COHOMOLOGY RING OF THE FINITE SPLIT METACYCLIC GROUP

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ABSTRACT. Let $G$ be the finite split metacyclic group of presentation $G = \langle A, B/A^m = B^n = e, BAB^{-1} = A^r, r^n \equiv 1(m), (n(r - 1), m) = 1 \rangle$. Huebschmann computed the integral cohomology ring of a large class of metacyclic groups using the mechanism of homological perturbation theory and determined the generators of the subring of even degree classes of any metacyclic group by means of the Chern classes of some of its representations. The purpose of this article is to investigate $H^*(G; \mathbb{Z})$ by localization methods. It also aims to determine explicit representations of $G$ whose Chern classes generate $H^*(G; \mathbb{Z})$ by induction techniques.

1. Introduction

Let $G$ be the finite split metacyclic group of presentation $G = \langle A, B/A^m = B^n = e, BAB^{-1} = A^r, r^n \equiv 1(m), (n(r - 1), m) = 1 \rangle$.

In [5, 6] Huebschmann determined the integral cohomology ring of a large class of metacyclic groups. The aim of this paper is to show by localization and induction techniques that the integral cohomology ring $H^*(G; \mathbb{Z})$ of $G$ is generated by Chern classes of complex linear representations of $G$. It is well known that the cohomology groups of finite groups are torsions (see Proposition 3.1) and the restriction homomorphisms $Res^G_{G_p} : H^k(G, \mathbb{Z})_p \rightarrow H^k(G_p, \mathbb{Z})^G$ for all $k \geq 1$ are isomorphisms between the $p$-primary parts of $H^k(G, \mathbb{Z})$ and $H^k(G_p, \mathbb{Z})^G$ the invariant elements of the $p$-Sylow subgroups $G_p$ of $G$, where $p$ runs over all prime numbers divinding the order of $G$ (see Proposition 3.3).

For our group we will be interested in two cases:
If $p/m : G_p$ is normal and $H^k(G, \mathbb{Z})_p \cong H^k(G_p, \mathbb{Z})^{G_p}$.
If $p/n : G_p$ is abelian and $H^k(G, \mathbb{Z})_p \cong H^k(G_p, \mathbb{Z})^{N_p}$.

Hence we get the group isomorphism Proposition 3.9: For every $k \geq 1$,

$$H^k(G, \mathbb{Z}) \rightarrow \prod_{p/m} H^k(G_p, \mathbb{Z})^{G_p} \times H^k(\langle B \rangle, \mathbb{Z})$$

In order to describe the cohomology ring $H^*(G; \mathbb{Z})$ by means of Chern classes of
representations of $G_p$ we establish the propositions below:

**Proposition 4.3**: For every $k \geq 1$, $H^{2k}(\langle B \rangle, \mathbb{Z}) = \frac{\mathbb{Z}}{n\mathbb{Z}}(c_1(\theta))^k$

$$\theta: \langle B \rangle \rightarrow \mathbb{C}^* \quad \frac{2i\pi}{n}$$

$$B \mapsto e \frac{2i\pi}{n}$$

where $c_1(\theta)$ is the first Chern class of $\theta$.

**Proposition 4.4**: For every $k \geq 1$,

$$H^{2k}(G_p, \mathbb{Z})^{G/G_p} = \begin{cases} \mathbb{Z} & \text{if } k \neq 0(d_p) \\ \frac{\mathbb{Z}}{p^n\mathbb{Z}}(c_1(\theta_p))^k & \text{if } k = qd_p \end{cases}$$

where

$$\theta_p: \langle A_p \rangle \rightarrow \mathbb{C}^* \quad \frac{2i\pi}{p^n}$$

$$A_p \mapsto e \frac{2i\pi}{p^n}$$

Which implies that $G$ has a periodic cohomology (see Proposition 4.8). Then by induction methods we obtain the two results stated as follows:

**Proposition 4.9**: Let $p$ be a prime divisor of $m$. There exists a representation $\zeta_p$ of $G$ of degree $d_p$ such that:

$$\zeta_p(A) = \begin{bmatrix} 2i\pi & 0 & \cdots & 0 \\ e \frac{2i\pi}{m} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 2i\pi d_p^{-1} \end{bmatrix}$$

$$\zeta_p(B) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ e \frac{2i\pi}{n} & 0 & \cdots & 0 \end{bmatrix}$$

where $\text{Res}_{G_p}^G(c_{d_p}(\zeta_p))$ is a generator of $H^{2d_p}(G_p, \mathbb{Z})^{G/G_p}$.

**Proposition 4.12**: There exists $\zeta \in R(G)$ where $\zeta(A)$ is a unitary matrix and $\zeta(B) = e \frac{2i\pi}{n}$ such that $\text{Res}_{\langle B \rangle}^G(c_1(\zeta))$ generates $H^2(\langle B \rangle, \mathbb{Z})$.

The arguments that have been carried out lead to Theorem 2.1:

There exists a graded ring isomorphism

$$H^*(G, \mathbb{Z}) = \mathbb{Z} \oplus \frac{\mathbb{Z}}{n\mathbb{Z}}[c_1(\zeta)]^+ \times \prod_{p/m} \frac{\mathbb{Z}}{p^n\mathbb{Z}}[c_{d_p}(\zeta_p)]^+$$
such that $c_1(\zeta)$ is the first Chern class of the representation $\zeta$ of $G$ where $\zeta(A)$ is a unitary matrix, $\zeta(B) = e^{2\pi i n}$ and $c_{d_p}(\zeta_p)$ are the $d_p$ Chern classes of representations $\zeta_p$ of $G$ given by:

$$
\zeta_p(A) = \begin{bmatrix}
2i\pi/m & 0 & \ldots & 0 \\
0 & 2i\pi r/m & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & 2i\pi r^{d_p-1}/m
\end{bmatrix}
$$

$$
\zeta_p(B) = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots \\
0 & \ldots & \ldots & 0 & 1 \\
2i\pi d_p/n & 0 & \ldots & \ldots & 0
\end{bmatrix}
$$

where $d_p$ is the order of $r$ in $(\mathbb{Z}/p^\alpha\mathbb{Z})^*$ and $p^\alpha$ stands for the highest power of $p$ dividing $m$. 

GROUP COHOMOLOGY OF $G = \langle A, B/A^m = B^n = e, BAB^{-1} = A^r \rangle$, $r^n \equiv 1(m), (n(r-1), m) = 1$
2. Statement of the main result and preliminaries

Let $G$ be the finite split metacyclic group of presentation $G = \langle A, B | A^m = B^n = e, BAB^{-1} = A^r, r^n \equiv 1(m), (n(r-1), m) = 1 \rangle$.

**Theorem 2.1.** There exists a graded ring isomorphism

$$H^*(G, \mathbb{Z}) = \mathbb{Z} \oplus \frac{\mathbb{Z}}{n\mathbb{Z}}[c_1(\zeta)]^+ \times \prod_{p/m\mathbb{Z}} \mathbb{Z}[c_{d_p}(\zeta_p)]^+$$

(2.1)

such that $c_1(\zeta)$ is the first Chern class of the representation $\zeta$ of $G$ where $\zeta(A)$ is a unitary matrix, $\zeta(B) = e^{2i\pi n}$ and $c_{d_p}(\zeta_p)$ are the $d_p$ Chern classes of representations $\zeta_p$ of $G$ given by:

$$\zeta_p(A) = \begin{bmatrix} 2i\pi & 0 & \ldots & 0 \\ e^m & 0 & \ldots & 0 \\ 2i\pi r & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 2i\pi r^{d_p-1} \end{bmatrix}$$

$$\zeta_p(B) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ e^m & 0 & \ldots & \ldots & 0 \end{bmatrix}$$

where $d_p$ is the order of $r$ in $(\frac{\mathbb{Z}}{p^{d_p}\mathbb{Z}})^*$ and $p^{d_p}$ stands for the highest power of $p$ dividing $m$.

This section is devoted to the description of the group we are dealing with. Several notations and reminders will be introduced in order to make the paper self contained.

Let $G$ be the finite group of presentation $G = \langle A, B | A^m = B^n = e, BAB^{-1} = A^r, r^n \equiv 1(m), (n(r-1), m) = 1 \rangle$.

Thus $G$ is a group of which all Sylow subgroups are cyclic. Moreover, $G$ is split and metacyclic. Its commutator $[G, G]$ is isomorphic to $\langle A \rangle \cong \mathbb{Z}/m\mathbb{Z}$ and its derived group $G/[G, G]$ is isomorphic to $\langle B \rangle \cong \mathbb{Z}/n\mathbb{Z}$. Furthermore, $G \cong \mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/n\mathbb{Z}$ is the semi-direct product where $\rho: \mathbb{Z}/n\mathbb{Z} \rightarrow Aut(\mathbb{Z}/m\mathbb{Z})$ denotes the conjugation morphism of elements of $\mathbb{Z}/m\mathbb{Z}$ by elements of $\mathbb{Z}/n\mathbb{Z}$.

In addition, $G$ is generated by the elements $A$ and $B$ of coprime order $m$ and $n$, therefore $|G| = mn$. An element $x$ of $G$ is written in the form $A^tB^s$ with $0 \leq t \leq m-1$ and $0 \leq s \leq n-1$. We also have $B^sA^t = A^{tr^s}B^s$ for all $t \in \mathbb{N}$ and $s \in \mathbb{N}$. 
Let now \( p \) be a prime number dividing \( |G| \) and \( G_p \) a p-Sylow subgroup of \( G \).

1. If \( p/m \), then \( G_p \) is isomorphic to the cyclic subgroup \( \langle A_p \rangle \) of \( G \) generated by \( A_p = A^{m/p^n} \). We have \( G_p \) is normal in \( G \), so its normalizer \( N_p = G \) and its centralizer \( C_p \) is isomorphic to \( \langle A, B^d \rangle \) where \( d_p \) is the smallest divisor of \( m \) satisfying \( r^n \equiv 1(p^a) \). The action of \( G \) on \( G_p \) boils down to the action of \( G/G_p \cong \langle \overline{B}^\ell \rangle \) where \( \overline{i} \) is the class of \( i \) modulo \( d_p \). We point out that \( p \) is different from 2 because \( (m, r(r - 1)) = 1 \).

2. If \( p/n \), then \( G_p \) is isomorphic to the cyclic subgroup \( \langle B_p \rangle \) of \( G \) generated by \( B_p = B^{n/p^a} \). In this case the normalizer of \( G_p \) coincides with the centralizer \( N_p = C_p \). Thus the action of \( N_p \) on \( G_p \) is trivial.

Recall that for each prime number \( p \) the \( p \)-Sylow subgroups of \( G \) are conjugates. We choose a representative \( G_p \) in this conjugation class. The collection \( \{ G_p; p/|G:1| \} \) is called a representative family of \( p \)-Sylow subgroups of \( G \).

3. Computation of \( H^*(G, \mathbb{Z}) \)

Let \( G \) be a finite group and \( \mathbb{Z} \) the trivial \( G \)-module. First of all, the cohomology of finite groups is torsion.

**Proposition 3.1.** [3] For every \( k \geq 1 \),

\[
|G : 1|H^k(G, \mathbb{Z}) = 0. \tag{3.1}
\]

Hence, if we want to determine the cohomology of a finite group \( G \), we only need to compute the \( p \)-primary parts for all primes \( p \) dividing \( |G| \).

**Corollary 3.2.** For every \( k \geq 1 \),

\[
H^k(G, \mathbb{Z}) = \prod_{p/|G:1|} H^k(G, \mathbb{Z})_p. \tag{3.2}
\]

where \( H^k(G, \mathbb{Z})_p = \{ z \in H^k(G, \mathbb{Z})/\exists n \in \mathbb{N} \; p^n.z = 0 \} \) is the \( p \)-primary part of \( H^k(G, \mathbb{Z}) \).

**Proposition 3.3.** There exists a graded ring isomorphism

\[
H^*(G, \mathbb{Z}) \cong \mathbb{Z} \oplus \prod_{p/|G|} H^*(G, \mathbb{Z})_p \tag{3.3}
\]

where \( H^*(G, \mathbb{Z})_p = \bigoplus_{k \geq 1} H^k(G, \mathbb{Z})_p \).

**Proof.** Since the set of prime divisors \( p \) of \( |G| \) is finite, the projections \( H^*(G, \mathbb{Z}) \rightarrow H^*(G, \mathbb{Z})_p \) yield a graded abelian group isomorphism

\[
H^*(G, \mathbb{Z}) \rightarrow \mathbb{Z} \oplus \bigoplus_{p/|G|} H^*(G, \mathbb{Z})_p \rightarrow \mathbb{Z} \oplus \prod_{p/|G|} H^*(G, \mathbb{Z})_p
\]

Moreover, these projections are ring homomorphisms because \( H^*(G, \mathbb{Z})_p \) is a graded ideal of \( H^*(G, \mathbb{Z}) \) then \( \bigoplus_{q/|G|, q \neq p} H^*(G, \mathbb{Z})_q \) is also a graded ideal of \( H^*(G, \mathbb{Z}) \). \( \square \)
In order to assemble the cohomology of the finite group $G$, we study the action of $G$ on the cohomology of some of its subgroups. Let $H$ be a subgroup of $G$ and $g$ an element of $G$, we denote by $^gH = gHg^{-1}$. The cochains complex map

$$c_g : \text{Hom}_{\mathbb{Z}}[H](P_*, \mathbb{Z}) \to \text{Hom}_{\mathbb{Z}}[gH](P_*, \mathbb{Z})$$

induces the isomorphisms for all $k \in \mathbb{N}$

$$c_g : H^k(H, \mathbb{Z}) \to H^k(^gH, \mathbb{Z})$$

where $P_*$ is a projective resolution of $\mathbb{Z}$.

**Definition 3.4.** Let $k \geq 1$. An element $\alpha \in H^k(H, \mathbb{Z})$ is said invariant if

$$\text{Res}_{H \cap gH} (\alpha) = \text{Res}_{gH \cap gH} (c_g(\alpha))$$

for all $g \in G$.

The following proposition connects the $p$-primary parts of $H^*(G, \mathbb{Z})$ to the cohomology of $p$-Sylow subgroups of $G$.

**Proposition 3.5.** [8] Let $\{G_p; p/|G|\}$ be a representative family of $p$-Sylow subgroups of $G$ then for all $k \geq 1$ :

$$\text{Res}_{G_p}^G : H^k(G, \mathbb{Z})_p \to H^k(G_p, \mathbb{Z})^G_p$$

is an isomorphism, where $H^k(G_p, \mathbb{Z})^G_p$ are the invariant elements of $H^k(G_p, \mathbb{Z})$.

**Proposition 3.6.** There exists a graded ring isomorphism

$$H^*(G, \mathbb{Z}) \cong \mathbb{Z} \oplus \prod_{p/|G|} H^*(G_p, \mathbb{Z})^G_p$$

where $H^*(G_p, \mathbb{Z})^G_p = \oplus_{k \geq 1} H^k(G_p, \mathbb{Z})^G_p$.

**Proof.** We have for all $p/|G|$, a graded abelian group isomorphism $H^*(G, \mathbb{Z})_p \to H^*(G_p, \mathbb{Z})^G_p$ induced by the ring homomorphism $\text{Res}_{G_p}^G$. Hence $H^*(G, \mathbb{Z})_p \to H^*(G_p, \mathbb{Z})^G_p$ is a ring isomorphism.

**Proposition 3.7.** [8] If $G_p$ is normal then for all $k \geq 1$,

$$\text{Res}_{G_p}^G : H^k(G, \mathbb{Z})_p \to H^k(G_p, \mathbb{Z})^{G/G_p}$$

is an isomorphism.

**Proposition 3.8.** [8] If $G_p$ is abelian then for all $k \geq 1$,

$$\text{Res}_{G_p}^G : H^k(G, \mathbb{Z})_p \to H^k(G_p, \mathbb{Z})^N_p$$

is an isomorphism.

For our group $G$, as enounced before, we will discuss the two particular cases:

1) If $p/m$ then $G_p$ is normal hence $H^k(G, \mathbb{Z})_p \cong H^k(G_p, \mathbb{Z})^{G/G_p}$.

2) If $p/n$, consequently $G_p$ is abelian then $H^k(G, \mathbb{Z})_p \cong H^k(G_p, \mathbb{Z})^N_p$.

So we have :
Proposition 3.9. For every \( k \geq 1 \),
\[
H^k(G, \mathbb{Z}) \longrightarrow \prod_{p/m} H^k(G_p, \mathbb{Z})^{G/G_p} \times H^k(\langle B \rangle, \mathbb{Z})
\] (3.8)
is a group isomorphism.

Proof. We know that \( H^k(G, \mathbb{Z}) \cong \prod_{p \mid |G|} H^k(G, \mathbb{Z})_p \) is an isomorphism .

- When \( p/n : G_p \) is abelian then \( H^k(G, \mathbb{Z})_p \cong H^k(G_p, \mathbb{Z})_{N_p} \). And since the action of \( N_p \) on \( G_p \) is trivial, it follows that \( H^k(G_p, \mathbb{Z})_{N_p} \cong H^k(G_p, \mathbb{Z}) \).

We also have \( H^k(\langle B \rangle, \mathbb{Z})_p \cong H^k(G_p, \mathbb{Z}) \), hence the following triangle is commutative :
\[
\begin{array}{ccc}
H^k(G, \mathbb{Z})_p & \xrightarrow{\text{Res}_{G_p}} & H^k(G_p, \mathbb{Z})_p \\
& \searrow & \searrow \\
& H^k(\langle B \rangle, \mathbb{Z})_p & \xrightarrow{\text{Res}_{G_p}} \\
\end{array}
\]

We deduce that \( \prod_{p/n} H^k(G, \mathbb{Z})_p \cong H^k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \) is a group isomorphism.

- When \( p/m : G_p = \langle A_p \rangle \) is a normal subgroup of \( G \), therefore \( H^k(G, \mathbb{Z})_p \cong H^k(G_p, \mathbb{Z})^{G/G_p} \) is an isomorphism. We deduce that \( \prod_{p/m} H^k(G, \mathbb{Z})_p \cong \prod_{p/m} H^k(G_p, \mathbb{Z})^{G/G_p} \) is an isomorphism.

\[ \square \]

As a result:

Corollary 3.10. For every \( k \in \mathbb{N} \),
\[
H^{2k+1}(G, \mathbb{Z}) = 0
\] (3.9)

Proof. Let \( k \in \mathbb{N} \). For every cyclic group \( H \), we have \( H^{2k+1}(H, \mathbb{Z}) = 0 \) then \( H^{2k+1}(G, \mathbb{Z}) = 0 \).

\[ \square \]

4. Chern classes of groups representations

Definition 4.1. [8] Let \( \rho : G \to GL_n(\mathbb{C}) \) be a representation of a discrete group \( G \) of degree \( n \) and \( p : EG \to BG \) the associated universal \( G \)-bundle .

The k-th Chern class \( c_k(\rho) \) is the k-th Chern class of the vector bundle \( EG \times_{\rho} \mathbb{C}^n \) of dimension \( n \) on \( BG \) defined by :
\[
c_k(\rho) = c_k(EG \times_{\rho} \mathbb{C}^n) \in H^{2k}(BG, \mathbb{Z}) = H^{2k}(G, \mathbb{Z}), \quad k=0,1,\ldots,n
\] (4.1)

Note that \( G \) acts on the right on \( EG \) by Deck transformations and on the left on \( \mathbb{C}^n \) by \( \rho \).

These classes verify :

Proposition 4.2. [8]

(1) \( c_k(\rho) = 0 \) if \( k > n \).
(2) If \( f : H \to G \) is a group homomorphism then \( c_k(\rho \circ f) = f^*(c_k(\rho)) \).

(3) If \( C(\rho) = 1 + c_1(\rho) + \ldots + c_n(\rho) \) is the total Chern class then \( C(\rho_1 \oplus \rho_2) = C(\rho_1)C(\rho_2) \).

(4) \( c_1 : \text{Hom}(G, \mathbb{C}^*) \xrightarrow{\cong} H^2(G, \mathbb{Z}) \) is an isomorphism.

**Proposition 4.3.** For every \( k \geq 1 \),

\[
H^{2k}(\langle B \rangle, \mathbb{Z}) = \frac{\mathbb{Z}}{n\mathbb{Z}} (c_1(\theta))^k
\]

where \( c_1(\theta) \) is the first Chern class of \( \theta \).

**Proof.** We know that \( \text{Hom}(\langle B \rangle, \mathbb{C}^*) \xrightarrow{\cong} H^2(\langle B \rangle, \mathbb{Z}) \) is an isomorphism, as \( \theta \) generates \( \text{Hom}(\langle B \rangle, \mathbb{C}^*) \) then \( c_1(\theta) \) generates \( H^2(\langle B \rangle, \mathbb{Z}) \).

The fact that \( \langle B \rangle \) is of period 2 implies that \( c_1(\theta)^k \) is the generator in dimension \( 2k \). \( \square \)

**Proposition 4.4.** For every \( k \geq 1 \),

\[
H^{2k}(G_p, \mathbb{Z})^{G/G_p} = \begin{cases}
0 & \text{if } k \not\equiv 0(d_p) \\
\frac{\mathbb{Z}}{p^\alpha \mathbb{Z}} (c_1(\theta))^k & \text{if } k = qd_p
\end{cases}
\]

where

\[
\theta_p : \langle A_p \rangle \to \mathbb{C}^*
\]

\[
\begin{array}{ccc}
\text{A}_p & \mapsto & e^{2i\pi p^\alpha} \\
\end{array}
\]

The proof of the Proposition 4.4 requires some technical lemmas:

**Lemma 4.5.** The generator \( \overline{B} \) of \( G/G_p \) acts on \( \theta_p \) sending it to \( \theta_p^r \).

**Proof.** If \( B\theta_p \) denotes this action we have \( (B\theta_p)(A_p) = \theta(A_p^r) \).

Remember that \( B \) acts on \( G_p \) by sending \( A_p \) to \( A_p^r \), by definition we get \( (B(\theta_p)(A_p) = (\theta_p(A_p^r)) = \theta_p^r(A_p) \). \( \square \)

**Lemma 4.6.** The generator \( \overline{B} \) of \( G/G_p \) acts on \( H^{2k}(G_p, \mathbb{Z}) \) by sending \( \lambda(c_1(\theta_p))^k \) to \( \lambda r^k(c_1(\theta_p))^k \) where \( \lambda \in \mathbb{Z}/p^\alpha \mathbb{Z} \).

**Proof.** Indeed, let \( x \in H^{2k}(\mathbb{Z}/p^\alpha \mathbb{Z}, \mathbb{Z}) \) and \( Bx \) the action \( \overline{B} \) on \( x \).
Proposition 4.8.

Let $x = \lambda(c_1(\theta_p))^k \in H^{2k}(G_p, \mathbb{Z})^{G/G_p}$.

Hence $x \in H^{2k}(G_p, \mathbb{Z})^{G/G_p} \Leftrightarrow \lambda(c_1(\theta_p))^k = \lambda r^k(c_1(\theta_p))^k \Leftrightarrow \lambda(r^k - 1) \equiv 0(p^\alpha)$.

We distinguish three cases:

First case: $(p, r^k - 1) = 1$ which means that $\lambda = 0(p^\alpha)$.

Second case: $p$ divides $r^k - 1$.

If $r^k - 1 = p^\alpha_1 p_1$ with $(p_1, p) = 1$ and $0 < \alpha_1 < \alpha$ then we have $\lambda p^\alpha_1 p_1 \equiv 0(p^\alpha)$, therefore $\lambda p_1 \equiv p^{\alpha - \alpha_1}$, which implies $\lambda \equiv 0(p)$.

Consider $x = (1 + r^k + \ldots + r^{k(d_p-1)})(c_1(\theta_p))^k$. Let us show that $x \in H^{2k}(G_p, \mathbb{Z})^{G/G_p}$.

If $r^k + \ldots + r^{k(d_p-1)} \equiv 0(p)$. On the other hand $r^k \equiv 1(p)$ which implies that $d_p \equiv 1 + r^k + \ldots + r^{k(d_p-1)}(p)$. Hence $d_p \equiv 0(p)$ which is impossible because $d_p/n$ and $(n, m) = 1$.

Third case: $r^k - 1 = p^\alpha_1 p_1$ with $(p_1, p) = 1$ and $\alpha_1 \geq \alpha$.

Therefore $r^k \equiv 1(p^\alpha)$ hence $k = q d_p$.

For any $\lambda \in \frac{\mathbb{Z}}{p^\alpha \mathbb{Z}}$ we have $\lambda(r^k - 1) \equiv 0(p^\alpha)$

then $H^{2k}(G_p, \mathbb{Z})^{G/G_p} = \frac{\mathbb{Z}}{p^\alpha \mathbb{Z}}(c_1(\theta_p))^q$.

Definition 4.7. [7]Let $G$ be a finite group of order $N$ and let $k$ be an integer $\geq 1$. We say that $G$ has a periodic cohomology with period $k$ if there exists an element $u \in H^{k}(G, \mathbb{Z})$ of order equal to $N$.

Proposition 4.8. The cohomological period of $G$ is $2.lcm_{p/m}(d_p)$, where $lcm_{p/m}(d_p)$ is the smallest common multiple of $d_p$ for $p/m$.

Proof. Let $|H^{2d}(G, \mathbb{Z})|$ be the order of $H^{2d}(G, \mathbb{Z})$ where $d = lcm_{p/m}(d_p)$.

$|H^{2d}(G, \mathbb{Z})| = |H^{2d}(\langle B \rangle, \mathbb{Z})| \times \prod_{p/m} |H^{2d}(G_p, \mathbb{Z})^{G/G_p}|$ since it is a direct product of groups. Since $|H^{2d}(\langle B \rangle, \mathbb{Z})| = \frac{\mathbb{Z}}{n \mathbb{Z}} = n$ due to the fact that 2 is the period of $\langle B \rangle$, and $|H^{2d}(G_p, \mathbb{Z})^{G/G_p}| = \frac{\mathbb{Z}}{p^\alpha \mathbb{Z}} = p^\alpha$ because $d$ is a multiple of $d_p$. Thus $|H^{2d}(G, \mathbb{Z})| = m \times n = |G|$. So $2d$ is a cohomological period of $G$. Furthermore this is the smallest period $|H^{2k}(G, \mathbb{Z})| = |G| \Leftrightarrow |H^{2k}(G_p, \mathbb{Z})^{G/G_p}| = p^\alpha \Leftrightarrow k$ is a multiple of $d_p$ for $p/m$, which shows that $2d$ is the smallest period. \[\square\]
By induction methods we get:

**Proposition 4.9.** Let $p$ be a prime divisor of $m$. There exists $\zeta_p$ a representation of $G$ of degree $d_p$ such that:

$$
\zeta_p(A) = \begin{bmatrix}
2i\pi & 0 & \ldots & 0 \\
e m & e & \ldots & 0 \\
2i\pi r & \ldots & \vdots & \\
0 & e & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & e m \\
\end{bmatrix}
$$

(4.10)

$$
\zeta_p(B) = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & 1 \\
2i\pi d_p & \ldots & 0 \\
e n & 0 & \ldots & 0 \\
\end{bmatrix}
$$

(4.11)

where $\text{Res}^{G}_{G_p}(c_{d_p}(\zeta_p))$ is a generator of $H^{2d_p}(G_p, \mathbb{Z})^G/G_p$.

To establish the latter proposition, we need the following lemmas:

**Lemma 4.10.** Let $C_p$ be the centralizer of $G_p = \langle A_p \rangle$ in $G$, and $\hat{\theta}_p$ the extension of $\theta_p$ to $C_p$.

If $\zeta_p = \text{Ind}^{C_p}_{G_p}(\hat{\theta}_p)$ is the induced representation from $\hat{\theta}_p$ then $\zeta_p$ has degree $d_p$ where

$$
\zeta_p(A) = \begin{bmatrix}
2i\pi & 0 & \ldots & 0 \\
e m & e & \ldots & 0 \\
2i\pi r & \ldots & \vdots & \\
0 & e & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & e m \\
\end{bmatrix}
$$

and

$$
\zeta_p(B) = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & 1 \\
2i\pi d_p & \ldots & 0 \\
e n & 0 & \ldots & 0 \\
\end{bmatrix}
$$

Proof. The representation $\theta_p$ has an extension $\hat{\theta}_p$ to $C_p$ defined by $\hat{\theta}_p(B^{d_p}) = e^{\frac{2i\pi d_p}{n}}$ and $\hat{\theta}_p(A) = e^{\frac{2i\pi}{m}}$. 
GROUP COHOMOLOGY OF $G = \langle A, B | A^n = B^n = e, BAB^{-1} = A', r^n \equiv 1(m), (n(r-1), m) = 1 \rangle$.

$G/C_p = \{ B^{1-i} \}_{1 \leq i \leq d_p} = \{ 1, B^{-1}, B^{-2}, ..., B^{-d_p+1} \}$.

Therefore by setting $\zeta_p = \text{ind}_{G_p}^G(\hat{\theta}_p)$ then, $\zeta_p$ is of degree $d_p$ and $\zeta_p(x) = (\hat{\theta}_p(B^{i-1}xB^{1-j}))_{1 \leq i \leq d_p}$ $1 \leq j \leq d_p$.

Using the equality of matrices in the basis $C \oplus C \oplus ... \oplus C$, with the convention

$\hat{\theta}_p(y) = 0$ if $y \not\in C_p$.

$\zeta_p(A) = (\hat{\theta}_p(B^{i-1}AB^{1-j}))_{1 \leq i \leq d_p} = (\hat{\theta}_p(A^{i-1}B^{1-j}))_{1 \leq i \leq d_p}$ $1 \leq j \leq d_p$ (4.12)

Hence

$$
\zeta_p(A) = \begin{bmatrix}
2i\pi \\
e m & 0 & \ldots & 0 \\
& 2i\pi r & \ldots & 0 \\
0 & e m & \ddots & \vdots \\
& \ddots & \ddots & 0 \\
& 0 & \ldots & 0 & e m
\end{bmatrix}
$$

$\zeta_p(B) = (\hat{\theta}_p(B^{i-1}BB^{1-j}))_{1 \leq i \leq d_p} = (\hat{\theta}_p(B^{i+j-1}))_{1 \leq i \leq d_p}$ $1 \leq j \leq d_p$ (4.13)

Hence

$$
\zeta_p(B) = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
& \ddots & \ddots & \ddots & \vdots \\
& \ddots & \ddots & \ddots & 0 \\
0 & \ldots & \ldots & 0 & 1 \\
& 2i\pi d_p & 0 & \ldots & 0
\end{bmatrix}
$$

Because $B^{1+i-j} \in C_p \Leftrightarrow 1 + i - j \equiv 0(d_p)$.

As $2 - d_p \leq 1 + i - j \leq d_p$, two cases arise: $1 + i - j = 0$ or $1 + i - j = d_p$.

If $1 + i - j = 0$, so $j = i + 1$ and $\hat{\theta}_p(1) = 1$.

If $1 + i - j = d_p$, from $2 \leq 1 + i \leq 1 + d_p$ and $1 + d_p \leq j + d_p \leq 2d_p$ we have

$i + 1 = j + d_p = 1 + d_p$ hence, $i = d_p$ and $j = 1$. So $\hat{\theta}_p(B^{d_p}) = e^{2i\pi d_p}$.

□

Lemma 4.11. Let $\eta_p = \text{Res}_{G_p}^G(\zeta_p)$ be the restriction of $\zeta_p$ to $G_p$. So

$$
\left\{ \begin{array}{ll}
\eta_p &= \theta_p \oplus \theta^{r} \oplus ... \oplus \theta^{r^{d_p-1}} \\
c_{d_p}(\eta_p) &= 1 \times r \times r^2 \times ... \times r^{d_p-1}(c_1(\theta_p))^{d_p}
\end{array} \right. (4.14)
$$
Proof. We deduce that \( \eta_p = \theta_p \oplus \theta_p^r \oplus \ldots \oplus \theta_p^{r_{dp}-1} \) from the equality

\[
\zeta_p(A_p) = \begin{bmatrix}
2i\pi & 0 & \ldots & 0 \\
e & m & 0 & \ldots & 0 \\
2i\pi r & e & p^\alpha & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & e & p^\alpha \\
\end{bmatrix}
\]

Hence \( c(\eta_p) = c(\theta_p) c(\theta_p^r) \ldots c(\theta_p^{r_{dp}-1}) \). In dimension \( 2d_p \) we have:

\[ c_{d_p}(\eta_p) = c_1(\theta_p) c_1(\theta_p^r) \ldots c_1(\theta_p^{r_{dp}-1}) = 1 \times r \times \ldots r_{dp}^{-1}(c_1(\theta_p))^{d_p}. \quad \square \]

Proof of Proposition 4.9. As \( 1 \times r \times \ldots r_{dp}^{-1} \) is coprime with \( p \), \( c_{d_p}(\eta_p) \) generates \( H^{2d_p}(G_p, \mathbb{Z})^{G/G_p} \).

Proposition 4.12. There exists \( \zeta \in R(G) \) defined by \( \zeta(A) \) a unitary matrix and \( \zeta(B) = e^{2i\pi/n} \) such that \( \text{Res}_{\langle B \rangle}^G(c_1(\zeta)) \) generates \( H^2(\langle B \rangle, \mathbb{Z}) \).

Proof. Consider \( \zeta : G \xrightarrow{p} \langle B \rangle \xrightarrow{\theta} \mathbb{C}^* \) such that \( G \xrightarrow{p} \langle B \rangle \) is the canonical projection and \( \theta : \langle B \rangle \to \mathbb{C}^* \) denotes the representation defined by \( \theta(B) = e^{2i\pi/n} \).

Thus \( \zeta \) is a representation of \( G \) with \( \zeta(A) \) is a unitary matrix and \( \zeta(B) = e^{2i\pi/n} \). Consequently, \( \text{Res}_{\langle B \rangle}^G(c_1(\zeta)) = \text{Res}_{\langle B \rangle}^G(c_1(\theta \circ p)) = \text{Res}_{\langle B \rangle}^G(p^*c_1(\theta)) = c_1(\theta) \) which is a generator of \( H^2(\langle B \rangle, \mathbb{Z}) \). \quad \square

5. Proof of main result

Theorem 5.1. There exists a graded ring isomorphism

\[
H^*(G, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} [c_1(\zeta)]^+ \times \prod_{p/m} \mathbb{Z}/p^\alpha\mathbb{Z} [c_{d_p}(\zeta_p)]^+
\]

such that \( c_1(\zeta) \) is the first Chern class of the representation \( \zeta \) of \( G \) where \( \zeta(A) \) is a unitary matrix, \( \zeta(B) = e^{2i\pi/n} \) and \( c_{d_p}(\zeta_p) \) are the \( d_p \) Chern classes of representations \( \zeta_p \) of \( G \) given by:

\[
\zeta_p(A) = \begin{bmatrix}
2i\pi & 0 & \ldots & 0 \\
e & m & 0 & \ldots & 0 \\
2i\pi r & e & m & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & e & m \\
\end{bmatrix}
\]
GROUP COHOMOLOGY OF $G = \langle A, B | A^m = B^n = e, BAB^{-1} = A^r, r^n \equiv 1(m), (n(r-1), m) = 1 \rangle$

$$\zeta_p(B) = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & \ldots & 0 & 1 \\
2i\pi d_p & e & n & 0 & \ldots & 0
\end{bmatrix}$$

where $d_p$ is the order of $r$ in $(\frac{\mathbb{Z}}{p^\alpha\mathbb{Z}})^*$ and $p^\alpha$ stands for the highest power of $p$ dividing $m$.

**Proof.** Step 1: We have the graded rings isomorphism

$$H^*(G, \mathbb{Z}) = \mathbb{Z} \oplus H^*(G, \mathbb{Z})^+$$

Step 2: According to Proposition 3.9

$$H^*(G, \mathbb{Z})^+ \cong H^*(\frac{\mathbb{Z}}{n\mathbb{Z}}, \mathbb{Z})^+ \times \prod_{p/m} H^*(G_p, \mathbb{Z})^{+G/G_p}$$

Step 3: It follows from Proposition 4.12

$$\mathbb{Z}_{n\mathbb{Z}}[c_1(\zeta)]^+ \cong H^*((B), \mathbb{Z})^+$$

Step 4: Proposition 4.9 yields to

$$\mathbb{Z}_{p^\alpha\mathbb{Z}}[c_p(\zeta_p)]^+ \cong H^*(G_p, \mathbb{Z})^{+G/G_p}$$

Hence the Theorem 2.1. \(\square\)

**Remark 5.2.** Let $BG$ be the classifying space of $G$. The previous computations of the cohomology ring $H^*(G; \mathbb{Z})$ allow the authors to describe some invariants of $BG$ such as its topological and connective $K$-theory rings (see [4]). In [1] Azi and Hamraoui computed the $k$- representation ring of the special group $SL_d(k)$, where $k$ a commutative field and they deduced in [2] a computation of the topological $K$-theory ring of its classifying space.

**References**

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