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SEQUENCES RELATED TO THE e NUMBER AND BERNOULLI NUMBERS

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Abstract. We study some sequences related with the e number and the Bernoulli numbers.

1. Introduction and Main Results

There are, in the literature, many articles where special numbers are studied. The Euler's constant γ appear, for instance, in certain sums of fractional parts [4] [5]. The values of the zeta function $\zeta(s)$ also appear, for instance, in certain sums of fractional parts [3] and the counting function of many sequences of positive integers [2]. In this note, we study some sequences related simultaneously with the e number and the Bernoulli numbers.

Lemma 1.1. *Let α be a positive real number. Let n and s be positive integers such that $n > s$. The following inequality holds.*

$$\sum_{i=1}^{n-s} i^{\alpha n} \leq \frac{(n-s)^{\alpha n+1}}{\alpha n+1} + (n-s)^{\alpha n}. \tag{1.1}$$

Proof. The function $x^{\alpha n}$ is positive and strictly increasing. Therefore

$$i^{\alpha n} \leq \int_i^{i+1} x^{\alpha n}\, dx$$

and consequently

$$\sum_{i=1}^{n-s-1} i^{\alpha n} \leq \int_1^{n-s} x^{\alpha n}\, dx \leq \int_0^{n-s} x^{\alpha n}\, dx = \frac{(n-s)^{\alpha n+1}}{\alpha n+1}.$$

The lemma is proved. □

Theorem 1.2. *Let α be an arbitrary but fixed positive real number. The following limit holds.*

$$\lim_{n\to\infty} \frac{1^{\alpha n} + 2^{\alpha n} + 3^{\alpha n} + \cdots + n^{\alpha n}}{n^{\alpha n}} = \frac{e^{\alpha}}{e^{\alpha}-1} = 1 + e^{-\alpha} + e^{-2\alpha} + e^{-3\alpha} + \cdots. \tag{1.2}$$

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Proof. We have

$$\begin{aligned} 1^{\alpha n}+2^{\alpha n}+\cdots+n^{\alpha n} &= 1^{\alpha n}+2^{\alpha n}+\cdots+(n-s)^{\alpha n} \\ + \quad & (n-(s-1))^{\alpha n}+(n-(s-2))^{\alpha n}+\cdots+(n-1)^{\alpha n}+n^{\alpha n}, \end{aligned} \tag{1.3}$$

where s is a fixed positive integer.

Equation (1.3) gives

$$\begin{aligned} \frac{1^{\alpha n}+2^{\alpha n}+\cdots+n^{\alpha n}}{n^{\alpha n}} &= \frac{1^{\alpha n}+2^{\alpha n}+\cdots+(n-s)^{\alpha n}}{n^{\alpha n}} \\ + \quad & \left(1-\frac{s-1}{n}\right)^{\alpha n}+\left(1-\frac{s-2}{n}\right)^{\alpha n}+\cdots+\left(1-\frac{1}{n}\right)^{\alpha n}+1. \end{aligned} \tag{1.4}$$

Let $\epsilon>0$ be. We have

$$\frac{e^{\alpha}}{e^{\alpha}-1}=\frac{1}{1-e^{-\alpha}}=1+e^{-\alpha}+e^{-2\alpha}+e^{-3\alpha}+\cdots. \tag{1.5}$$

Therefore, we can choose s such that

$$e^{-s\alpha}+e^{-(s+1)\alpha}+e^{-(s+2)\alpha}+\cdots<\epsilon. \tag{1.6}$$

If we choose s sufficiently large (see (1.4) and (1.1)) then we obtain

$$\begin{aligned} \frac{1^{\alpha n}+2^{\alpha n}+\cdots+(n-s)^{\alpha n}}{n^{\alpha n}} &\leq \frac{\frac{(n-s)^{\alpha n+1}}{\alpha n+1}+(n-s)^{\alpha n}}{n^{\alpha n}} \\ = \quad & \left(1-\frac{s}{n}\right)^{\alpha n}\frac{n-s}{\alpha n+1}+\left(1-\frac{s}{n}\right)^{\alpha n} \\ = \quad & \left(e^{-s}\right)^{\alpha}\left(1+\frac{1}{\alpha}\right)(1+o(1))<\epsilon. \end{aligned} \tag{1.7}$$

Since

$$\lim_{n\to\infty}\left(\left(1-\frac{s}{n}\right)^{\alpha n}\frac{n-s}{\alpha n+1}+\left(1-\frac{s}{n}\right)^{\alpha n}\right)=\left(e^{-s}\right)^{\alpha}\left(1+\frac{1}{\alpha}\right).$$

On the other hand, we have

$$\lim_{n\to\infty}\left(\left(1-\frac{h}{n}\right)^{\alpha n}-e^{-h\alpha}\right)=0 \qquad (h=1,2,\ldots,s-1)$$

Therefore, if n is sufficiently large then we have

$$\left|\left(\left(1-\frac{h}{n}\right)^{\alpha n}-e^{-h\alpha}\right)\right|<\frac{\epsilon}{s-1} \qquad (h=1,2,\ldots,s-1) \tag{1.8}$$

Equations (1.4), (1.5), (1.6), (1.7) and (1.8) give

$$\begin{aligned} & \left|\frac{1^{\alpha n}+2^{\alpha n}+3^{\alpha n}+\cdots+n^{\alpha n}}{n^{\alpha n}}-\frac{e^{\alpha}}{e^{\alpha}-1}\right| \\ = \quad & \left|\frac{\sum_{h=1}^{n-s}h^{\alpha n}}{n^{\alpha n}}+(1-1)+\sum_{h=1}^{s-1}\left(\left(1-\frac{h}{n}\right)^{\alpha n}-e^{-h\alpha}\right)-\sum_{h=s}^{\infty}e^{-h\alpha}\right| \\ < \quad & 3\epsilon \qquad (n\geq n_{\epsilon}). \end{aligned}$$

That is, limit (1.2). Since $\epsilon>0$ can be arbitrarily small. The theorem is proved. □

The Bernoulli numbers B_k are defined by the following power series.

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}x^k=1-\frac{1}{2}x+\sum_{k=2}^{\infty}\frac{B_k}{k!}x^k \quad (|x|<2\pi) \tag{1.9}$$

The following formulas are well-known (h is a positive integer)

$$P_1(h)=1+2+\cdots+h=\frac{h^2}{2}+\frac{h}{2}.$$

$$P_2(h)=1^2+2^2+\cdots+h^2=\frac{h^3}{3}+\frac{h^2}{2}+\frac{h}{6}.$$

$$P_3(h)=1^3+2^3+\cdots+h^3=\frac{h^4}{4}+\frac{h^3}{2}+\frac{h^2}{4}.$$

$$P_4(h)=1^4+2^4+\cdots+h^4=\frac{h^5}{5}+\frac{h^4}{2}+\frac{h^3}{3}-\frac{h}{30}.$$

Etc.

These polynomials $P_k(h)$, where k is a positive integer, are related with the Bernoulli numbers. More precise, we have the following theorem.

Theorem 1.3. *Let k be an arbitrary but fixed positive integer. The following formula holds.*

$$1^k+2^k+\cdots+h^k=\frac{h^{k+1}}{k+1}+\frac{1}{2}h^k+\sum_{r=2}^{k}\frac{1}{k+1-r}\binom{k}{r}B_r h^{k+1-r}=P_k(h). \tag{1.10}$$

Proof. See [1] (page 90). □

Equation (1.9) gives

$$\frac{xe^x}{e^x-1}=x+\frac{x}{e^x-1}=1+\frac{1}{2}x+\sum_{k=2}^{\infty}\frac{B_k}{k!}x^k \quad (|x|<2\pi) \tag{1.11}$$

and equation (1.11) gives

$$\frac{e^x}{e^x-1}=\frac{1}{x}+\frac{1}{2}+\sum_{k=2}^{\infty}\frac{B_k}{k!}x^{k-1} \quad (0<|x|<2\pi)$$

Hence, if $0<\alpha<2\pi$ then we have (see (1.2))

$$\frac{e^\alpha}{e^\alpha-1}=\frac{1}{\alpha}+\frac{1}{2}+\sum_{k=2}^{\infty}\frac{B_k}{k!}\alpha^{k-1} \quad (0<\alpha<2\pi) \tag{1.12}$$

and if $k=1,2,3,4,5,6$ then (1.12) becomes

$$\frac{e^k}{e^k-1}=\frac{1}{k}+\frac{1}{2}+k\frac{B_2}{2!}+\sum_{r=3}^{\infty}\frac{B_r}{r!}k^{r-1} \quad (k=1,2,3,4,5,6)$$

Consequently, the sequence

$$P(n)=\frac{1}{k}+\frac{1}{2}+k\frac{B_2}{2!}+\sum_{r=3}^{kn}\frac{B_r}{r!}k^{r-1}\quad(k=1,2,3,4,5,6)\tag{1.13}$$

has limit

$$\lim_{n\to\infty}P(n)=\frac{e^k}{e^k-1}=1+e^{-k}+e^{-2k}+\cdots\qquad(k=1,2,3,4,5,6)\tag{1.14}$$

In the following theorem, we obtain a sequence related with sequence (1.13) such that limit (1.14) holds for every positive integer k.

Theorem 1.4. *Let k be an arbitrary but fixed positive integer. Let us consider the sequence (compare with (1.13))*

$$\begin{aligned}S(n)&=\frac{1}{k}+\frac{1}{2}+k\frac{B_2}{2!}\\&+\sum_{r=3}^{kn}\frac{B_r}{r!}k^{r-1}\left(1-\frac{1}{kn}\right)\left(1-\frac{2}{kn}\right)\cdots\left(1-\frac{r-2}{kn}\right).\end{aligned}$$

We have

$$\lim_{n\to\infty}S(n)=\frac{e^k}{e^k-1}=1+e^{-k}+e^{-2k}+\cdots.$$

Proof. Theorem 1.3 gives

$$\begin{aligned}Q(n)&=\frac{1^{kn}+2^{kn}+\cdots+n^{kn}}{n^{kn}}=\frac{\frac{n^{k+1}}{kn+1}+\frac{1}{2}n^{kn}+\sum_{r=2}^{kn}\frac{1}{kn+1-r}\binom{kn}{r}B_rn^{kn+1-r}}{n^{kn}}\\&=\frac{n}{kn+1}+\frac{1}{2}+k\frac{B_2}{2!}\\&+\sum_{r=3}^{kn}\frac{B_r}{r!}k^{r-1}\left(1-\frac{1}{kn}\right)\left(1-\frac{2}{kn}\right)\cdots\left(1-\frac{r-2}{kn}\right)\end{aligned}$$

and Theorem 1.2 gives

$$\lim_{n\to\infty}Q(n)=\frac{e^k}{e^k-1}$$

Now, note that

$$\lim_{n\to\infty}(S(n)-Q(n))=\frac{1}{k}-\frac{n}{kn+1}=0.$$

The theorem is proved. □

Theorem 1.5. *Let α be an arbitrary but fixed positive real number. The following limit holds.*

$$\lim_{n\to\infty}\frac{(-1)^{n-1}1^{\alpha n}+\cdots+(-1)^2(n-2)^{\alpha n}+(-1)^1(n-1)^{\alpha n}+n^{\alpha n}}{n^{\alpha n}}=\frac{e^\alpha}{e^\alpha+1}.$$

Proof. The proof is the same as the proof of Theorem 1.2. Note that in this case

$$\frac{e^{\alpha}}{e^{\alpha}+1}=\frac{1}{1+e^{-\alpha}}=1+(-1)e^{-\alpha}+(-1)^2e^{-2\alpha}+(-1)^3e^{-3\alpha}+\cdots$$

The theorem is proved. □

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