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STRUCTURAL STABILITY FOR NONLINEAR $P(U)$-LAPLACIAN PROBLEM WITH FOURIER BOUNDARY CONDITION

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Abstract. We study a nonlinear elliptic $p(u)-$Laplacian problem with Fourier boundary conditions and L^1- data. The existence and uniqueness of weak solutions and the structural stability result are established.

1. Introduction

In this paper, we consider the following nonlinear Fourier boundary value problem

$$\begin{cases} b(u) - \mathrm{div}a(x,u,\nabla u) = f \ \text{ in } \ \Omega \\ a(x,u,\nabla u).\eta + \lambda u = g \quad \text{ on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ is a bounded open domain with Lipschitz boundary $\partial\Omega$, η is the outer unit normal vector on $\partial\Omega$ and $\lambda > 0$.
The operator $\mathrm{div}a(x,u,\nabla u)$ is called $p(u)$-Laplacian. It is more complicated than $p(x)$-Laplacian in the term of nonlinearity. A prototype of this operator is $\mathrm{div}\big(|\nabla u|^{p(u)-2}.\nabla u\big)$. The variable exponent p depend both on the space variable x and on the unknown solution u.

This paper is inspired by recent works of Andreianov, Bendahmane and Ouaro (see [2]). These authors studied the existence and uniqueness of weak solutions for the following problem

$$\begin{cases} b(u) - \mathrm{div}a(x,u,\nabla u) = f \ \text{ in } \ \Omega \\ u = 0 \quad \text{ on } \ \partial\Omega, \end{cases}$$

and the structural stability of weak solutions u_n for the problem

$$\begin{cases} b(u_n) - \mathrm{div}a_n(x,u_n,\nabla u_n) = f_n \ \text{ in } \ \Omega \\ u_n = 0 \quad \text{ on } \ \partial\Omega, \end{cases}$$

where $(a_n(x,z,\xi))_{n\in\mathbb{N}}$ verifies the classical Leray-Lions hypotheses with variable exponent $p_n(x,z)$ such that

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$1 < p_- \leq p_n(.,.) \leq p_+ < +\infty$ and f_n is a sequence of data weakly convergent to f in $L^1(\Omega)$. In this paper, we consider the following basic assumptions on the data for the study of the problem (1.1).
(A_1) f and g are some functions such as $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$ and $g \not\equiv 0$.
(A_2) b is nondecreasing surjective and continuous function defined on $\mathbb{R}$ such that $b(0) = 0$.
Problem (1.1) is adapted into a generalized Leray-Lions framework under the assumption that $a : \Omega \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}^N$ is a Carathéodory function with:
(A_3) $a(x, z, 0) = 0$ for all $z \in \mathbb{R}$, and a.e. $x \in \Omega$;
(A_4) $\big(a(x, z, \xi) - a(x, z, \eta)\big).(\xi - \eta) > 0$ for all $\xi, \eta \in \mathbb{R}^N, \xi \neq \eta$, as well as the growth and the coercivity assumptions with variable exponent
(A_5) $\big|a(x, z, \xi)\big|^{p'(x,z)} \leq C_1\big(|\xi|^{p(x,z)} + \mathcal{M}(x)\big)$ and
(A_6) $a(x, z, \xi).\xi \geq \dfrac{1}{C_2}|\xi|^{p(x,z)}$.
Here, C_1 and C_2 are positive constants and $\mathcal{M}$ is a positive function such that $\mathcal{M} \in L^1(\Omega)$.
$p : \Omega \times \mathbb{R} \to [p_-, p_+]$ is a Carathéodory function, $1 < p_- \leq p_+ < +\infty$ and $p'(x, z) = \dfrac{p(x, z)}{p(x, z) - 1}$ is the conjugate exponent of $p(x, z)$, with

$$p_- := ess \inf_{(x,z)\in\Omega\times\mathbb{R}} p(x, z) \text{ and } p_+ := ess \sup_{(x,z)\in\Omega\times\mathbb{R}} p(x, z).$$

We assume that $p_- > N$ and for all $M > 0$

$$p \text{ is log-Hölder continuous in } (x, z) \text{ uniformly on } \overline{\Omega} \times [-M, M]. \tag{1.2}$$

This paper is interested in Fourier type boundary conditon, so we cannot work in the space $W_0^{1,p(.,u(.))}(\Omega)$, but in the space $W^{1,p(.,u(.))}(\Omega)$. We are facing serious difficulties when we want to apply the theory of monotone operators in the Banach space $W^{1,p(.,u(.))}(\Omega)$ to prove the existence of weak solutions. Therefore, we use the technic of pseudo-monotone operators in Banach spaces, the Poincaré Wirtinger inequality and Young measure associated to a weakly convergence method of sequence of gradients of solution (cf [9, 11]) to obtain some useful convergence results. Indeed, we define an approximation problem, and we prove that this problem has a solution u_n which converges to u, a weak solution of problem (1.1).

The interest of the study of this kind of problem is due to the fact that they can model various phenomena which arise in the study of elastic mechanic (see [4]), electrorheological fluids (see [16]) or image restoration (see [8]).
Here, we consider a Fourier boundary condition which bring some difficulties to treat the term at the boundary.
We were inspired by the work of Ouaro and Tchousso (see [14]), where the authors defined for the first time a new space by taking into account the boundary.
For the next part of the paper (section 2), we introduce some preliminaries results. In section 3, we study the existence and uniqueness of weak solution with L^1-data f and g. In section 4, we study the continuous dependence of weak solutions.

2. Preliminaries

We will use the so-called truncation function

$$T_k(s) := \begin{cases} s & \text{if } |s| \le k \\ k sign_0(s) & \text{if } |s| > k \end{cases}, \quad \text{where } sign_0(s) := \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0. \end{cases}$$

The truncation function possesses the following properties.

$$T_k(-s) = -T_k(s), |T_k(s)| = \min\{|s|, k\},$$
$$\lim_{k\to+\infty} T_k(s) = s \text{ and } \lim_{k\to 0} \frac{1}{k} T_k(s) = sign_0(s).$$

We also need to truncate vector valued-function with the help of the mapping

$$h_m : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \ h_m(\lambda) = \begin{cases} \lambda, & \text{if } |\lambda| \le m \\ m\dfrac{\lambda}{|\lambda|} & \text{if } |\lambda| > m, \end{cases} \quad \text{where } m > 0.$$

For a Lebesgue measurable set $A \subset \Omega$, χ_A denotes its characteristic function and $meas(A)$ denotes its Lebesgue measure. Let $u : \Omega \to \mathbb{R}$ be a function and $k \in \mathbb{R}$, we write $\{|u| \le k\}$ or $[|u| \le k]$ for the set $\{x \in \Omega : \ |u(x)| \le k\}$, (respectively, $\ge, =, <, >$).
As the exponent appearing in (1.1) depend on x and on $u(x)$, we must work in Lebesgue and Sobolev spaces with variable exponent.
For the study of problem (1.1) we need the Sobolev spaces $W^{1,\pi(.)}(\Omega)$, where $\pi(.) = p(., u(.))$.

Definition 2.1. Let $\pi : \Omega \longrightarrow [1, +\infty)$ be a measurable function.

(1) $L^{\pi(.)}(\Omega)$ is the space of all measurable function $f : \Omega \longrightarrow \mathbb{R}$ such that the modular
$$\rho_{\pi(.)}(f) := \int_\Omega |f|^{\pi(x)} dx < +\infty.$$
If p_+ is finite, this space is equipped with the Luxembourg norm
$$\|f\|_{L^{\pi(.)}(\Omega)} := \inf\left\{\lambda > 0; \quad \rho_{\pi(.)}\left(\frac{f}{\lambda}\right) \le 1\right\}.$$
In the sequel, we will use the same notation $L^{\pi(.)}(\Omega)$ for the space $(L^{\pi(.)}(\Omega))^N$ of vector-valued functions.

(2) $W^{1,\pi(.)}(\Omega)$ is the space of all functions $f \in L^{\pi(.)}(\Omega)$ such that the gradient of f (taken in the sense of distributions) belongs to $L^{\pi(.)}(\Omega)$. The space $W^{1,\pi(.)}(\Omega)$ is equipped with the norm
$$\|u\|_{W^{1,\pi(.)}(\Omega)} := \|u\|_{L^{\pi(.)}(\Omega)} + \|\nabla u\|_{L^{\pi(.)}(\Omega)}.$$
When $1 < p_- \le \pi(.) \le p_+ < +\infty$, all the above spaces are separable and reflexive Banach spaces.

We denote $\pi_n(x) := p(x, u_n(x))$.

Proposition 2.2. *(See* [1, Proposition 2.3]*)*
For all measurable function $\pi : \Omega \to [p_-, p_+]$, the following properties hold.

(i) *$L^{\pi(.)}(\Omega)$ and $W^{1,\pi(.)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *$L^{\pi'(.)}(\Omega)$ can be identified with the dual space of $L^{\pi(.)}(\Omega)$, and the following Hölder type inequality holds:*

$$\forall f \in L^{\pi(.)}(\Omega), g \in L^{\pi'(.)}(\Omega), \quad \left| \int_\Omega f g dx \right| \leq 2\|f\|_{L^{\pi(.)}(\Omega)} \|g\|_{L^{\pi'(.)}(\Omega)}.$$

(iii) *One has $\rho_{\pi(.)}(f) = 1$ if and only if $\|f\|_{L^{\pi(.)}(\Omega)} = 1$; further,*

$$\text{if } \rho_{\pi(.)}(f) \leq 1, \text{ then } \|f\|^{p_+}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq \|f\|^{p_-}_{L^{\pi(.)}(\Omega)};$$

$$\text{if } \rho_{\pi(.)}(f) \geq 1, \text{ then } \|f\|^{p_-}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq \|f\|^{p_+}_{L^{\pi(.)}(\Omega)}.$$

In particular, if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^{\pi(.)}(\Omega)$, then $\|f_n\|_{L^{\pi(.)}(\Omega)}$ tends to zero (resp., to infinity) if and only if $\rho_{\pi(.)}(f_n)$ tends to zero (resp., to infinity), as $n \to +\infty$.

For a measurable function $f \in W^{1,\pi(.)}(\Omega)$ we introduce the following notation:

$$\rho_{1,\pi(.)}(f) = \int_\Omega |f|^{\pi(.)} dx + \int_\Omega |\nabla f|^{\pi(.)} dx.$$

Replacing $p(x)$ by $\pi(x)$ in [7, Proposition 2.2], we obtain the following result that is fundamental in this paper.

Proposition 2.3. *(See* [18, 19] *) If $f \in W^{1,\pi(.)}(\Omega)$, the following properties hold:*

(i) $\|f\|_{W^{1,\pi(.)}(\Omega)} > 1 \Rightarrow \|f\|^{p_-}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < \|f\|^{p_+}_{W^{1,\pi(.)}(\Omega)}$;

(ii) $\|f\|_{W^{1,\pi(.)}(\Omega)} < 1 \Rightarrow \|f\|^{p_+}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < \|f\|^{p_-}_{W^{1,\pi(.)}(\Omega)}$;

(iii) $\|f\|_{W^{1,\pi(.)}(\Omega)} < 1$ *(respectively* $= 1; > 1$*)* $\Leftrightarrow \rho_{1,\pi(.)}(f) < 1$ *(respectively* $= 1; > 1$*)*.

The following lemma prove that the space $W^{1,\pi(.)}(\Omega)$ is stable by truncation.

Lemma 2.4. *If $u \in W^{1,\pi(.)}(\Omega)$ then $T_k(u) \in W^{1,\pi(.)}(\Omega)$.*

Now, we give the embedding results.

Proposition 2.5. *(See* [1, Proposition 2.4]*)*
Assume that $\pi : \Omega \to [p_-, p_+]$ has a representative which can be extended into a continuous function up to the boundary $\partial\Omega$ and satisfying the log-Hölder continuity assumption:

$$\exists L > 0, \ \forall x, y \in \overline{\Omega}, x \neq y, \quad -\big(\log|x-y|\big)|\pi(x) - \pi(y)| \leq L. \tag{2.1}$$

(i) *Then, $C^\infty(\overline{\Omega})$ is dense in $W^{1,\pi(.)}(\Omega)$.*

(ii) *$W^{1,\pi(.)}(\Omega)$ is embedded into $L^{\pi^*(.)}(\Omega)$, where $\pi^*(.)$ is the Sobolev embedding exponent defined as in (2.2) below. If q is a measurable variable exponent such that ess $\inf_{x\in\Omega}(\pi^*(.) - q(.)) > 0$, then the embedding of $W^{1,\pi(.)}(\Omega)$ into $L^{q(.)}(\Omega)$ is compact.*

For a given $\pi(.)$, a function taking values in $[p_-, p_+]$, $\pi^*(.)$ denotes the optimal Sobolev embedding defined for any $x \in \Omega$ by

$$\pi^*(x) = \begin{cases} \dfrac{N\pi(x)}{N-\pi(x)} & \text{if} \quad \pi(x) < N \\ \text{any real value} & \text{if} \quad \pi(x) = N \\ +\infty & \text{if} \quad \pi(x) > N. \end{cases} \tag{2.2}$$

Put

$$\pi^\partial(x) := \big(\pi(x)\big)^\partial := \begin{cases} \dfrac{(N-1)\pi(x)}{N-\pi(x)} & \text{if} \quad \pi(x) < N \\ +\infty & \text{if} \quad \pi(x) \geq N. \end{cases} \tag{2.3}$$

Proposition 2.6. *(See* [14, Proposition 2.3]*)*
Let $\pi(.) \in C(\overline{\Omega})$ and $p_- > 1$. If $q \in C(\partial\Omega)$ satisfies the condition:

$$1 \leq q(x) < \pi^\partial(x), \quad \forall x \in \partial\Omega,$$

then, there is a compact embedding

$$W^{1,\pi(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega).$$

In particular there is compact embedding

$$W^{1,\pi(.)}(\Omega) \hookrightarrow L^{\pi(.)}(\partial\Omega).$$

Young measures and nonlinear weak-* convergence.
Throughout the paper, we denote by δ_c the Dirac measure on $\mathbb{R}^d$ ($d \in \mathbb{N}$), concentrated at the point $c \in \mathbb{R}^d$.
In the following theorem, we gather the results of Ball [6], Pedregal [15] and Hungerbühler [12] which will be needed for our purposes (we limit the statement to the case of a bounded domain Ω). Let us underline that the results of (ii),(iii), expressed in terms of the convergence in measure, are very convenient for the applications that we have in mind.

Theorem 2.7.

(i) *Let $\Omega \subset \mathbb{R}^N, N \in \mathbb{N}$, and a sequence $(v_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^d$-valued functions, $d \in \mathbb{N}$, such that $(v_n)_{n\in\mathbb{N}}$ is equi-integrable on Ω. Then, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a parametrized family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^d$ ($d \in \mathbb{N}$) , weakly measurable in x with respect to the Lebesgue measure in Ω, such that for all Carathéodory function*
$F : \Omega \times \mathbb{R}^d \to \mathbb{R}^t$, $t \in \mathbb{N}$, we have

$$\lim_{k\to+\infty} \int_\Omega F(x, v_{n_k})dx = \int_\Omega \int_{\mathbb{R}^d} F(x,\lambda)d\nu_x(\lambda)dx, \tag{2.4}$$

whenever the sequence $(F(., v_n(.)))_{n\in\mathbb{N}}$ is equi-integrable on Ω. In particular,

$$v(x) := \int_{\mathbb{R}^d} \lambda d\nu_x(\lambda) \tag{2.5}$$

is the weak limit of the sequence $(v_{n_k})_{k\in\mathbb{N}}$ *in* $L^1(\Omega)$.
The family $(\nu_x)_{x\in\Omega}$ *is called the Young measure generated by the subsequence* $(v_{n_k})_{k\in\mathbb{N}}$.

(ii) *If* Ω *is of finite measure, and* $(\nu_x)_{x\in\Omega}$ *is the Young measure generated by a sequence* $(v_n)_{n\in\mathbb{N}}$, *then* $\nu_x = \delta_{v(x)}$ *for a.e.* $x \in \Omega \Leftrightarrow v_n$ *converges in measure in* Ω *to* v *as* $n \to +\infty$.

(iii) *If* Ω *is of finite measure,* $(u_n)_{n\in\mathbb{N}}$ *generates a Dirac Young measure* $(\delta_{u(x)})_{x\in\Omega}$ *on* $\mathbb{R}^{d_1}$, *and* $(v_n)_{n\in\mathbb{N}}$ *generates a Young measure* $(\nu_x)_{x\in\Omega}$ *on* $\mathbb{R}^{d_2}$, *then the sequence* $(u_n, v_n)_{n\in\mathbb{N}}$ *generates the Young measure* $(\delta_{u(x)} \otimes \nu_x)_{x\in\Omega}$ *on* $\mathbb{R}^{d_1+d_2}$.

Whenever a sequence $(v_n)_{n\in\mathbb{N}}$ generates a Young measure $(\nu_x)_{x\in\Omega}$, following the terminology of [10] we will say that $(v_n)_{n\in\mathbb{N}}$ nonlinear weak-* converges, and $(\nu_x)_{x\in\Omega}$ is the nonlinear weak-* limit of the sequence $(v_n)_{n\in\mathbb{N}}$. In the case where $(v_n)_{n\in\mathbb{N}}$ possesses a nonlinear weak-* convergent subsequence, we will say that it is nonlinear weak-* compact. ([1, Theorem 2.10 (i)]) It means that any equi-integrable sequence of measurable functions is nonlinear weak-* compact on Ω.

Lemma 2.8. *(See* [1, Theorem 3.11] *and* [2, Step 2 of proof of Theorem 2.6]*). Assume that* $(u_n)_{n\in\mathbb{N}}$ *converges a.e. on* Ω *to some function* u, *then*

$$\begin{aligned}&|p(x,u_n(x)) - p(x,u(x))| \text{ converges in measure to } 0 \text{ on } \Omega,\\ &\text{and for all bounded subset } K \text{ of } \mathbb{R}^N,\\ &\sup_{\xi\in K}|a(x,u_n(x),\xi) - a(x,u(x),\xi)| \text{ converges in measure to } 0 \text{ on } \Omega. \end{aligned} \tag{2.6}$$

We recall some notations. For any $u \in W^{1,\pi(.)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense. We will identify at boundary u and $\tau(u)$.

3. Weak solution

Let $f_n = T_n(f)$, then $(f_n)_{n\in\mathbb{N}^\star}$ is a sequence of bounded function which strongly converges to f in $L^1(\Omega)$ such that $||f_n||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)}$. We consider the following problem

$$\begin{cases} T_n(b(u_n)) - \operatorname{div} a(x,u_n,\nabla u_n) - \dfrac{1}{n}\triangle_{p_+}u_n + \dfrac{1}{n}|u_n|^{p_+-2}u_n = f_n \quad \text{in} \quad \Omega \\ \\ \Big(a(x,u_n,\nabla u_n) + \dfrac{1}{n}|\nabla u_n|^{p_+-2}\nabla u_n\Big).\eta + \lambda T_n(u_n) = g_n \qquad\qquad \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

where

$$-\triangle_{p_+}u_n := -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_+-2}\frac{\partial u_n}{\partial x_i}\right).$$

In this part, we prove that the problem (3.1) admits at least one weak solution u_n. We define the following reflexive space

$$E = W^{1,p_+}(\Omega) \times L^{p_+}(\partial\Omega).$$

Let

$$X_0 = \{(u, v) \in E : \ v = \tau(u)\}.$$

In the sequel, we will identify an element $(u, v) \in X_0$ with its representative $u \in W^{1,p_+}(\Omega)$ (since $W^{1,p_+}(\Omega) \hookrightarrow\hookrightarrow L^{p_+}(\partial\Omega)$).

Theorem 3.1. *There exists at least one weak solution u_n for the problem (3.1) in the sense that $u_n \in X_0$ and for all $v \in X_0$,*

$$\begin{aligned}\int_\Omega T_n(b(u_n))v dx \ &+ \ \int_\Omega a(x, u_n, \nabla u_n)\nabla v dx + \int_{\partial\Omega} \lambda T_n(u_n)v d\sigma \\ &+ \ \frac{1}{n}\int_\Omega \left[|u_n|^{p_+-2}u_n v + |\nabla u_n|^{p_+-2}\nabla u_n \nabla v\right] dx \\ &= \ \int_\Omega f_n v dx + \int_{\partial\Omega} g_n v d\sigma. \end{aligned} \tag{3.2}$$

For the proof of Theorem 3.1, we apply the following Lemmas.

Lemma 3.2. *(see [13, Remark 2.12]). Let V be a separable reflexive Banach space and A, $M: V \to V'$ such that*

(i) *A is an operator pseudo-monotone;*
(ii) *M is a bounded hemicontinuous and monotone operator;*

then $A + M$ is pseudo-monotone.

Lemma 3.3. *(See [17, Corollary 2.2]). If an operator $\mathcal{A}$ is of type (M), bounded and coercive on a separable Banach space to its dual, then $\mathcal{A}$ is surjective.*

Let

$$< A_n u, v >=< Au, v > + < G_n u, v >,$$

where

$$< Au, v >= \int_\Omega a(x, u, \nabla u).\nabla v dx, \ \ < G_n u, v >= \frac{1}{n}\int_\Omega |\nabla u|^{p_+-2}\nabla u \nabla v dx$$

and

$$< B_n u, v >= \int_\Omega T_n(b(u))v dx + \lambda \int_{\partial\Omega} T_n(u)v d\sigma + \frac{1}{n}\int_\Omega |u|^{p_+-2}uv dx,$$

with $u, v \in X_0$. Set $C_n = A_n + B_n$.

Proof. The proof of Theorem 3.1 is done in three steps.

Step 1: C_n is bounded.
By using Hölder type inequality and assumption (A_5) with constant exponent p_+, we deduce that A is bounded. Moreover,

$$\begin{aligned}\left| < G_n u, v > \right| = \left|\frac{1}{n}\int_\Omega |\nabla u|^{p_+-2}\nabla u \nabla v dx\right| \ &\leq \ \int_\Omega |\nabla u|^{p_+-1}|\nabla v| dx \\ &\leq \ \|\nabla u\|^{\frac{p_+}{p'_+}}_{L^{p_+}(\Omega)}\|\nabla v\|_{L^{p_+}(\Omega)} \\ &\leq \ C'_2\|v\|_{W^{1,p_+}(\Omega)}.\end{aligned}$$

So, A_n is bounded. Additionnaly, using the same argument as in [5, Proof of Lemma 4.2], we prove that the operator B_n is bounded. Therefore, C_n is bounded.

Step 2: C_n **is of type** (M).

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in X_0 such that $\begin{cases} u_k \rightharpoonup u \quad \text{in} \quad X_0 \\ C_n u_k \rightharpoonup \chi \quad \text{in} \quad X_0' \\ \limsup\limits_{k\to+\infty} \langle C_n u_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases}$

We will prove that $\chi = C_n u$.
As $T_n(b(u_k))u_k \geq 0$, $\lambda T_n(u_k)u_k \geq 0$ and $|u_k|^{p_+} \geq 0$ by Fatou's Lemma, we deduce that

$$\begin{aligned} &\liminf_{k\to\infty}\left(\int_\Omega T_n(b(u_k))u_k dx + \lambda\int_{\partial\Omega} T_n(u_k)u_k d\sigma + \frac{1}{n}\int_\Omega |u_k|^{p_+}dx\right) \\ \geq\ &\int_\Omega T_n(b(u))u dx + \lambda\int_{\partial\Omega} T_n(u)u d\sigma + \frac{1}{n}\int_\Omega |u|^{p_+}dx. \end{aligned}$$

One the other hand, thanks to Lebesgue dominated convergence Theorem and the fact that $|u_k|^{p_+-2}u_k \rightharpoonup |u|^{p_+-2}u$ in $L^{p_+'}(\Omega)$, we have

$$\begin{aligned} &\lim_{k\to\infty}\left(\int_\Omega T_n(b(u_k))v dx + \lambda\int_{\partial\Omega} T_n(u_k)v d\sigma + \frac{1}{n}\int_\Omega |u_k|^{p_+-2}u_k v dx\right) \\ &\qquad = \int_\Omega T_n(b(u))v dx + \lambda\int_{\partial\Omega} T_n\big(u\big)v d\sigma + \frac{1}{n}\int_\Omega |u|^{p_+-2}uv dx, \end{aligned}$$

for any $v \in X_0$. Since, $u_k \to u$ a.e. in $\partial\Omega$ and $|T_n(u_k)v| \leq n|v|$. Therefore, for k goes to ∞,

$$B_n u_k \rightharpoonup T_n(b(u)) + \lambda T_n(u) + \frac{1}{n}|u|^{p_+-2}u \text{ in } X_0'.$$

Thus, it follows that

$$A_n u_k \rightharpoonup \chi - \left(T_n(b(u)) + \lambda T_n(u) + \frac{1}{n}|u|^{p_+-2}u\right) \text{ in } X_0', \text{ as } k\to\infty.$$

It remains to prove that A_n is of type (M). For this, we first show that G_n is bounded monotone and hemicontinuous.
From Step 1, G_n is bounded. Let us prove that G_n is monotone.
For all $u, v \in W^{1,p_+}(\Omega)$, we have

$$< G_n u - G_n v, u - v >= \frac{1}{n}\int_\Omega \big(|\nabla u|^{p_+-2}\nabla u - |\nabla v|^{p_+-2}\nabla v\big)\big(\nabla u - \nabla v\big)dx \geq 0,$$

since $\xi \mapsto |\xi|^{p_+-2}\xi$ is increasing for $p_+ > 2$.
Moreover, G_n is hemicontinuous. Indeed, let $f : t \in \mathbb{R} \mapsto f(t) =< G_n(u+tv), v >$ and $t, t_0 \in \mathbb{R}$ such that $t \to t_0$.
Let us set $w = u + tv \in W^{1,p_+}(\Omega)$ and $w_0 = u + t_0 v \in W^{1,p_+}(\Omega)$. Then,

$$\|w - w_0\|_{W^{1,p_+}(\Omega)} = \|(t-t_0)v\|_{W^{1,p_+}(\Omega)} = |t-t_0|\|v\|_{W^{1,p_+}(\Omega)} \to 0.$$

So $w \to w_0$ in $W^{1,p_+}(\Omega)$, as $t \to t_0$, which implies that $\nabla w \to \nabla w_0$ in $L^{p_+}(\Omega)$ and we infer that

$$|\nabla w|^{p_+-2}\nabla w \to |\nabla w_0|^{p_+-2}\nabla w_0 \text{ in } L^{p'_+}(\Omega) \text{ as } t \to t_0.$$

Therefore,

$$\begin{aligned} |f(t)-f(t_0)| &= |< G_n(u+tv), v > - < G_n(u+t_0v), v >| \\ &\leq \frac{1}{n}\int_\Omega \big||\nabla w|^{p_+-2}\nabla w - |\nabla w_0|^{p_+-2}\nabla w_0\big||\nabla v|dx \\ &\leq \||\nabla w|^{p_+-2}\nabla w - |\nabla w_0|^{p_+-2}\nabla w_0\|_{L^{p'_+}(\Omega)}\|\nabla v\|_{L^{p_+}(\Omega)} \to 0. \end{aligned}$$

Then, we deduce that f is continuous, namely the operator G_n is hemi-continuous. Now, we are going to prove that A is pseudo-monotone.

Let us set

$$a_1(u,v,w) = \int_\Omega a(x,u,\nabla v)\nabla w dx.$$

Then, $w \mapsto a_1(u,v,w)$ is continuous on $W^{1,p_+}(\Omega)$, thus

$$a_1(u,v,w) = \big\langle A(u,v), w\big\rangle, \;\; A(u,v) \in (W^{1,p_+}(\Omega))',$$

and verify

$$A(u,u) = Au.$$

Let us prove that A is of type of Calculus of variation.

• As $A(u,.)$ is bounded, we prove that $v \mapsto A(u,v)$ is hemi-continuous from $W^{1,p_+}(\Omega) \to (W^{1,p_+}(\Omega))'$.
Since $a(x,u,\nabla(v_1+\lambda v_2) \to a(x,u,\nabla v_1)$ in $L^{p'_+}(\Omega)$ as $\lambda \to 0$ and $u, v_1, v_2 \in W^{1,p_+}(\Omega)$ then, $a_1(u, v_1+\lambda v_2, w) \to a_1(u,v_1,w)$ as $\lambda \to 0$.
In the same manner we prove that $u \mapsto A(u,v)$ is hemi-continuous from $W^{1,p_+}(\Omega) \to (W^{1,p_+}(\Omega))'$.
Moreover, for all $u, v \in W^{1,p_+}(\Omega)$, we have

$$\begin{aligned} < A(u,u)-A(u,v), u-v > &= < A(u,u), u-v > - < A(u,v), u-v > \\ &= a_1(u,u,u-v) - a_1(u,v,u-v) \\ &= \int_\Omega a(x,u,\nabla u)\nabla(u-v)dx \\ &- \int_\Omega a(x,u,\nabla v)\nabla(u-v)dx \\ &= \int_\Omega \big(a(x,u,\nabla u) - a(x,u,\nabla v)\big)\nabla(u-v)dx \geq 0. \end{aligned}$$

• Let us suppose that $u_k \rightharpoonup u$ in $W^{1,p_+}(\Omega)$ and $< A(u_k,u_k) - A(u_k,u), u_k - u > \to 0$. We prove that

$$\forall v \in \; W^{1,p_+}(\Omega), \;\; A(u_k,v) \rightharpoonup A(u,v) \text{ in } (W^{1,p_+}(\Omega))'.$$

Let's set

$$\int_\Omega F_k dx = \big\langle A(u_k,u_k) - A(u_k,u), u_k - u\big\rangle.$$

As $u_k \rightharpoonup u$, we have

$$a(x,u_k,\nabla v) \rightharpoonup a(x,u,\nabla v) \text{ in } L^{p_+'}(\Omega),$$

(see [13, Lemma 2.2 with $m=1$]). Therefore, $A(u_k,v) \rightharpoonup A(u,v)$ in $(W^{1,p_+}(\Omega))'$.

• Now, we suppose that $u_k \rightharpoonup u$ in $W^{1,p_+}(\Omega)$ and $A(u_k,v) \rightharpoonup \Theta$ in $(W^{1,p_+}(\Omega))'$. We prove that

$$\big\langle A(u_k,v), u_k\big\rangle \to \big\langle \Theta, u\big\rangle.$$

Then, by using [13, Lemma 2.1], we obtain that $a(x,u_k,\nabla v) \to a(x,u,\nabla v)$ in $L^{p_+'}(\Omega)$ and thus, $a_1(u_k,v,u_k) \to a_1(u,v,u)$.
Therefore,

$$< A(u_k,v), u_k >= a_1(u_k,v,u_k) \to < A(u,v), u > \text{ and } \Theta = A(u,v).$$

Hence, A is of Calculus of variation type. Finally, by using [13, Proposition 2.6], we deduce that A is pseudo-monotone.
Therefore, it follows from Lemma 3.2 that $A_n = A + G_n$ is pseudo-monotone. So the operator A_n is of type (M) (see, [13, Proposition 2.5]) and we immediately have

$$Au + G_n u = \chi - \big(T_n(b(u)) + \lambda T_n(u) + \frac{1}{n}|u|^{p_+-2}u\big).$$

Therefore, we obtain $C_n u = \chi$.

For more understanding regarding operator of "type (M)", "calculus of variation type" and "pseudo-monotone", see [17, 13].

Step 3: C_n is coercive.
Using (A_6) with constant exponent p_+, we get

$$\begin{aligned} < C_n u, u > &= \int_\Omega a(x,u,\nabla u).\nabla u dx + \int_\Omega T_n(b(u))u dx \\ &+ \lambda \int_{\partial\Omega} T_n(u) u d\sigma + \frac{1}{n}\int_\Omega \big[|u|^{p_+} + |\nabla u|^{p_+}\big] dx \\ &\geq \frac{1}{C_2}\int_\Omega |\nabla u|^{p_+} dx + \frac{1}{n}\int_\Omega |u|^{p_+} dx \\ &\geq C_3 ||u||^{p_+}_{W^{1,p_+}(\Omega)}, \text{ where } C_3 = \min\left\{\frac{1}{C_2}, \frac{1}{n}\right\}. \end{aligned}$$

We deduce that

$$\frac{< C_n u, u >}{||u||_{W^{1,p_+}(\Omega)}} \to \infty, \quad \text{as } ||u||_{W^{1,p_+}(\Omega)} \to \infty.$$

Then, according to Lemma 3.3, C_n is surjective.
Thus, for any $F_n =< T_n(f), T_n(g) > \subset E' \subset X_0'$, there exists at least one solution

$u_n \in X_0$ of the problem

$$< C_n u_n, v >=< F_n, v > \quad \text{for all } v \in X_0.$$

Therefore, u_n is a weak solution of the problem (3.1). □

Remark 3.4. If u_n is a weak solution of the problem (3.1), then $u_n \in W^{1,\pi_n(.)}(\Omega)$, since $W^{1,p_+}(\Omega) \hookrightarrow W^{1,\pi_n(.)}(\Omega)$ continuously. Moreover, $a(x, u_n, \nabla u_n)$ satisfies $(A_3) - (A_6)$ with variable exponent $\pi_n(x) := p(x, u_n(x))$.

Now, we can introduce the notion of weak solution.

Definition 3.5. A measurable function $u \in W^{1,\pi(.)}(\Omega)$ for $\pi(.) = p(., u(.))$ is called weak solution of the problem (1.1) if $b(u) \in L^1(\Omega)$, $u \in L^1(\partial\Omega)$ and for all $\varphi \in W^{1,\pi(.)}(\Omega)$,

$$\int_\Omega b(u)\varphi dx + \int_\Omega a(x, u, \nabla u)\nabla\varphi dx + \int_{\partial\Omega} u\varphi d\sigma = \int_\Omega f\varphi dx + \int_{\partial\Omega} g\varphi d\sigma. \quad (3.3)$$

These integrals are well defined. For the first term of the left-hand side and the first term of the right-hand side of the above equality, we use the fact that $\varphi \in L^\infty(\Omega)$, since $\varphi \in W^{1,\pi(.)}(\Omega) \subset W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$, for $p_- > N$. For the second term of the left-hand side of (3.3), we use the growth assumption (A_5) to prove that $a(x, u, \nabla u)$ belongs to $L^{\pi'(.)}(\Omega)$. Moreover, as $\varphi \in C(\overline{\Omega})$, then $\varphi \in L^\infty(\partial\Omega)$, so, the third term of the left-hand side and the second term of the right-hand side of (3.3) are well defined.

One of the main Theorem of this paper is the following.

Theorem 3.6. *Assume that* $(A_1) - (A_6)$ *and* (1.2) *hold. Then, there exists at least one weak solution to the problem* (1.1).

To prove the above theorem, we need the two following Lemmas.

Lemma 3.7. *Assume that* $(A_2) - (A_6)$ *hold with variable exponent* $\pi_n(.)$. *If* u_n *is a weak solution of* (3.1), *then we have*

$$\int_\Omega \big|T_n(b(u_n))\big| dx \le \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}, \quad (3.4)$$

$$\int_{\partial\Omega} \lambda\big|T_n(u_n)\big| d\sigma \le \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)} \quad (3.5)$$

and

$$\|u_n\|_{W^{1,p_-}(\Omega)} \le const\big(p_-, \Omega, f, g\big). \quad (3.6)$$

Proof. By taking $v = T_k(u_n)$, for all $k > 0$, in the weak formulation (3.2), we obtain

$$\begin{aligned}\int_\Omega T_n(b(u_n))T_k(u_n)dx &+ \int_\Omega a(x,u_n,\nabla u_n).\nabla T_k(u_n)dx + \int_{\partial\Omega}\lambda T_n(u_n)T_k(u_n)d\sigma \\ &+ \frac{1}{n}\int_\Omega \left[|u_n|^{p_+-2}u_nT_k(u_n) + |\nabla u_n|^{p_+-2}\nabla u_n\nabla T_k(u_n)\right]dx \\ &= \int_\Omega f_nT_k(u_n)dx + \int_{\partial\Omega} g_nT_k(u_n)d\sigma. \qquad (3.7)\end{aligned}$$

Since all the terms of the left hand-side of (3.7) are nonnegative, we deduce that

$$\begin{aligned}\int_\Omega T_n(b(u_n))T_k(u_n)dx &\le \int_\Omega f_nT_k(u_n)dx + \int_{\partial\Omega} g_nT_k(u_n)d\sigma \\ &\le k\big(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}\big) \qquad (3.8)\end{aligned}$$

and

$$\begin{aligned}\int_{\partial\Omega}\lambda T_n(u_n)T_k(u_n)d\sigma &\le \int_\Omega f_nT_k(u_n)dx + \int_{\partial\Omega} g_nT_k(u_n)d\sigma \\ &\le k\big(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}\big). \qquad (3.9)\end{aligned}$$

Dividing (3.8) and (3.9) by k and letting k goes to 0, we have

$$\int_\Omega T_n(b(u_n))sign_0(u_n)dx \le \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}$$

and

$$\int_{\partial\Omega}\lambda T_n(u_n)sign_0(u_n)d\sigma \le \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}.$$

Therefore,

$$\int_\Omega |T_n(b(u_n))|dx \le \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)} \qquad (3.10)$$

and

$$\int_{\partial\Omega}\lambda|T_n(u_n)|d\sigma \le \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}. \qquad (3.11)$$

From the relation (3.10), the sequence $(T_n(b(u_n)))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^1(\Omega)$. Thus, we deduce that $(b(u_n))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^1(\Omega)$. But b is continuous nondecreasing and surjective, then $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^1(\Omega)$. So, there exists a positive constant C_4 such that

$$\int_\Omega |u_n|dx \le C_4.$$

Hence,

$$\tilde{u}_n := \frac{1}{meas(\Omega)}\int_\Omega u_ndx \le const(\Omega).$$

Moreover, from Poincaré-Wirtinger inequality:

$$\int_\Omega |u_n - \tilde{u}_n|^{p_-}dx \le const(p_-,\Omega)\int_\Omega |\nabla u_n|^{p_-}dx,$$

where $\tilde{u}_n$ is given by the above inequality, we deduce that

$$\int_\Omega |u_n|^{p_-} dx \leq const(p_-, \Omega) \int_\Omega |\nabla u_n|^{p_-} dx + \left| \int_\Omega u_n dx \right|^{p_-}.$$

Thus

$$\int_\Omega |u_n|^{p_-} dx \leq C_5 \int_\Omega |\nabla u_n|^{p_-} dx + C_6, \tag{3.12}$$

where $C_5 = const(p_-, \Omega)$ and $C_6 = const(p_-, C_4, \Omega)$ are the positive constants. Furthermore, as $W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$, there exists a positive constant C_7 such that

$$\|u_n\|^{p_-}_{L^\infty(\overline{\Omega})} \leq C_7 \|u_n\|^{p_-}_{W^{1,p_-}(\Omega)},$$

which implies that

$$\|u_n\|^{p_-}_{L^\infty(\Omega)} \leq C_7 \|u_n\|^{p_-}_{W^{1,p_-}(\Omega)} \text{ and } \|u_n\|^{p_-}_{L^\infty(\partial\Omega)} \leq C_7 \|u_n\|^{p_-}_{W^{1,p_-}(\Omega)}. \tag{3.13}$$

Using (A_6) with variable exponent $p(x, u_n(x))$ on $a(x, u_n, \nabla u_n)$ and Theorem 3.1, the sequence u_n satisfies

$$\begin{aligned} \int_\Omega T_n(b(u_n)) u_n dx + \int_{\partial\Omega} \lambda T_n(u_n) u_n d\sigma &+ \frac{1}{C_2} \int_\Omega |\nabla u_n|^{\pi_n(.)} dx \\ &+ \frac{1}{n} \left(\int_\Omega |\nabla u_n|^{p_+} dx + \int_\Omega |u_n|^{p_+} dx \right) \\ &\leq \int_\Omega f_n u_n dx + \int_{\partial\Omega} g_n u_n d\sigma. \end{aligned} \tag{3.14}$$

Applying Young's inequality on the right hand side of (3.14) and using (3.13), we get

$$\begin{aligned}
\int_\Omega f_n u_n dx &+ \int_\Omega g_n u_n d\sigma \\
&\leq \int_\Omega |f||u_n|dx + \int_{\partial\Omega} |g||u_n|d\sigma \\
&\leq \|f\|_{L^1(\Omega)}\|u_n\|_{L^\infty(\Omega)} + \|g\|_{L^1(\partial\Omega)}\|u_n\|_{L^\infty(\partial\Omega)} \\
&= \left(\frac{4C_2C_7(C_5+1)}{p_-}\right)^{\frac{1}{p_-}}\|f\|_{L^1(\Omega)}\cdot\left(\frac{p_-}{4C_2C_7(C_5+1)}\right)^{\frac{1}{p_-}}\|u_n\|_{L^\infty(\Omega)} \\
&+ \left(\frac{4C_2C_7(C_5+1)}{p_-}\right)^{\frac{1}{p_-}}\|g\|_{L^1(\partial\Omega)}\cdot\left(\frac{p_-}{4C_2C_7(C_5+1)}\right)^{\frac{1}{p_-}}\|u_n\|_{L^\infty(\partial\Omega)} \\
&\leq \frac{1}{p_-'}\left(\frac{4C_2C_7(C_5+1)}{p_-}\right)^{\frac{p_-'}{p_-}}\|f\|^{p_-'}_{L^1(\Omega)} + \frac{1}{p_-}\frac{p_-}{4C_2C_7(C_5+1)}\|u_n\|^{p_-}_{L^\infty(\Omega)} \\
&+ \frac{1}{p_-'}\left(\frac{4C_2C_7(C_5+1)}{p_-}\right)^{\frac{p_-'}{p_-}}\|g\|^{p_-'}_{L^1(\partial\Omega)} + \frac{1}{p_-}\frac{p_-}{4C_2C_7(C_5+1)}\|u_n\|^{p_-}_{L^\infty(\partial\Omega)} \\
&\leq \frac{1}{p_-'}\left(\frac{4C_2C_7(C_5+1)}{p_-}\right)^{\frac{p_-'}{p_-}}\|f\|^{p_-'}_{L^1(\Omega)} \\
&+ \frac{1}{p_-'}\left(\frac{4C_2C_7(C_5+1)}{p_-}\right)^{\frac{p_-'}{p_-}}\|g\|^{p_-'}_{L^1(\partial\Omega)} \\
&+ \frac{1}{2C_2(C_5+1)}\|u_n\|^{p_-}_{W^{1,p_-}(\Omega)}. \qquad (3.15)
\end{aligned}$$

Moreover, as $p_- < \pi_n(.)$, we have

$$\int_\Omega |\nabla u_n|^{p_-}dx \leq meas(\Omega) + \int_\Omega |\nabla u_n|^{\pi_n(.)}dx. \qquad (3.16)$$

Combining (3.12) and (3.16), we get

$$\int_\Omega |u_n|^{p_-}dx \leq C_5 meas(\Omega) + C_5\int_\Omega |\nabla u_n|^{\pi_n(.)}dx + C_6.$$

We infer from above inequality and (3.16) that

$$\begin{aligned}
\|u_n\|^{p_-}_{W^{1,p_-}(\Omega)} &= \int_\Omega \left[|u_n|^{p_-} + |\nabla u_n|^{p_-}\right]dx \\
&\leq (C_5+1)meas(\Omega) + C_6 + (C_5+1)\int_\Omega |\nabla u_n|^{\pi_n(.)}dx. \qquad (3.17)
\end{aligned}$$

Furthermore, using (3.15) and (3.17), we obtain

$$\begin{aligned}\int_\Omega f_n u_n dx + \int_{\partial\Omega} g_n u_n d\sigma &\leq \frac{1}{p'_-}\left(\frac{4C_2C_7(C_5+1)}{p_-}\right)^{\frac{p'_-}{p_-}} \|f\|^{p'_-}_{L^1(\Omega)} \\ &+ \frac{1}{p'_-}\left(\frac{4C_2C_7(C_5+1)}{p_-}\right)^{\frac{p'_-}{p_-}} \|g\|^{p'_-}_{L^1(\partial\Omega)} \\ &+ \frac{meas(\Omega)}{2C_2} + \frac{C_6}{2C_2(C_5+1)} + \frac{1}{2C_2}\int_\Omega |\nabla u_n|^{\pi_n(.)} dx. \end{aligned} \tag{3.18}$$

Combining (3.14) and (3.18), we get

$$\begin{aligned}\int_\Omega T_n(b(u_n))u_n dx + \int_{\partial\Omega} \lambda T_n(u_n)u_n d\sigma &+ \frac{1}{2C_2}\int_\Omega |\nabla u_n|^{\pi_n(.)} dx \\ &+ \frac{1}{n}||u_n||^{p_+}_{W^{1,p_+}(\Omega)} \leq C_8, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned}C_8 = \frac{1}{p'_-}\left(\frac{4C_2C_7(C_5+1)}{p_-}\right)^{\frac{p'_-}{p_-}} \|f\|^{p'_-}_{L^1(\Omega)} &+ \frac{1}{p'_-}\left(\frac{4C_2C_7(C_5+1)}{p_-}\right)^{\frac{p'_-}{p_-}} \|g\|^{p'_-}_{L^1(\partial\Omega)} \\ &+ \frac{meas(\Omega)}{2C_2} + \frac{C_6}{2C_2(C_5+1)}. \end{aligned}$$

Thus, we deduce from (3.19) that

$$\int_\Omega |\nabla u_n|^{\pi_n(.)} dx \leq C_9. \tag{3.20}$$

Now, using (3.17) and (3.20), we infer

$$\|u_n\|_{W^{1,p_-}(\Omega)} \leq const\big(p_-, \Omega, f, g\big). \tag{3.21}$$

□

Lemma 3.8. *$(u_n)_{n\in\mathbb{N}^*}$ converges a.e. in $\partial\Omega$ to v.*

Proof. Since $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $W^{1,p_-}(\Omega)$, then, up to extraction of a subsequence still denoted $(u_n)_{n\in\mathbb{N}^*}$, it converges a.e. in Ω (and also weakly in $W^{1,p_-}(\Omega)$) to a limit u.
We know that the trace operator is compact from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$. Obviously, $W^{1,p_-}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ because $p_- > 1$. Therefore, $u_n \to u$ in $L^1(\partial\Omega)$ and a.e. in $\partial\Omega$. Thus, $v = u_{|\partial\Omega}$ has definite meaning. □

The following assertions are based on the Young measure and nonlinear weak$-^*$ convergence results (see [6, 12, 15]).

Assertion 1.

The sequence $(\nabla u_n)_{n\in\mathbb{N}^*}$ converges to a Young measure $\nu_x(\lambda)$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-* convergence, and

$$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda). \tag{3.22}$$

Proof. As $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $W^{1,p_-}(\Omega)$, then, up to extraction of a subsequence still denoted $(u_n)_{n\in\mathbb{N}^*}$, u_n converges a.e. in Ω (and also weakly in $W^{1,p_-}(\Omega)$) to a limit u.
Therefore, ∇u_n weakly converges to ∇u in $L^{p_-}(\Omega)$. Now, we prove that $(\nabla u_n)_{n\in\mathbb{N}}$ is equi-integrable.
$(|\nabla u_n|^{p_-})_{n\in\mathbb{N}}$ being bounded, is equi-integrable. As $p_- > 1$, then for all subset $E \subset \Omega$, we obtain

$$\int_E |\nabla u_n| dx \leq \int_E (1 + |\nabla u_n|^{p_-}) dx.$$

Therefore, for $meas(E)$ small enough, $(\nabla u_n)_{n\in\mathbb{N}}$ is equi-integrable on Ω. Then, using the representation of weakly convergent sequences in $L^1(\Omega)$ in terms of Young measures (See, Theorem 2.7-property (2.5)), we can write

$$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda).$$

□

Assertion 2.
$|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ on $\mathbb{R}^N \times \Omega$, moreover, $u \in W^{1,\pi(.)}(\Omega)$.

Proof. We know that π_n converges in measure to π. Using Theorem 2.7-[(ii),(iii)], $(\pi_n, \nabla u_n)_{n\in\mathbb{N}}$ converges on $\mathbb{R} \times \mathbb{R}^N$ to the Young measure $\mu_x = \delta_{\pi(x)} \otimes \nu_x$. Thus, we can apply the weak convergence property (2.4) of Theorem 2.7 to the Carathéodory function

$$F_m : (x, \lambda_0, \lambda) \in \Omega \times (\mathbb{R} \times \mathbb{R}^N) \mapsto |h_m(\lambda)|^{\lambda_0},$$

with $m \in \mathbb{N}$, where h_m is defined in the preliminaries. We have

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{\pi(.)} d\nu_x(\lambda)dx &= \int_{\Omega\times(\mathbb{R}\times\mathbb{R}^N)} |h_m(\lambda)|^{\lambda_0} d\mu_x(\lambda_0,\lambda)dx \\
&= \int_\Omega \int_{\mathbb{R}\times\mathbb{R}^N} F_m(x,\lambda_0,\lambda) d\mu_x(\lambda_0,\lambda)dx \\
&= \lim_{n\to\infty} \int_\Omega F_m(x,\pi_n(x),\nabla u_n(x))dx \\
&= \lim_{n\to\infty} \int_\Omega |h_m(\nabla u_n)|^{\pi_n(.)} dx \\
&\leq \lim_{n\to\infty} \int_\Omega |\nabla u_n|^{\pi_n(.)} dx \\
&\leq C_9.
\end{aligned}$$

Since $h_m(\lambda) \to \lambda$, as $m \to \infty$ and as $m \mapsto h_m(\lambda)$ is increasing, using Lebesgue convergence Theorem, we deduce from the above inequality that

$$\int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(.)} d\nu_x(\lambda)dx \leq C_9.$$

Hence, $|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ on $\mathbb{R}^N \times \Omega$.
Now, we prove that $\nabla u \in L^{\pi(.)}(\Omega)$. Using (3.22), Jensen inequality and the last inequality, we get

$$\int_\Omega |\nabla u|^{\pi(.)} dx = \int_\Omega \left| \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda) \right|^{\pi(.)} dx \leq \int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(.)} d\nu_x(\lambda)dx < \infty.$$

Thus, $\nabla u \in L^{\pi(.)}(\Omega)$. Moreover, $u \in L^{\pi(.)}(\Omega)$.
Indeed, $u \in W^{1,p_-}(\Omega) \subset L^\infty(\Omega) \subset L^{\pi(.)}(\Omega)$ for $p_- > N$. Hence, $u \in W^{1,\pi(.)}(\Omega)$. □

Assertion 3
The sequence Φ_n defined by $\Phi_n := a(x,u_n,\nabla u_n)$ is equi-integrable on Ω.

Proof. By using (A_5) with exponent $\pi_n(.)$, we obtain

$$|a(x,u_n,\nabla u_n)|^{\pi_n'(.)} \leq C_1(|\nabla u_n|^{\pi_n(.)} + \mathcal{M}(x)).$$

The above inequality give us

$$\begin{aligned}
|a(x,u_n,\nabla u_n)| &\leq C\big((1+|\nabla u_n|^{\pi_n(.)}) + \mathcal{M}(x)\big)^{\frac{1}{\pi_n'(.)}} \\
&\leq C\big((1+\mathcal{M}(x))^{\frac{1}{\pi_n'(.)}} + |\nabla u_n|^{\frac{\pi_n(.)}{\pi_n'(.)}}\big) \\
&\leq C(1+\mathcal{M}(x)+|\nabla u_n|^{\pi_n(.)-1}).
\end{aligned}$$

For all set $E \subset \Omega$,

$$\int_E |a(x,u_n,\nabla u_n)|dx \leq C\int_E (1+\mathcal{M}(x))dx + C_{10}\left|\left||\nabla u_n|^{\pi_n(.)-1}\right|\right|_{L^{\pi_n'(.)}(\Omega)} ||\chi_E||_{L^{\pi_n(.)}(\Omega)},$$

where $C_{10} = const(p_-)$. The first term of the right hand side of the above inequality is small for $meas(E)$ small enough, since $1 + \mathcal{M} \in L^1(\Omega)$.
According to Proposition 2.2, we obtain

$$\begin{aligned} ||\chi_E||_{L^{\pi_n(.)}(\Omega)} &\leq \max\left\{\rho_{\pi_n(.)}(\chi_E)^{\frac{1}{p_+}}; \rho_{\pi_n(.)}(\chi_E)^{\frac{1}{p_-}}\right\} \\ &= \max\left\{(meas(E))^{\frac{1}{p_-}}, (meas(E))^{\frac{1}{p_+}}\right\}. \end{aligned}$$

Analogously,

$$\begin{aligned} &\left|\left||\nabla u_n|^{\pi_n(.)-1}\right|\right|_{L^{\pi_n'(.)}(\Omega)} \\ \leq\ & \max\left\{\left(\rho_{\pi_n'(.)}(|\nabla u_n|^{\pi_n(.)-1})^{\frac{1}{(p')_+}}\right), \left(\rho_{\pi_n'(.)}(|\nabla u_n|^{\pi_n(.)-1})^{\frac{1}{(p')_-}}\right)\right\} \\ =\ & \max\left\{\left(\int_\Omega |\nabla u_n|^{\pi_n(.)}dx\right)^{\frac{1}{(p')_+}}, \left(\int_\Omega |\nabla u_n|^{\pi_n(.)}dx\right)^{\frac{1}{(p')_-}}\right\}. \end{aligned}$$

Using (3.20), $\int_E |a(x, u_n, \nabla u_n)|dx$ become small for $meas(E)$ small enough.
Hence, $(\Phi_n)_{n\in\mathbb{N}}$ is equi-integrable. □

Assertion 4
The weak limit Φ of Φ_n (or a subsequence) belongs to $L^{\pi'(.)}(\Omega)$ and we have

$$\Phi(x) = \int_{\mathbb{R}^N} a(x, u(x), \lambda)d\nu_x(\lambda). \tag{3.23}$$

Proof. Set $\tilde{\Phi}_n = a(x, u(x), \nabla v_n)$ with $\nabla v_n = \nabla u_n \chi_{S_n}$, where

$$S_n = \left\{x \in \Omega, \quad |\pi(x) - \pi_n(x)| < \frac{1}{2}\right\}.$$

We prove that $\tilde{\Phi}_n$ is equi-integrable on Ω.

We applied (A_5) with variable exponent $\pi(.)$ on $\tilde{\Phi}_n$.
Let $E \subset \Omega$, we have

$$\begin{aligned} \int_E |a(x, u(x), \nabla v_n)|dx &\leq C\int_E (1 + \mathcal{M}(x) + |\nabla v_n|^{\pi(.)-1})dx \\ &\leq C\left(\int_E (1 + \mathcal{M}(x))dx + \int_{E\cap S_n} |\nabla u_n|^{\pi(.)-1}dx\right). \end{aligned}$$

The first term of the right hand side of the last inequality is small for $meas(E)$ small enough.
For all $x \in S_n$, $\pi(x) < \pi_n(x) + \frac{1}{2}$, thus

$$\int_{E\cap S_n} |\nabla u_n|^{\pi(.)-1}dx \leq \int_E \left(1 + |\nabla u_n|^{\pi_n(.)-\frac{1}{2}}\right)dx$$

and

$$\int_\Omega |\nabla u_n|^{(\pi_n(.)-\frac{1}{2})(2\pi_n(.))'}dx = \int_\Omega |\nabla u_n|^{\pi_n(.)}dx < \infty,$$

which is equivalent to saying $|\nabla u_n|^{\pi_n(.)-\frac{1}{2}} \in L^{(2\pi_n(.))'}(\Omega)$. Now, using Hölder type inequality,

$$\begin{aligned}\int_{E\cap S_n} |\nabla u_n|^{\pi(.)-1}dx &\leq \int_E \left(1+|\nabla u_n|^{\pi_n(.)-\frac{1}{2}}\right)dx \\ &\leq meas(E) + 2\big|\big|\nabla u_n\big|\big|_{L^{\pi_n(.)}(\Omega)}||\chi_E||_{L^{2\pi_n(.)}(\Omega)}. \quad (3.24)\end{aligned}$$

According to Proposition 2.2,

$$\begin{aligned}||\chi_E||_{L^{2\pi_n(.)}(\Omega)} &\leq \max\left\{\big(\rho_{2\pi_n(.)}(\chi_E)\big)^{\frac{1}{2p_-}}, \big(\rho_{2\pi_n(.)}(\chi_E)\big)^{\frac{1}{2p_+}}\right\} \\ &= \max\left\{\big(meas(E)\big)^{\frac{1}{2p_-}}, \big(meas(E)\big)^{\frac{1}{2p_+}}\right\}.\end{aligned}$$

The right-hand side of (3.24) is uniformly small for $meas(E)$ small, and the equi-integrability of $\tilde{\Phi}_n$ follows. Therefore, (up to a subsequence) $\tilde{\Phi}_n$ weakly converges in $L^1(\Omega)$ to $\tilde{\Phi}$, as $n \to \infty$.
Now, we prove that $\tilde{\Phi} = \Phi$; more precisely, we show that $\tilde{\Phi}_n - \Phi_n$ strongly converges in $L^1(\Omega)$ to 0.
From (3.20), $\int_\Omega |\nabla u_n|^{\pi_n(.)}dx$ is uniformly bounded, which implies that $\int_\Omega |\nabla u_n|dx$ is finite, since

$$\int_\Omega |\nabla u_n|dx \leq \int_\Omega (1+|\nabla u_n|^{\pi_n(x)})dx.$$

By Chebyschev inequality, we have

$$meas(\{|\nabla u_n| > L\}) \leq \frac{\int_\Omega |\nabla u_n|dx}{L}.$$

Therefore, $\sup_{n\in\mathbb{N}} meas(\{|\nabla u_n| > L\})$ tends to 0 for L large enough. Since $\tilde{\Phi}_n - \Phi_n$ is equi-integrable, then for all $\beta > 0$, there exists $\delta = \delta(\beta)$ such that for all $A \subset \Omega$, $meas(A) < \delta$ and $\int_A |\tilde{\Phi}_n - \Phi_n|dx < \frac{\beta}{4}$.
Therefore, if we choose L large enough, we get $\frac{\int_\Omega |\nabla u_n|dx}{L} < \delta$, so $meas(\{|\nabla u_n| > L\}) < \delta$. Hence,

$$\int_{\{|\nabla u_n|>L\}} |\tilde{\Phi}_n - \Phi_n|dx < \frac{\beta}{4}.$$

By lemma 2.8, we also have

$$meas\left(\left\{x\in\Omega;\ \sup_{\lambda\in K}|a(x,u_n(x),\lambda)-a(x,u(x),\lambda)| \geq \sigma\right\}\right) \longrightarrow 0,$$

as $n \to \infty$.
Thus, by the above equi-integrability, for all $\sigma > 0$, there exists $n_0 = n_0(\sigma, L) \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\int_{\left\{x\in\Omega;\ \sup_{|\lambda|\leq L}|a(x,u_n(x),\lambda)-a(x,u(x),\lambda)|\geq\sigma\right\}} |\tilde{\Phi}_n - \Phi_n|dx < \frac{\beta}{4}.$$

Using the definition of Φ_n and $\tilde{\Phi}_n$, we have

$$\tilde{\Phi}_n - \Phi_n = a(x, u_n(x), \nabla u_n) - a(x, u(x), \nabla u_n) \ \text{ on } \ S_n.$$

Now, we reason on

$$S_{n,L,\sigma} := \left\{ x \in \Omega;\ \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| < \sigma, |\nabla u_n| \leq L \right\}.$$

We get

$$\begin{aligned} \int_{S_n\cap S_{n,L,\sigma}} |\tilde{\Phi}_n - \Phi_n|dx &\leq \int_{S_{n,L,\sigma}} \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)|dx \\ &\leq \sigma meas(\Omega). \end{aligned}$$

We observe that

$$\int_{S_n} |\tilde{\Phi}_n - \Phi_n|dx = \int_{S_n\cap S_{n,L,\sigma}} |\tilde{\Phi}_n - \Phi_n|dx + \int_{S_n\setminus S_{n,L,\sigma}} |\tilde{\Phi}_n - \Phi_n|dx$$

and

$$S_n \setminus S_{n,L,\sigma} \subset \left\{ x \in \Omega;\ \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| \geq \sigma \right\} \cup \left\{ |\nabla u_n| > L \right\}.$$

Consequently, by choosing $\sigma = \sigma(\beta) < \dfrac{\beta}{4meas(\Omega)}$, we get

$$\int_{S_n} |\tilde{\Phi}_n - \Phi_n|dx < \frac{\beta}{4} + \frac{\beta}{4} + \frac{\beta}{4} = \frac{3\beta}{4},$$

for all $n \geq n_0(\sigma, L)$. By lemma 2.8, we also have
$meas(\{x \in \Omega, |\pi(x) - \pi_n(x)| \geq \frac{1}{2}\}) \to 0$ for n large enough; which means that $meas(\Omega \setminus S_n)$ converges to 0 for n large enough. Thus,

$$\int_{\Omega\setminus S_n} |\tilde{\Phi}_n - \Phi_n|dx = \int_{\Omega\setminus S_n} |\Phi_n|dx \leq \frac{\beta}{4}.$$

Therefore, for all $\beta > 0$ there exists $n_0 = n_0(\beta)$ such that for all $n \geq n_0$, $\displaystyle\int_\Omega |\tilde{\Phi}_n - \Phi_n|dx \leq \beta$.
Hence, $\tilde{\Phi}_n - \Phi_n$ strongly converges to 0 in $L^1(\Omega)$. We prove that

$$\Phi(x) = \int_{\mathbb{R}^N} a(x, u(x), \lambda)d\nu_x(\lambda) \quad a.e.\ x \in \Omega \ \text{ and } \ \Phi \in L^{\pi'(\cdot)}(\Omega).$$

Notice that

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n|(1 - \chi_{S_n})dx = \lim_{n\to+\infty} \int_{\Omega\setminus S_n} |\nabla u_n|dx = 0,$$

since $(\nabla u_n)_{n\in\mathbb{N}}$ is equi-integrable and $meas(\Omega \setminus S_n)$ converges to 0 for n large enough.
Therefore, $(\nabla u_n)_{n\in\mathbb{N}}$ and $\nabla u_n \chi_{S_n}$ converge to the same Young measure $\nu_x(\lambda)$. Moreover, by applying the Theorem 2.7-(i) to the Carathéodory function $F(x,(\lambda_0,\lambda)) := a(x,\lambda_0,\lambda)$, we infer that

$$\tilde{\Phi}(x) = \Phi(x) = \int_{\mathbb{R}^N} a(x,u(x),\lambda) d\nu_x(\lambda) \text{ a.e. } x \in \Omega.$$

Using (A_5), it follows that $|a(x,u(x),\lambda)|^{\pi'(.)} \leq C(\mathcal{M}(x) + |\lambda|^{\pi(.)})$. Thus, with Jensen inequality, it follows that

$$\begin{aligned}
\int_\Omega |\Phi(x)|^{\pi'(.)} dx &= \int_\Omega \left| \int_{\mathbb{R}^N} a(x,u(x),\lambda) d\nu_x(\lambda) \right|^{\pi'(.)} dx \\
&\leq \int_{\Omega\times\mathbb{R}^N} \left| a(x,u(x),\lambda) \right|^{\pi'(.)} d\nu_x(\lambda) dx \\
&\leq C \int_{\Omega\times\mathbb{R}^N} \left(\mathcal{M}(x) + |\lambda|^{\pi(.)} \right) d\nu_x(\lambda) dx < \infty.
\end{aligned}$$

Hence, $\Phi \in L^{\pi'(.)}(\Omega)$. □

Assertion 5

$$\int_\Omega \Phi.\nabla u dx \geq \int_{\Omega\times\mathbb{R}^N} a(x,u(x),\lambda).\lambda d\nu_x(\lambda) dx. \tag{3.25}$$

Proof. For all $\varphi \in C^\infty(\overline{\Omega})$, we have

$$\begin{aligned}
\int_\Omega T_n(b(u_n))\varphi dx &+ \int_{\partial\Omega} \lambda T_n(u_n)\varphi d\sigma + \int_\Omega a(x,u_n,\nabla u_n).\nabla\varphi dx \\
&+ \frac{1}{n}\int_\Omega \left[|\nabla u_n|^{p_+-2}\nabla u_n \nabla\varphi + |u_n|^{p_+-2}u_n\varphi \right] dx \\
&= \int_\Omega f_n\varphi dx + \int_{\partial\Omega} g_n\varphi d\sigma.
\end{aligned} \tag{3.26}$$

Letting n goes to ∞ in (3.26), we obtain

$$\int_\Omega b(u)\varphi dx + \int_{\partial\Omega} \lambda u\varphi d\sigma + \int_\Omega \Phi.\nabla\varphi dx = \int_\Omega f\varphi dx + \int_{\partial\Omega} g\varphi d\sigma. \tag{3.27}$$

Indeed, $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in the space $W^{1,p_-}(\Omega)$ and $p_- > N$. Then, by embedding result $(W^{1,p_-}(\Omega) \hookrightarrow L^\infty(\Omega))$, $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^\infty(\Omega)$. Thus, as $b(.)$ is continuous, then $(b(u_n))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^\infty(\Omega)$. Otherwise, $T_n(b(u_n))$ converges to $b(u)$ a.e. in Ω. Therefore, thanks to Lebesgue dominated convergence Theorem, $T_n(b(u_n))$ converges to $b(u)$ in $L^1(\Omega)$.
For the third term of the left hand side of (3.26), we use the fact that $a(x,u_n,\nabla u_n) = \Phi_n$ weakly converges to Φ and $\nabla\varphi \in L^\infty(\Omega)$.
The fourth term of the left hand side of (3.26) goes to 0 for n large enough, since

$$\frac{1}{n}\|u_n\|^{p_+}_{W^{1,p_+}(\Omega)} \leq C_8.$$

We know that $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $W^{1,p_-}(\Omega)$ and $W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $p_- > N$. Therefore, $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^\infty(\partial\Omega)$ and

$$\lambda|T_n(u_n)\varphi| \leq \lambda|u_n||\varphi|.$$

We also recall that $\lambda T_n(u_n)\varphi \to \lambda u\varphi$ a.e. in $\partial\Omega$, which implies that

$$\lim_{n\to\infty}\int_{\partial\Omega}\lambda T_n(u_n)\varphi d\sigma = \int_{\partial\Omega}\lambda u\varphi d\sigma,$$

thanks to Lebesgue dominated convergence Theorem.
We are now interested to the right-hand side of (3.26). For the first term of the right-hand side of (3.26), we have

$$\lim_{n\to\infty}\int_{\Omega} f_n\varphi dx = \int_{\Omega} f\varphi dx,$$

since f_n strongly converges to f in $L^1(\Omega)$ and $\varphi \in L^\infty(\Omega)$. In same manner, for the second term of the right-hand side of (3.26), we get

$$\lim_{n\to\infty}\int_{\partial\Omega} g_n\varphi d\sigma = \int_{\Omega} g\varphi d\sigma.$$

The relation (3.27) is then justified.
One the other hand, by the density argument we can replace φ with u_n in (3.26) to get

$$\begin{aligned}\int_\Omega T_n(b(u_n))u_n dx + \int_{\partial\Omega}\lambda T_n(u_n)u_n d\sigma \quad &+ \quad \int_\Omega a(x,u_n,\nabla u_n).\nabla u_n dx \\ &+ \quad \frac{1}{n}\int_\Omega \left[|\nabla u_n|^{p_+} + |u_n|^{p_+}\right]dx \\ &= \quad \int_\Omega f_n u_n dx + \int_{\partial\Omega} g_n u_n d\sigma. \quad (3.28)\end{aligned}$$

$u \in W^{1,\pi(.)}(\Omega) \subset W^{1,p_-}(\Omega) \subset C^{0,\alpha}(\overline{\Omega})$ and $p(.,.)$ is locally uniformly log-Hölder continuous, then the exponent $\pi(.)$ verifies (2.1). Therefore, $C^\infty(\overline{\Omega})$ is dense in $W^{1,\pi(.)}(\Omega)$, so, we change φ by u in (3.27) to obtain

$$\int_\Omega b(u)u dx + \int_{\partial\Omega}\lambda u^2 d\sigma + \int_\Omega \Phi.\nabla u dx = \int_\Omega f u dx + \int_{\partial\Omega} g u d\sigma. \quad (3.29)$$

The sequence $(T_n(b(u_n))u_n)_{n\in\mathbb{N}}$ is nonnegative. $T_n(b(u_n))u_n$ is also measurable and converges a.e. in Ω to $b(u)u$. By Fatou's Lemma, we get

$$\liminf_{n\to\infty}\int_\Omega T_n(b(u_n))u_n dx \geq \int_\Omega b(u)u dx. \quad (3.30)$$

In the same manner,

$$\liminf_{n\to\infty}\int_{\partial\Omega}\lambda T_n(u_n)u_n d\sigma \geq \int_{\partial\Omega}\lambda u^2 d\sigma. \quad (3.31)$$

Moreover, the sequence $(f_n u_n)_{n\in\mathbb{N}*}$ converges a.e. in Ω to fu and

$$|f_n u_n| \leq |f||u_n||_{L^\infty(\Omega)}.$$

$(u_n)_{n\in\mathbb{N}^*}$ is also uniformly bounded in $L^\infty(\Omega)$. Applying Lebesgue dominated convergence Theorem, we get

$$\lim_{n\to\infty}\int_\Omega f_n u_n dx = \int_\Omega f u dx. \tag{3.32}$$

By adopting the same approach as before, we have

$$\lim_{n\to\infty}\int_{\partial\Omega} g_n u_n d\sigma = \int_{\partial\Omega} g u d\sigma. \tag{3.33}$$

By combining (3.30), (3.31) (3.32) and (3.33), we obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\left(\int_\Omega f_n u_n dx + \int_{\partial\Omega} g_n u_n d\sigma\right) - \int_\Omega b(u)u dx - \int_{\partial\Omega}\lambda u^2 d\sigma \\
\geq\ &\liminf_{n\to\infty}\left(\int_\Omega f_n u_n dx + \int_{\partial\Omega} g_n u_n d\sigma - \int_\Omega T_n(b(u_n))u_n dx - \int_{\partial\Omega}\lambda T_n(u_n)u_n d\sigma\right).
\end{aligned}$$

By using (3.28), (3.29), the above inequality and the definition of Φ_n, we get

$$\int_\Omega \Phi.\nabla u dx \geq \liminf_{n\to\infty}\int_\Omega\left(\Phi_n.\nabla u_n + \frac{1}{n}\left[|\nabla u_n|^{p_+}+|u_n|^{p_+}\right]\right)dx \geq \liminf_{n\to\infty}\int_\Omega \Phi_n.\nabla u_n dx.$$

Hence,

$$\int_\Omega \Phi.\nabla u dx \geq \liminf_{n\to\infty}\int_\Omega \Phi_n.\nabla u_n dx. \tag{3.34}$$

By [1, Lemma 2.1], $m\mapsto a(x,u_n,h_m(\nabla u_n)).h_m(\nabla u_n)$ is increasing and converges to $a(x,u_n,\nabla u_n).\nabla u_n$ for m large enough. Then,

$$a(x,u_n,h_m(\nabla u_n)).h_m(\nabla u_n) \leq a(x,u_n,\nabla u_n).\nabla u_n.$$

Therefore, by using (3.34) and Theorem 2.7, we have

$$\begin{aligned}
\int_\Omega \Phi.\nabla u dx &\geq \liminf_{n\to\infty}\int_\Omega \Phi_n.\nabla u_n dx \\
&\geq \lim_{n\to\infty}\int_\Omega a(x,u_n,h_m(\nabla u_n)).h_m(\nabla u_n)dx \\
&= \int_{\Omega\times\mathbb{R}^N} a(x,u,h_m(\lambda)).h_m(\lambda)d\nu_x(\lambda)dx.
\end{aligned} \tag{3.35}$$

Using Lebesgue dominated convergence Theorem in (3.35), as m goes to ∞, we get (3.25). □

Assertion 6
The "div-curl" inequality holds.

$$\int_{\Omega\times\mathbb{R}^N}\big(a(x,u(x),\lambda)-a(x,u(x),\nabla u(x))\big)(\lambda-\nabla u(x))d\nu_x(\lambda)dx \leq 0.$$

Proof. We have

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^N}(a(x,u(x),\lambda)-a(x,u(x),\nabla u(x)))(\lambda-\nabla u(x))d\nu_x(\lambda)dx\\
=&\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx-\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\nabla u(x)d\nu_x(\lambda)dx\\
-&\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\nabla u(x)).\lambda d\nu_x(\lambda)dx\\
+&\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\nabla u(x)).\nabla u(x)d\nu_x(\lambda)dx\\
=&\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx-\int_\Omega\left(\int_{\mathbb{R}^N}a(x,u(x),\lambda)d\nu_x(\lambda)\right)\nabla u(x)dx\\
-&\int_\Omega a(x,u(x),\nabla u(x)).\left(\int_{\mathbb{R}^N}\lambda d\nu_x\right)dx\\
+&\int_\Omega a(x,u(x),\nabla u(x)).\nabla u(x)\left(\int_{\mathbb{R}^N}d\nu_x\right)dx\\
=&\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx-\int_\Omega\Phi.\nabla u dx\le 0.
\end{aligned}$$

We pass from the first equality to the second equality by using Fubini Theorem and from the second inequality to the third one, by using (3.22) and the fact that ν_x is a probability measure on $\mathbb{R}^N$. Finally, (3.23) and (3.25) give us the desired inequality. □

Assertion 7
$\Phi(x)=a(x,u(x),\nabla u(x))$ a.e. $x\in\Omega$ and ∇u_n converges to ∇u in measure on Ω, as $n\to\infty$.

Proof. From Assertion 6 and relation (A_4), we deduce that

$$\big(a(x,u(x),\lambda)-a(x,u(x),\nabla u(x))\big)\big(\lambda-\nabla u(x)\big)=0\quad a.e.\quad x\in\Omega,\quad \lambda\in\mathbb{R}^N.$$

Thus, $\lambda=\nabla u(x)$ a.e. $x\in\Omega$ wrt ν_x on $\mathbb{R}^N$; therefore $\nu_x(\nabla u(x))=1$ and $\delta_{\nabla u}=\nu_x$. By using (3.23), we get

$$\Phi(x)=\int_{\mathbb{R}^N}a(x,u(x),\lambda)d\nu_x(\lambda)=a(x,u(x),\nabla u(x))\ \text{ a.e. } x\in\Omega.$$

Thus, the Theorem 2.7-(ii) show that ∇u_n converges in measure to ∇u. □

Lemma 3.9. *u is a weak solution of the problem (1.1).*

Proof. We first prove that $b(u)\in L^1(\Omega)$ and $u\in L^1(\partial\Omega)$.
Using Lemma 3.7 and Lemma 3.8, it follows from Fatou's Lemma that

$$\int_\Omega|b(u)|dx\le\liminf_{n\to\infty}\int_\Omega\big|T_n(b(u_n))\big|dx\le\|f\|_{L^1(\Omega)}+\|g\|_{L^1(\partial\Omega)}$$

and

$$\int_{\partial\Omega}|u|d\sigma\le\liminf_{n\to\infty}\int_{\partial\Omega}\big|T_n(u_n)\big|d\sigma\le\frac{\|f\|_{L^1(\Omega)}+\|g\|_{L^1(\partial\Omega)}}{\lambda}.$$

Hence, $b(u) \in L^1(\Omega)$ and $u \in L^1(\partial\Omega)$.
Now, by using Assertion 4, Assertion 7 and thanks to the density argument ($C^\infty(\overline{\Omega})$ is dense in the space $W^{1,\pi(.)}(\Omega)$), we can take φ in $W^{1,\pi(.)}(\Omega)$ as test function in (3.27) to get

$$\int_\Omega b(u)\varphi dx + \int_{\partial\Omega} \lambda u\varphi d\sigma + \int_\Omega a(x,u,\nabla u)\nabla\varphi dx = \int_\Omega f\varphi dx + \int_{\partial\Omega} g\varphi\sigma.$$

Hence, u is a weak solution of (1.1). □

Now we state the uniqueness result of weak solution. This result uses the same arguments that [2, Theorem 2.8].

Theorem 3.10. *Assume that b is strictly increasing.*
Assume that $(A_3), (A_4), (A_5), (A_6)$ and (1.2) hold, and $\mathcal{M}$ taken constant in (A_5).
Moreover, $\boldsymbol{a}$ satisfies:
for all bounded subset K of $\mathbb{R}\times\mathbb{R}^N$, there exists a constant $C(K)$ such that

$$\begin{aligned} &\text{a.e. } x\in\Omega, \text{ for all } (z,\eta),(\tilde{z},\eta)\in K,\\ &|a(x,z,\eta)-a(x,\tilde{z},\eta)|\le C(K)|z-\tilde{z}|. \end{aligned} \tag{3.36}$$

Finally, suppose the following regularity property:

$$\begin{aligned} &\text{for all } f\in L^\infty(\Omega) \text{ and } g\in L^\infty(\partial\Omega)\\ &\text{there exists a weak solution of (1.1),}\\ &\text{which is Lipchitz continuous on } \overline{\Omega}. \end{aligned} \tag{3.37}$$

Then, for all $f\in L^1(\Omega)$ and $g\in L^1(\partial\Omega)$ the problem (1.1) admits a unique weak solution.

Remark 3.11. As in [2, Theorem 2.8], the condition (3.37) goes back to idea of [3]. Moreover, in the Theorem 3.10 the relation (3.36) is used to obtain the inequality (3.43) below.

Proof. The existence has already been proved. Now, we show the uniqueness. For more details see [2, Proof of Theorem 2.8].
Let u be a Lipschitz continuous weak solution of (1.1) with $f\in L^\infty(\Omega), g\in L^\infty(\partial\Omega)$ and v be a weak solution, with $\hat{f}\in L^1(\Omega), \hat{g}\in L^1(\partial\Omega)$.
The function $\phi := \frac{1}{k}T_k(u-v)$ is an admissible test function in the weak formulation of u and v. Indeed, as Ω is open bounded domain with smooth boundary $\partial\Omega$, the space of Lipschitz functions $C^{0,1}(\overline{\Omega})$ and $W^{1,\infty}(\Omega)$ are homeomorphic and they can be identified. Moreover, ϕ belongs to $W^{1,1}(\Omega)\cap L^\infty(\Omega)$ and even $\phi\in L^\infty(\partial\Omega)$. As u is bounded, we have

$$\phi := \frac{1}{k}T_k(u-v) = \frac{1}{k}T_k(u - T_{k+||u||_{L^\infty}}(v)) \tag{3.38}$$

with

$$|\phi|\le 1 \text{ and } \nabla\phi = \frac{1}{k}\nabla(u-v)\chi_{[|u-v|<k]}. \tag{3.39}$$

Firstly, we use the fact that ∇u is bounded and assumption (A_4) of this theorem to get

$$|a(x,u,\nabla u)| \leq C\big(|\nabla u|^{p(.,u(.))}+1\big) \in L^\infty(\Omega).$$

Thus, $\phi \in W^{1,1}(\Omega)\cap L^\infty(\Omega)$ is admissible as a test function in the weak formulation for the solution u which belongs to $W^{1,\infty}(\Omega)$. Secondly, if $v \in W^{1,p(.,v(.))}(\Omega)$, then by (3.38), Lemma 2.4 and because $u \in W^{1,p_+}(\Omega) \subset W^{1,p(.,v(.))}(\Omega)$, we obtain $\phi \in W^{1,p(.,v(.))}(\Omega)\cap L^\infty(\Omega)$ as test function in the weak formulation for the solution v.
Hence, ϕ is necessary both an admissible test function for u and v. Thus, with this test function defined in (3.38) and (3.39), we have

$$\begin{aligned}\int_\Omega b(u)\frac{1}{k}T_k(u-v)dx &+ \int_{\partial\Omega}\lambda u\frac{1}{k}T_k(u-v)d\sigma\\ &+ \frac{1}{k}\int_\Omega a(x,u,\nabla u).\nabla(u-v)\chi_{[0<|u-v|<k]}dx\\ &= \int_\Omega f\frac{1}{k}T_k(u-v)dx+\int_{\partial\Omega}g\frac{1}{k}T_k(u-v)d\sigma \quad (3.40)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega b(v)\frac{1}{k}T_k(u-v)dx &+ \int_{\partial\Omega}\lambda v\frac{1}{k}T_k(u-v)d\sigma\\ &+ \frac{1}{k}\int_\Omega a(x,v,\nabla v).\nabla(u-v)\chi_{[0<|u-v|<k]}dx\\ &= \int_\Omega \hat{f}\frac{1}{k}T_k(u-v)dx+\int_{\partial\Omega}\hat{g}\frac{1}{k}T_k(u-v)d\sigma. \quad (3.41)\end{aligned}$$

Subtracting (3.40) and (3.41), we get

$$\begin{aligned}&\int_\Omega \big(b(u)-b(v)\big)\frac{1}{k}T_k(u-v)dx+\int_{\partial\Omega}\lambda(u-v)\frac{1}{k}T_k(u-v)d\sigma\\ &+ \frac{1}{k}\int_\Omega \big(a(x,u,\nabla u)-a(x,v,\nabla v)\big).\nabla(u-v)\chi_{[0<|u-v|<k]}dx\\ &= \int_\Omega \big(f-\hat{f}\big)\frac{1}{k}T_k(u-v)dx\\ &+ \int_{\partial\Omega}\big(g-\hat{g}\big)\frac{1}{k}T_k(u-v)d\sigma. \quad (3.42)\end{aligned}$$

Let's denote by I the third term of left hand side of (3.42). We know that

$$\begin{aligned}(a(x,u,\nabla u)-a(x,v,\nabla v))\nabla(u-v) &= (a(x,u,\nabla u)-a(x,v,\nabla u))\nabla(u-v)\\ &+ \underbrace{(a(x,v,\nabla u)-a(x,v,\nabla v))\nabla(u-v)}_{\geq 0}.\end{aligned}$$

We have

$$I = I_k + \int_\Omega \big(a(x,v,\nabla u)-a(x,v,\nabla v)\big)\frac{1}{k}\nabla(u-v)\chi_{[0<|u-v|<k]}dx,$$

where

$$I_k = \int_\Omega \big(a(x,u,\nabla u) - a(x,v,\nabla u)\big)\frac{1}{k}\nabla(u-v)\chi_{[0<|u-v|<k]}dx.$$

Let's show that $I_k \to 0$ as $k \to 0$. Since u is bounded, then v is also bounded on the set $[0<|u-v|<k]$. Thus, it follows from (3.36) that

$$\begin{aligned} |I_k| &\leq \frac{1}{k}\int_{[0<|u-v|<k]} |a(x,u,\nabla u) - a(x,v,\nabla u)||\nabla u - \nabla v|dx \\ &\leq \frac{1}{k}\int_{[0<|u-v|<k]} C\big(||u||_{L^\infty(\Omega)}, ||\nabla u||_{L^\infty(\Omega)}\big)|u-v||\nabla u - \nabla v|dx \\ &\leq C\big(||u||_{L^\infty(\Omega)}, ||\nabla u||_{L^\infty(\Omega)}\big)\int_{[0<|u-v|<k]} |\nabla u - \nabla v|dx \to 0, \end{aligned} \tag{3.43}$$

as $k \to 0$. Note that $\lim_{k\to 0} meas([0<|u-v|<k]) = 0$ and $|\nabla u - \nabla v| \in L^1(\Omega)$. For the first term of the left hand side of (3.42), one has

$$\begin{aligned} \lim_{k\to 0}\int_\Omega \big(b(u)-b(v)\big)\frac{1}{k}T_k(u-v)dx &= \int_\Omega (b(u)-b(v))sign_0(u-v)dx \\ &= \int_\Omega |b(u)-b(v)|dx. \end{aligned} \tag{3.44}$$

In the same manner

$$\begin{aligned} \lim_{k\to 0}\int_{\partial\Omega} \lambda(u-v)\frac{1}{k}T_k(u-v)d\sigma &= \int_{\partial\Omega} \lambda\big(u-v\big)sign_0(u-v)d\sigma \\ &= \int_{\partial\Omega} \lambda\big|u-v\big|d\sigma, \end{aligned} \tag{3.45}$$

$$\lim_{k\to 0}\int_\Omega \big(f-\hat{f}\big)\frac{1}{k}T_k(u-v)dx = \int_\Omega |f-\hat{f}|dx \tag{3.46}$$

and

$$\lim_{k\to 0}\int_{\partial\Omega} \big(g-\hat{g}\big)\frac{1}{k}T_k(u-v)d\sigma = \int_{\partial\Omega} |g-\hat{g}|d\sigma. \tag{3.47}$$

Finally, one makes k goes to 0 in (3.42) and taking into account inequalities (3.43), (3.44), (3.45), (3.46) and (3.47) , we get

$$\begin{aligned} \int_\Omega |b(u)-b(v)|dx &+ \int_{\partial\Omega}\lambda\big|u-v\big|d\sigma \\ &+ \lim_{k\to 0}\int_\Omega \big(a(x,v,\nabla u) - a(x,v,\nabla v)\big)\frac{1}{k}\nabla(u-v)\chi_{[0<|u-v|<k]}dx \\ &= \int_\Omega |f-\hat{f}|dx + \int_{\partial\Omega}|g-\hat{g}|d\sigma. \end{aligned} \tag{3.48}$$

Since, the three integral of left-hand in (3.48) are nonnegative, we deduce that

$$\int_\Omega |b(u)-b(v)|dx + \lambda\int_{\partial\Omega}|u-v|d\sigma \leq \int_\Omega |f-\hat{f}\big|dx + \int_{\partial\Omega}\big|g-\hat{g}\big|d\sigma. \tag{3.49}$$

Let us take a sequence $(f_i)_{i\in\mathbb{N}} \subset L^\infty(\Omega)$ and $(g_i)_{i\in\mathbb{N}} \subset L^\infty(\partial\Omega)$ and let $(u_i)_{i\in\mathbb{N}}$ be the corresponding sequence of Lipschitz continuous weak solutions. By (3.49), we have

$$\begin{aligned}\int_\Omega |b(u)-b(v)|dx \;+\; \lambda\int_{\partial\Omega}|u-v|d\sigma &\le \int_\Omega \big[|b(u)-b(u_i)|+|b(v)-b(u_i)|\big]dx\\ &+\; \lambda\int_{\partial\Omega}\big[|u-u_i|+|v-u_i|\big]d\sigma\\ &\le \int_\Omega \big[|f-f_i|+|\hat f-f_i|\big]dx\\ &+\; \int_{\partial\Omega}\big[|g-g_i|+|\hat g-g_i|\big]d\sigma, \end{aligned}\tag{3.50}$$

so that at the limit as $i\to\infty$ in left hand-side of (3.50) and using the density argument ($L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$ are dense (respectively) in $L^1(\Omega)$ and $L^1(\partial\Omega)$), we infer that

$$b(u)=b(v) \text{ a.e. in } \Omega \text{ and } u=v \text{ a.e. on } \partial\Omega.$$

Hence,

$$u=v \text{ a.e. in } \Omega \text{ and } u=v \text{ a.e. on } \partial\Omega.$$

Since b is assumed strictly increasing. □

4. Continuous dependence for the weak solution

Here, we are interested to the stability result of weak solutions to the problems

$$(P_n)\begin{cases} b(u_n)-\mathrm{div}a_n(x,u_n,\nabla u_n)=f_n \text{ in } \Omega\\ \\ a_n(x,u_n,\nabla u_n).\eta+\lambda u_n=g_n \quad \text{on } \partial\Omega. \end{cases}$$

$(a_n)_{n\in\mathbb{N}}$ is a sequence of diffusion flux function such that $a_n(x,z,\xi)$ satisfies $(A_3)-(A_6)$ with variable exponent $p_n : \Omega\times\mathbb{R}\to[p_-,p_+]$ and $(\mathcal{M}_n)_{n\in\mathbb{N}}$ is equi-integrable on Ω. In the sequel, we make the log-Hölder continuous and convergence hypothesis.

$$\left[\begin{array}{l} p_n : \overline{\Omega}\times\mathbb{R}\longrightarrow[p_-,p_+], \text{with} \quad p_->N \text{ and } \quad \forall M>0,\\ p_n \text{ is log-Hölder continuous in } (x,z) \text{ uniformly on } \overline{\Omega}\times[-M,M]. \end{array}\right.\tag{4.1}$$

$$\left[\begin{array}{l} \text{For all bounded subset } K \text{ of } \mathbb{R}\times\mathbb{R}^N,\\ \sup_{(z,\xi)\in K}|a_n(.,z,\xi)-a(.,z,\xi)|\to 0 \text{ in measure on } \Omega. \end{array}\right.\tag{4.2}$$

$$\left[\begin{array}{l} \text{For all bounded subset } K \text{ of } \mathbb{R},\\ \sup_{z\in K}|p_n(.,z)-p(.,z)| \text{ converges to zero in measure on } \Omega. \end{array}\right.\tag{4.3}$$

Finally, assume that

$$(f_n)_{n\in\mathbb{N}} \text{ is a sequence of data weakly convergent to } f \text{ in } L^1(\Omega)\tag{4.4}$$

and

$$(g_n)_{n\in\mathbb{N}} \text{ is a sequence of data weakly convergent to } g \text{ on } L^1(\partial\Omega).\tag{4.5}$$

The following structural stability result for weak solutions holds.

Theorem 4.1. *Let u_n be a weak solution of (P_n).*
Assume that $a(x,z,\xi)$ satisfies $(A_3)-(A_6)$ with a variable exponent $p(x,z)$. Assume that (A_2) and (1.2) hold. Assume that $(a_n)_{n\in\mathbb{N}}$ is a sequence of diffusion flux functions of the form $a_n(x,z,\xi)$ such that $(A_3)-(A_6)$ hold with $p_n(x,z)$, C independent of n and with a sequence $(\mathcal{M}_n)_{n\in\mathbb{N}}$ equi-integrable on Ω. Furthermore, let us consider the assumptions (4.1)-(4.5).
Then, there exists a measurable function $u\in C(\overline{\Omega})$ such that u_n converges to u a.e. in Ω and a.e. on $\partial\Omega$, and ∇u_n converges to ∇u a.e. in Ω, as $n\to\infty$ (up to extraction of subsequence). The function u is a weak solution of the problem (1.1) associated to $a(.,.,.)$, f and g.

The proof is organized in several steps and we reason up to an extracted subsequence of $(u_n)_{n\in\mathbb{N}}$ still denoted $(u_n)_{n\in\mathbb{N}}$.

Claim 1. Let u_n be a weak solution of (P_n). Then,
(i) the sequence $(b(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $L^1(\Omega)$;
(ii) there exists a positive constant $C(f,g,p_-)$ such that

$$\int_\Omega |\nabla u_n|^{p_n(.,u_n(.))}dx\le C(f,g,p_-).$$

Moreover, $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,p_-}(\Omega)$.

Proof. (i) Taking $\varphi=T_k(u_n)$ as a test function in a weak formulation of the problem (P_n), where a_n replace a, we get

$$\begin{aligned}\int_\Omega b(u_n)T_k(u_n)dx+\int_{\partial\Omega}\lambda u_nT_k(u_n)d\sigma \;&+\; \int_\Omega a_n(x,u_n,\nabla u_n)\nabla T_k(u_n)dx\\ &= \int_\Omega f_nT_k(u_n)dx+\int_{\partial\Omega}g_nT_k(u_n)d\sigma.\end{aligned}$$

Since all the terms of the left-hand side of the above equality are nonnegative, we deduce that

$$\int_\Omega b(u_n)T_k(u_n)dx\le\int_\Omega|f_nT_k(u_n)|dx+\int_\Omega|g_nT_k(u_n)|d\sigma\le k\int_\Omega|f_n|dx+k\int_{\partial\Omega}|g_n|d\sigma.$$

Since the sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ converge, respectively weakly in $L^1(\Omega)$ and in $L^1(\partial\Omega)$, then the right-hand side of the above inequality is uniformly bounded. So, there exists a positive constant C such that

$$\int_\Omega b(u_n)T_k(u_n)dx\le kC.$$

Dividing the above inequality by k and letting k goes to 0, we get

$$\int_\Omega|b(u_n)|dx\le C. \tag{4.6}$$

(ii) From above inequality, we deduce that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^1(\Omega)$, since b is continuous and onto. Thus, from Poincaré-Wirtinger inequality, we obtain

$$\int_\Omega|u_n|^{p_-}dx\le const(p_-,\Omega)\int_\Omega|\nabla u_n|^{p_-}dx+C_0, \tag{4.7}$$

where C_0 depends on $meas(\Omega)$ and p_-.
Moreover, using (A_6) with variable exponent $p_n(x,u_n(x))$ on $a_n(x,u_n,\nabla u_n)$ (See the reasoning that lead to (3.19) in the proof of Assertion 1.), the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the standard estimation

$$\int_\Omega b(u_n)u_n dx + \int_{\partial\Omega} \lambda u_n^2 d\sigma + \frac{1}{2C_2}\int_\Omega |\nabla u_n|^{p_n(x,u_n(x))}dx \leq C, \tag{4.8}$$

with C that depends on f, g, $meas(\Omega)$ and p_- but no on n. From the above inequality we deduce that

$$\int_\Omega |\nabla u_n|^{p_n(.,u_n(.))}dx \leq C(f,g,\Omega,p_-). \tag{4.9}$$

Moreover, using (4.7), (4.9) and the fact that

$$\int_\Omega |\nabla u_n|^{p_-}dx \leq meas(\Omega) + \int_\Omega |\nabla u_n|^{p_n(.,u_n(.))}dx,$$

we obtain

$$\begin{aligned}
\|u_n\|^{p_-}_{W^{1,p_-}(\Omega)} &= \int_\Omega \left[|u_n|^{p_-} + |\nabla u_n|^{p_-}\right]dx \\
&\leq \big(const(p_-,\Omega)+1\big)\left[meas(\Omega) + \int_\Omega |\nabla u_n|^{p_n(.,u_n(.))}dx\right] + C_0 \\
&\leq const(p_-,\Omega,f,g). \qquad (4.10)
\end{aligned}$$

Therefore, $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,p_-}(\Omega)$ □

From (ii), up to a subsequence, u_n converges a.e in Ω (and also weakly in $W^{1,p_-}(\Omega)$) to u and a.e on $\partial\Omega$. For this, see the Lemma 3.7 and Lemma 3.8.

Claim 2. The sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges to a Young measure $\nu_x(\lambda)$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-* convergence, and one has

$$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda). \tag{4.11}$$

Moreover, $|\lambda|^{\pi(x)}$ is integrable with respect to the measure $\nu_x dx$ on $\mathbb{R}^N\times\Omega$ and $\nabla u \in L^{\pi(.)}(\Omega)$.

Proof. Writing

$$|p_n(x,u_n(x))-p(x,u(x))| \leq |p_n(x,u_n(x))-p(x,u_n(x))|+|p(x,u_n(x))-p(x,u(x))|,$$

from a.e. convergence of u_n to u, from assumption (4.3) and from the Lusin Theorem applied to the map

$$p:\Omega\mapsto p(x,.)\in C(\mathbb{R}),$$

we deduce that $p_n(.,u_n(.)) \to p(.,u(.)) = \pi(.)$ in measure on Ω. Now, using the Theorem 2.7-[(ii),(iii)] $(p_n(.,u_n(.)),\nabla u_n)_{n\in\mathbb{N}}$ converges on $\mathbb{R}\times\mathbb{R}^N$ to Young measure $\mu_x = \delta_\pi(x)\otimes\nu_x$.
Thus, we can apply the weak convergence properties of the Theorem 2.7-(i) to the Carathéodory function

$F_m(x,\lambda_0,\lambda)\in\Omega\times(\mathbb{R}\times\mathbb{R}^N)\mapsto|h_m(\lambda)|^{\lambda_0}$ with $m\in\mathbb{N}$, where h_m is defined in the preliminaries. Then, we obtain

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^N}|h_m(\lambda)|^{\pi(x)}d\nu_x(\lambda)dx &= \int_{\Omega\times(\mathbb{R}\times\mathbb{R}^N)}|h_m(\lambda)|^{\lambda_0}d\mu_x(\lambda_0,\lambda)dx\\ &= \int_\Omega\int_{\mathbb{R}\times\mathbb{R}^N}F_m(x,\lambda_0,\lambda)d\mu_x(\lambda_0,\lambda)dx\\ &= \lim_{n\to\infty}\int_\Omega F_m(x,p_n(x,u_n(x)),\nabla u_n(x))dx\\ &= \lim_{n\to\infty}\int_\Omega|h_m(\nabla u_n)|^{p_n(x,u_n(x))}dx\\ &\le \lim_{n\to\infty}\int_\Omega|\nabla u_n|^{p_n(x,u_n(x))}dx\\ &\le C(f,g,\Omega,p_-)\quad(\text{using (4.9)}).\end{aligned}$$

Since $h_m(\lambda)\to\lambda$, as $m\to\infty$, using Lebesgue convergence Theorem, as $m\mapsto h_m(\lambda)$ is increasing, we deduce from last inequality that

$$\int_{\Omega\times\mathbb{R}^N}|\lambda|^{\pi(x)}d\nu_x(\lambda)dx\le C(f,g,\Omega,p_-).$$

Hence, $|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ on $\mathbb{R}^N\times\Omega$. Moreover, from the last inequality and Jensen inequality, we get

$$\int_\Omega|\nabla u|^{\pi(x)}dx=\int_\Omega\left|\int_{\mathbb{R}^N}\lambda d\nu_x(\lambda)\right|^{\pi(x)}dx\le\int_{\Omega\times\mathbb{R}^N}|\lambda|^{\pi(x)}d\nu_x(\lambda)dx<\infty.$$

Thus, $\nabla u\in L^{\pi(.)}(\Omega)$. □

Claim 3.

(a) The sequence $(A_n)_{n\in\mathbb{N}}$ defined by $A_n(x):=a_n(x,u_n(x),\nabla u_n(x))$ is equi-integrable.

(b) The weak limit A of (A_n) belongs to $L^{\pi'(.)}(\Omega)$ and one has

$$A(x)=\int_{\mathbb{R}^N}a(x,u(x),\lambda)d\nu_x(\lambda)\quad\text{a.e.}\quad x\in\Omega.\tag{4.12}$$

Proof. (a) Using (A_5), we have

$$|a_n(x,u_n,\nabla u_n)|^{p_n'(.,u_n(.))}\le C\big(\mathcal{M}_n+|\nabla u_n|^{p_n(.,u_n(.))}\big).$$

The sequence $\big(|\nabla u_n|^{p_n(.,u_n(.))}\big)_{n\in\mathbb{N}}$ is uniformly bounded in $L^1(\Omega)$ and $\mathcal{M}_n$ is also equi-integrable on Ω, hence $\big(|a_n(x,u_n,\nabla u_n)|^{p_n'(.,u_n(.))}\big)_{n\in\mathbb{N}}$ is equi-integrable. Otherwise, as $p_n'(.,u_n(.))>1$, we have

$$|a_n(x,u_n,\nabla u_n)|\le 1+|a_n(x,u_n,\nabla u_n)|^{p_n'(.,u_n(.))}.$$

Thus, for all subset $E\subset\Omega$, we have

$$\int_E|a_n(x,u_n,\nabla u_n)|dx\le meas(E)+\int_E|a_n(x,u_n,\nabla u_n)|^{p_n'(.,u_n(.))}dx.$$

Thus, for $meas(E)$ small enough, we deduce that $(A_n)_{n\in\mathbb{N}}$ is equi-integrable.

(b) Set $\nabla v_n := \nabla u_n \chi_{S'_n}$ and consider auxiliary functions $\tilde{A}_n := a(x, u, \nabla v_n)$, where

$$S'_n := \left\{ x \in \Omega, \quad |p(x, u(x)) - p_n(x, u_n(x))| < \frac{1}{2} \right\}.$$

Let us prove that for all $\sigma > 0$,

$$meas\big(\{x \in \Omega, \sup_{\lambda\in K} |a_n(x, u_n, \lambda) - a(x, u, \lambda)| \geq \sigma\}\big) \to 0, \text{ as } n \to \infty,$$

where K is a bounded subset of $\mathbb{R}^N$. We know that

$$|a_n(x, u_n, \lambda) - a(x, u, \lambda)| \leq |a_n(x, u_n, \lambda) - a(x, u_n, \lambda)| + |a(x, u_n, \lambda) - a(x, u, \lambda)|.$$

Thus, it follows from Lemma 2.8 and (4.2) that

$$\sup_{\lambda\in K} |a_n(x, u_n, \lambda) - a(x, u, \lambda)| \to 0 \text{ in measure, as } n \to \infty.$$

Now, using the same argument as in the proof of Assertion 4 to A_n and $\tilde{A}_n$ instead of Φ_n and $\tilde{\Phi}_n$, we get a wished result. □

Claim 4.

(a) $\displaystyle\int_\Omega A.\nabla u dx \geq \int_\Omega a(x, u(x), \lambda).\lambda d\nu_x(\lambda).$

(b) The "div-curl" inequality holds:

$$\int_{\Omega\times\mathbb{R}^N} \big(a(x, u(x), \lambda) - a(x, u(x), \nabla u(x))\big)(\lambda - \nabla u(x)) d\nu_x(\lambda) dx \leq 0. \tag{4.13}$$

(c) $A(x) = a(x, u(x), \nabla u(x))$ for a.e. $x \in \Omega$, and ∇u_n converges to ∇u in measure on Ω, as $n \to +\infty$.

Proof. We only give the proof of (a). The proofs of (b) and (c) are exactly the same as the proofs of Assertion 6 and Assertion 7.
Let $\varphi \in C^\infty(\overline{\Omega})$. For n large enough, φ is an admissible test function in the weak formulation of u_n and we have

$$\begin{aligned} \int_\Omega b(u_n)\varphi dx + \int_{\partial\Omega} \lambda u_n \varphi d\sigma \quad &+ \quad \int_\Omega a_n(x, u_n, \nabla u_n).\nabla\varphi dx \\ &= \int_\Omega f_n \varphi dx + \int_{\partial\Omega} g_n \varphi d\sigma. \end{aligned} \tag{4.14}$$

$(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in the space $W^{1,p_-}(\Omega)$, so, it is uniformly bounded in $L^\infty(\Omega)$. Then, $(b(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$. Moreover, $b(u_n)$ converges a.e. in Ω to $b(u)$. Therefore, thanks to Lebesgue dominated convergence Theorem, we have

$$\lim_{n\to\infty} \int_\Omega b(u_n)\varphi dx = \int_\Omega b(u)\varphi dx. \tag{4.15}$$

One the other hand, $W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $p_- > N$, so, $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\partial\Omega)$. We also note that u_n converges to u a.e. on Ω by Lemma 3.8

and $\lambda|u_n\varphi| \leq \lambda\|u_n\|_{L^\infty(\partial\Omega)}|\varphi| \in L^1(\partial\Omega)$. So, the Lebesgue dominated convergence theorem give us

$$\lim_{n\to\infty}\int_{\partial\Omega}\lambda u_n\varphi d\sigma = \int_{\partial\Omega}\lambda u\varphi d\sigma. \tag{4.16}$$

Moreover, we have

$$\lim_{n\to\infty}\int_{\Omega} f_n\varphi dx = \int_{\Omega} f\varphi dx, \tag{4.17}$$

since $f_n \rightharpoonup f$ in $L^1(\Omega)$, as n tends to ∞ and $\varphi \in L^\infty(\Omega)$. Similary way with (4.17), we obtain

$$\lim_{n\to\infty}\int_{\partial\Omega} g_n\varphi d\sigma = \int_{\partial\Omega} g\varphi d\sigma. \tag{4.18}$$

It remains to prove that

$$\lim_{n\to\infty}\int_{\Omega} a_n(x,u_n,\nabla u_n)\nabla\varphi dx = \int_{\Omega} A\nabla\varphi dx.$$

Let us prove that $(a_n(x,u_n,\nabla u_n)\nabla\varphi)_{n\in\mathbb{N}}$ is equi-integrable. Let E be a subset of Ω, by using Young's inequality, we have

$$\begin{aligned}
\int_E a_n(x,u_n,\nabla u_n).\nabla\varphi dx &\leq \int_E |a_n(x,u_n,\nabla u_n)||\nabla\varphi| dx\\
&\leq \int_E \frac{1}{p_n'(.,u_n(.))}|a_n(x,u_n,\nabla u_n)|^{p_n'(.,u_n(.))}dx\\
&+ \int_E \frac{1}{p_n(.,u_n(.))}|\nabla\varphi|^{p_n(.,u_n(.))}dx\\
&\leq C\int_E \big(\mathcal{M}_n(x) + |\nabla u_n|^{p_n(.,u_n(.))}\big)dx\\
&+ \int_E |\nabla\varphi|^{p_n(.,u_n(.))}dx\\
&\leq C\int_E \big(\mathcal{M}_n(x) + |\nabla u_n|^{p_n(,u_n(.))}\big)dx\\
&+ \int_{E\cap\{|\nabla\varphi|\leq 1\}} |\nabla\varphi|^{\pi_n(.)}dx + \int_{E\cap\{|\nabla\varphi|>1\}} |\nabla\varphi|^{p_n(.,u_n(.))}dx\\
&\leq C\int_E \big(\mathcal{M}_n(x) + |\nabla u_n|^{p_n(,u_n(.))}\big)dx\\
&+ meas(E) + \int_E |\nabla\varphi|^{p_+}dx
\end{aligned} \tag{4.19}$$

By using Claim 1-(ii) and the fact that $(\mathcal{M}_n)_{n\in\mathbb{N}}$ is equi-integrable, $\big(\mathcal{M}_n + |\nabla u_n|^{p_n(.,u_n(.))}\big)_{n\in\mathbb{N}}$ is equi-integrable. Moreover, $|\nabla\varphi|^{p_+} \in L^1(\Omega)$, since $\nabla\varphi$ is bounded. Thus, we deduce from (4.19) that

$$\lim_{meas(E)\to 0}\int_E a_n(x,u_n,\nabla u_n).\nabla\varphi dx = 0.$$

Furthermore,

$$a_n(x,u_n,\nabla u_n).\nabla\varphi \to A.\nabla\varphi \quad \text{a.e.} \quad \text{in } \Omega.$$

By applying Vitali's Theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega a_n(x,u_n,\nabla u_n).\nabla\varphi dx=\int_\Omega A.\nabla\varphi dx. \tag{4.20}$$

We can pass to the limit as n goes to ∞ in (4.14) and taking into account the inequality (4.15), (4.16), (4.17), (4.18) and (4.20) to obtain

$$\int_\Omega b(u)\varphi dx+\int_{\partial\Omega}\lambda u\varphi d\sigma+\int_\Omega A.\nabla\varphi dx=\int_\Omega f\varphi dx+\int_{\partial\Omega}g\varphi d\sigma. \tag{4.21}$$

By density argument, we can replace φ with u_n in (4.14) to get

$$\begin{aligned}\int_\Omega b(u_n)u_n dx+\int_{\partial\Omega}\lambda u_n^2 d\sigma+\int_\Omega a_n(x,u_n,\nabla u_n).\nabla u_n dx &= \int_\Omega f_n u_n dx\\ &+\int_{\partial\Omega}g_n u_n d\sigma.\end{aligned} \tag{4.22}$$

Since, $C^\infty(\overline{\Omega})$ is dense in $W^{1,\pi(.)}(\Omega)$, so we change φ by u in (4.21) and we have

$$\int_\Omega b(u)udx+\int_{\partial\Omega}\lambda u^2 d\sigma+\int_\Omega A.\nabla udx=\int_\Omega fudx+\int_{\partial\Omega}gud\sigma. \tag{4.23}$$

By Fatou's Lemma, we deduce

$$\liminf_{n\to\infty}\int_\Omega b(u_n)u_n dx\geq\int_\Omega b(u)udx \tag{4.24}$$

and

$$\liminf_{n\to\infty}\int_{\partial\Omega}\lambda u_n^2 d\sigma\geq\int_{\partial\Omega}\lambda u^2 d\sigma. \tag{4.25}$$

Moreover,

$$\lim_{n\to\infty}\int_\Omega f_n u_n dx=\int_\Omega fudx. \tag{4.26}$$

Indeed,

$$\int_\Omega f_n u_n dx=\int_\Omega f_n udx+\int_\Omega f_n(u_n-u)dx. \tag{4.27}$$

For the first term of the right-hand side of the above equality, we have

$$\lim_{n\to\infty}\int_\Omega f_n udx=\int_\Omega fudx,$$

since $f_n\rightharpoonup f$ in $L^1(\Omega)$ and $u\in L^\infty(\Omega)$. For the second term of the right-hand side of (4.27), we have, for all $R>0$

$$\begin{aligned}\int_\Omega|f_n(u_n-u)|dx &\leq \int_{\{|f_n|>R\}}|f_n|\|u_n-u\|_{L^\infty(\Omega)}dx+R\int_{\{|f_n|\leq R\}}|u_n-u|dx\\ &\leq C\int_{\{|f_n|>R\}}|f_n|dx+R\int_\Omega|u_n-u|dx.\end{aligned} \tag{4.28}$$

For all R fixed, the second term of the right-hand side of above inequality tends to zero as $n\to\infty$. Since by Chebyschev inequality,

$$\sup_n meas\big(\{|f_n|>R\}\big)\leq\frac{\sup_n\|f_n\|_{L^1(\Omega)}}{R}\leq\frac{C}{R}\to 0,\ \text{ as }\ R\to\infty$$

and because a weakly convergent in $L^1(\Omega)$ sequence is equi-integrable on Ω, by the choice of R the first term in the right-hand side of (4.28) can be made small as desired. Hence, we deduce that $f_n(u_n-u)$ goes to zero in $L^1(\Omega)$. Thus, (4.26) is justified. Analogously with (4.26), we obtain

$$\lim_{n\to\infty}\int_{\partial\Omega} g_n u_n d\sigma = \int_{\partial\Omega} g u d\sigma. \tag{4.29}$$

Combining (4.24), (4.25), (4.26) and (4.29), we have

$$\begin{aligned}&\liminf_{n\to\infty}\left(\int_\Omega f_n u_n dx+\int_{\partial\Omega} g_n u_n d\sigma\right)-\int_\Omega b(u)u dx-\int_{\partial\Omega}\lambda u^2 d\sigma\\ \geq\ &\liminf_{n\to\infty}\left(\int_\Omega f_n u_n dx+\int_{\partial\Omega} g_n u_n d\sigma-\int_\Omega b(u_n)u_n dx-\int_{\partial\Omega}\lambda u_n^2 d\sigma\right).\end{aligned}$$

By using (4.22), (4.23), the above inequality and the definition of A_n, we get

$$\int_\Omega A.\nabla u dx \geq \liminf_{n\to\infty}\int_\Omega A_n.\nabla u_n dx. \tag{4.30}$$

By [1, Lemma 2.1], $m\mapsto a_n(x,u_n,h_m(\nabla u_n)).h_m(\nabla u_n)$ is increasing and converges to $a_n(x,u_n,\nabla u_n).\nabla u_n$ for m large enough. Then,

$$a_n(x,u_n,h_m(\nabla u_n)).h_m(\nabla u_n)\leq a_n(x,u_n,\nabla u_n).\nabla u_n.$$

Therefore, by using (4.30) and the property (2.4) of the Theorem 2.7, we have

$$\begin{aligned}\int_\Omega A.\nabla u dx &\geq \liminf_{n\to\infty}\int_\Omega A_n.\nabla u_n dx\\ &\geq \lim_{n\to\infty}\int_\Omega a_n(x,u_n,h_m(\nabla u_n)).h_m(\nabla u_n)dx\\ &= \int_{\Omega\times\mathbb{R}^N} a(x,u,h_m(\lambda)).h_m(\lambda)d\nu_x(\lambda)dx.\end{aligned} \tag{4.31}$$

Using Lebesgue convergence Theorem in (4.31), as m goes to ∞, we get (a). □

Lemma 4.2. *u is a weak solution of the problem (1.1).*

Proof. Let u_n be a weak solution of the problem (P_n). By using (4.1) and the fact that $u_n\in W^{1,p_n(.,u_n(.))}(\Omega)\hookrightarrow W^{1,p_-}(\Omega)\hookrightarrow C^{0,\alpha}(\overline{\Omega})$, $C^\infty(\overline{\Omega})$ is dense in the space $W^{1,p_n(.,u_n(.))}(\Omega)$. Therefore, for n large enough, we can choose φ in $C^\infty(\overline{\Omega})$ as a test function in the weak formulation of the problem (P_n). Then, we obtain

$$\begin{aligned}\int_\Omega b(u_n)\varphi dx+\int_{\partial\Omega}\lambda u_n\varphi d\sigma\ &+\ \int_\Omega a_n(x,u_n,\nabla u_n).\nabla\varphi dx\\ &= \int_\Omega f_n\varphi dx+\int_{\partial\Omega} g_n\varphi d\sigma.\end{aligned} \tag{4.32}$$

Firstly, we can prove that $u\in W^{1,p(.,u(.))}(\Omega)$, thanks to Claim 2 and the fact that $u\in W^{1,p_-}(\Omega)\hookrightarrow L^\infty(\Omega)\hookrightarrow L^{p(.,u(.))}(\Omega)$. Moreover, $b(u)\in L^1(\Omega)$ and $u\in L^1(\partial\Omega)$, since $b(u_n)$ strongly converges to $b(u)$ in $L^1(\Omega)$ and u_n strongly converges to u in $L^1(\partial\Omega)$, thanks to the reasoning that lead to (4.15) and (4.16).
Currently, we are looking at the third term of the left-hand side of the above

equality. We know that $(a_n(x,u_n,\nabla u_n)\nabla\varphi)_{n\in\mathbb{N}}$ is equi-integrable, by (4.19). In addition, it follows from Claim 3-(b) and Claim 4-(c) that

$$a_n(x,u_n,\nabla u_n)\nabla\varphi \to a(x,u,\nabla u)\nabla\varphi \text{ a.e. in } \Omega.$$

Therefore, by using Vitali's Theorem, we get

$$\lim_{n\to\infty}\int_\Omega a_n(x,u_n,\nabla u_n)\nabla\varphi dx = \int_\Omega a(x,u,\nabla u)\nabla\varphi dx.$$

Finally using (4.15), (4.16), (4.17), (4.18), (4.32) and the above equality, we get (3.3) for u with all test function in $C^\infty(\overline{\Omega})$. □

Remark 4.3. Under the assumptions (3.36) and (3.37), the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges to u a.e. in Ω and a.e. on $\partial\Omega$ and the whole sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges to ∇u a.e. on Ω, as $n\to\infty$.
Indeed, by Claim 1, we deduce that $(u_n)_{n\in\mathbb{N}}$ converges to u a.e. in Ω and a.e. on $\partial\Omega$, up to extraction of subsequence. From Claim 4-(c), $(\nabla u_n)_{n\in\mathbb{N}}$ converges to ∇u a.e. on Ω, up to extraction of subsequence.
Now, by Lemma 4.2 and the uniqueness of the weak solution to (1.1) assured in Theorem 3.10, we conclude that all convergent subsequences of $(u_n)_{n\in\mathbb{N}}$, $(\nabla u_n)_{n\in\mathbb{N}}$ converge to the same limits u, ∇u respectively.

References

1. B. Andreianov, M. Bendahmane, S. Ouaro; *Structural stability for variable exponent elliptic problems. I. The $p(x)$-Laplacian kind problems*, Nonlinear Analysis, **73** (2010), 2 – 24.
2. B. Andreianov, M. Bendahmane, S. Ouaro; *Structural stability for variable exponent elliptic problems, II: The $p(u)$-Laplacian and coupled problems*, Nonlinear Analysis, **72** (2010), 4649 – 4660.
3. B. Andreianov, F. Bouhsiss; *Uniqueness for an elliptic parabolic problem with Neumann boundary condition*, J. Evol. Equ. 4 (2) (2004) 273-295.
4. S.N. Antontsev, J.F. Rodrigues; *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 52 (1) (2006), 19 – 36.
5. E. Azroul, A. Barbara, M.B. Benboubker and S. Ouaro; *Renormalized solutions for a $p(x)$-Laplacian equation with Neumann nonhomogeneous boundary conditions and L^1-data* Ann. Univ. Craiova. Math. Inform., 40 (1) (2013), 9 – 22.
6. J.M. Ball; *A version of the fundamental theorem for Young measures. PDEs and continuum models of phase transitions* (Nice, 1988), 207 – 215, Lecture Notes in Phys., 344, Springer, 1989.
7. M.B. Benboubker, S. Ouaro, U. Traoré; *Entropy solutions for nonhomogeneous Neumann problems involving the generalized $p(x)$-laplacian operators and measure data*, Nonlinear Evolution Equation and Application. Volume 2014, Number 5, pp. 53 – 76.
8. Y. Chen, S. Levine, M. Rao; *Variable exponent, linear growth functionals in image restoration*. SIAM. J.Appl. Math., 66 (2006), 1383 – 1406.
9. G. Dolzmann, N. Hungerbühler, S. Müller; *The p-harmonic system with measure-valued right hand side*, Ann. Inst. H. Poincaré. Anal. Non Linéaire 14 (3) (1997), 353 – 364.
10. R. Eymard, T. Galloüet and R. Herbin; *Finite volume methods. Handbook of numerical analysis*, Vol. VII, 713 – 1020, North-Holland, 2000.
11. N. Hungerbühler; *Quasi-linear parabolic systems in divergence form with weak monotonicity*, Duke Math. J. 107 (3) (2001), 497 – 520.
12. N. Hungerbühler; *A refinement of Ball's theorem on Young measures*. New York J. Math. **3** (1997), 48 – 53.

13. J.-L. Lions; Quelques méthodes de résolution de problèmes aux limites nonlinéaires, Dunod. Gauthier-Villars, Paris, 1969.
14. S. Ouaro, A Tchousso; *Well-posedness result for a nonlinear elliptic problem involving variable exponent and Robin type boundary condition*, African Diaspora Journal of Mathematics **11**, No. 2, 36 – 64 (2011).
15. P. Pedregal; Parametrized measures and variational principles. Progress in Nonlinear Differential Equations and their Applications, 30. Birkhäuser, Basel, 1997.
16. M. Ruzicka; *Electrorheological fluids: modelling and mathematical theory*. Lecture Notes in Mathematics 1748, Springer-Verlag, Berlin, 2002.
17. R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical surveys and monographs, 49, American Mathematical Society.
18. L. Wang, Y. Fan, W. Ge; *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)-$Laplace operator* . Nonlinear Anal. **71** (2009), 4259 – 4270.
19. J. Yao; Solutions for Neumann boundary value problems involving $p(x)$-Laplace operator. Nonlinear Anal: (TMA) **68**(2008), 1271 – 1283.

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