 > 0$ and $\eta > 0$ satisfying

$$|x| \leq \eta \Rightarrow |j_\alpha(x) - 1| \geq C_1|x|^2. \tag{2.4}$$

Lemma 2.1. *The following inequalities are valids for Jacobi functions* $\varphi_\mu^{\alpha,\beta}(x)$
(c) $|\varphi_\mu^{\alpha,\beta}(x)| \leq 1,$
(d) $|1 - \varphi_\mu^{\alpha,\beta}(x)| \leq x^2(\mu^2 + \rho^2).$

Proof. (See [8], Lemma 3.1, Lemma 3.2). □

Lemma 2.2. *Let* $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$. *Then for* $|\nu| \leq \rho$, *there exists a positive constant* C_2 *such that*

$$|1 - \varphi_{\mu+i\nu}^{\alpha,\beta}(x)| \geq C_2|1 - j_\alpha(\mu x)|.$$

Proof. (See [4], Lemma 9). □

Denote by $W_2^m(\Lambda_{\alpha,\beta})$, $m = 0, 1, 2...$, the class of functions $f \in L^2_{\alpha,\beta}(\mathbb{R})$ that have on $\mathbb{R}$ generalized derivatives belong to $L^2_{\alpha,\beta}(\mathbb{R})$ with $\Lambda^m_{\alpha,\beta} f \in L^2_{\alpha,\beta}(\mathbb{R})$, i.e.,

$$W_2^m(\Lambda_{\alpha,\beta}) = \left\{f \in L^2_{\alpha,\beta}(\mathbb{R}), \Lambda^m_{\alpha,\beta} f \in L^2_{\alpha,\beta}(\mathbb{R})\right\},$$

where $\Lambda^0_{\alpha,\beta} f = f$, $\Lambda^m_{\alpha,\beta} f = \Lambda_{\alpha,\beta}(\Lambda^{m-1}_{\alpha,\beta} f), m = 0, 1, 2...$.

3. Main results

In this section we give the main result of this paper. We need first to define the ϕ-Jacobi-Dunkl Lipschitz class.
Denote N_h by

$$N_h = T_h + T_{-h} - 2I,$$

where I is the unit operator in the space $L^2_{\alpha,\beta}(\mathbb{R})$.

Definition 3.1. A function $f \in W_2^m(\Lambda_{\alpha,\beta})$ is said to be in the ϕ-Jacobi-Dunkl Lipschitz class, denoted by $Lip(\phi, 2, \alpha, \beta)$, if

$$\|N_h \Lambda^m_{\alpha,\beta} f\| = O(\phi(h)), \quad as \quad h \to 0, m = 0, 1, 2, ...,$$

where
(e) ϕ is a continuous increasing function on $[0, \infty)$.

(f) $\phi(0)=0$,$\phi(ts)=\phi(t)\phi(s)$ for all $t,s\in[0,\infty)$.
(g) and

$$\int_0^{1/h} s\phi(s^{-2})ds = O(\frac{1}{h^2}\phi(h^2)) \quad \text{as} \quad h\to 0.$$

Lemma 3.2. *For $f\in W_2^m(\Lambda_{\alpha,\beta})$, then*

$$\|N_h\Lambda^m_{\alpha,\beta}f\|^2 = 4\int_{\mathbb{R}} \lambda^{2m}|\varphi_\mu^{\alpha,\beta}(h)-1|^2|\mathcal{F}_{\alpha,\beta}f(\lambda)|^2 d\sigma(\lambda).$$

Proof. Since $\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta}f)(\lambda)=i\lambda\mathcal{F}_{\alpha,\beta}(f)(\lambda)$, we have

$$\mathcal{F}_{\alpha,\beta}(\Lambda^m_{\alpha,\beta}f)(\lambda) = i^m\lambda^m\mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad m=0,1,2,.... \tag{3.1}$$

We us formulas 2.2 and 3.1, we conclude that

$$\mathcal{F}_{\alpha,\beta}(N_h\Lambda^m_{\alpha,\beta}f)(\lambda) = i^m(\psi_\lambda^{(\alpha,\beta)}(h)+\psi_\lambda^{(\alpha,\beta)}(-h)-2)\lambda^m\mathcal{F}_{\alpha,\beta}(f)(\lambda),$$

Since

$$\psi_\lambda^{(\alpha,\beta)}(h) = \varphi_\mu^{\alpha,\beta}(h) + i\frac{\lambda}{4(\alpha+1)}\sinh(2h)\varphi_\mu^{\alpha+1,\beta+1}(h),$$

and $\varphi_\mu^{\alpha,\beta}$ is even, then

$$\mathcal{F}_{\alpha,\beta}(N_h\Lambda^m_{\alpha,\beta}f)(\lambda) = 2i^m(\varphi_\mu^{\alpha,\beta}(h)-1)\lambda^m\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

By Parseval's identity (formula 2.1), we have the result. □

Theorem 3.3. Let $f\in W_2^m(\Lambda_{\alpha,\beta})$. Then the following are equivalents

(i) $f\in Lip(\phi,2,\alpha,\beta)$,

(ii) $\displaystyle\int_{|\lambda|\geq r}\lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) = O(\phi(r^{-2})), \quad as \quad r\to\infty.$

Proof. $(i)\Rightarrow(ii)$. Assume that $f\in Lip(\phi,2,\alpha,\beta)$, then we have

$$\|N_h\Lambda^m_{\alpha,\beta}f\| = O(\phi(h)), \quad as \quad h\to 0.$$

From Lemma 3.2, we have

$$\|N_h\Lambda^m_{\alpha,\beta}f\|^2 = 4\int_{\mathbb{R}}\lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda).$$

By 2.4 and Lemma 2.2, we get

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}}\lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda) \geq C_1^2C_2^2\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}}|\mu h|^4\lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma(\lambda).$$

From $\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}$ we have

$$\left(\frac{\eta}{2h}\right)^2 - \rho^2 \leq \mu^2 \leq \left(\frac{\eta}{h}\right)^2 - \rho^2$$
$$\Rightarrow \mu^2h^2 \geq \frac{\eta^2}{4} - \rho^2h^2.$$

Take $h \leq \frac{\eta}{3\rho}$, then we have $\mu^2 h^2 \geq C_3 = C_3(\eta)$.
So,

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \geq C_1^2 C_2^2 C_3^2 \int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

There exists then a positive constant C such that

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq C \int_{\mathbb{R}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq C\phi(h^2).$$

For all $0 < h < \frac{\eta}{3\rho}$. Then we have,

$$\int_{r\leq|\lambda|\leq 2r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq C\phi(2^{-2}\eta r^{-2}), \quad r\to\infty.$$

Thus there exists $K > 0$ such that

$$\int_{r\leq|\lambda|\leq 2r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq K\phi(r^{-2}), \quad r\to\infty.$$

Furthermore, we obtain

$$\begin{aligned}\int_{|\lambda|\geq r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \sum_{i=0}^{\infty}\int_{2^i r\leq|\lambda|\leq 2^{i+1}r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda)\\ &= O(\phi(r^{-2}) + \phi(2^{-2}r^{-2}) + ...)\\ &= O(\phi(r^{-2}) + \phi(r^{-2}) + ...)\\ &= O(\phi(r^{-2})).\end{aligned}$$

This proves that

$$\int_{|\lambda|\geq r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(\phi(r^{-2})), \quad as \quad r\to\infty.$$

$(ii) \Rightarrow (i)$. Suppose now that

$$\int_{|\lambda|\geq r} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(\phi(r^{-2})), \quad as \quad r\to\infty,$$

and write

$$\begin{aligned}\int_{\mathbb{R}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \int_{|\lambda|<\frac{1}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda)\\ &+ \int_{|\lambda|\geq\frac{1}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).\end{aligned}$$

Using the inequality (c) of Lemma 2.1, we get

$$\int_{|\lambda|\geq\frac{1}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq 4\int_{|\lambda|\geq\frac{1}{h}} \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

Then

$$\int_{|\lambda|\geq\frac{1}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = \phi(h^2), \quad as \quad h\to 0. \tag{3.2}$$

Set

$$g(x) = \int_x^\infty \lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

An integration by parts gives

$$\begin{aligned}
\int_0^{\frac{1}{h}} \lambda^2\lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \int_0^{\frac{1}{h}} -\lambda^2 g'(\lambda)d\lambda \\
&= -\frac{1}{h^2}g(\frac{1}{h}) + 2\int_0^{\frac{1}{h}} \lambda\phi(\lambda)d\lambda \\
&\leq 2\int_0^{\frac{1}{h}} \lambda\phi(\lambda^{-2})d\lambda \\
&= O(\frac{1}{h^2}\phi(h^2)).
\end{aligned}$$

From Lemma 2.1, we get

$$\begin{aligned}
\int_{|\lambda|\leq\frac{1}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &\leq \int_{|\lambda|\leq\frac{1}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)||\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \\
&\leq \int_{|\lambda|\leq\frac{1}{h}} \lambda^{2m}(\mu^2+\rho^2)h^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \\
&\leq h^2\int_{|\lambda|\leq\frac{1}{h}} \lambda^2\lambda^{2m}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \\
&= O(h^2\frac{1}{h^2}\phi(h^2)) \\
&= O(\phi(h^2)).
\end{aligned}$$

Hence,

$$\int_{|\lambda|\leq\frac{1}{h}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(\phi(h^2)). \tag{3.3}$$

Finally, we conclude from 3.2 and 3.3 that

$$\begin{aligned}
\int_{\mathbb{R}} \lambda^{2m}|1-\varphi_\mu^{\alpha,\beta}(h)|^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) &= \int_{|\lambda|<\frac{1}{h}} + \int_{|\lambda|\geq\frac{1}{h}} \\
&= O(\phi(h^2)) + O(\phi(h^2)) \\
&= O(\phi(h^2)).
\end{aligned}$$

And this ends the proof. □

Corollary 3.3. *Let $f \in W_2^m(\Lambda_{\alpha,\beta})$, and let*

$$\|N_h\Lambda_{\alpha,\beta}^m f\| = O(\phi(h)), \quad as \quad h \to 0,$$

then

$$\int_{|\lambda|\geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2m}\phi(r^{-2})), \quad as \quad r \to \infty.$$

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