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COHOMOLOGY RING OF THE FINITE SPLIT METACYCLIC GROUP

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Abstract. Let G be the finite split metacyclic group of presentation $G = \langle A, B/A^m = B^n = e, BAB^{-1} = A^r, r^n \equiv 1(m), (n(r-1), m) = 1\rangle$. Huebschmann computed the integral cohomology ring of a large class of metacyclic groups using the mechanism of homological perturbation theory and determined the generators of the subring of even degree classes of any metacyclic group by means of the Chern classes of some of its representations. The purpose of this article is to investigate $H^*(G;\mathbb{Z})$ by localization methods. It also aims to determine explicit representations of G whose Chern classes generate $H^*(G;\mathbb{Z})$ by induction techniques.

1. Introduction

Let G be the finite split metacyclic group of presentation $G = \langle A, B/A^m = B^n = e, BAB^{-1} = A^r, r^n \equiv 1(m), (n(r-1), m) = 1\rangle$.
In [5, 6] Huebschmann determinated the integral cohomology ring of a large class of metacyclic groups. The aim of this paper is to show by localization and induction techniques that the integral cohomology ring $H^*(G;\mathbb{Z})$ of G is generated by Chern classes of complex linear representations of G. It is well known that the cohomology groups of finite groups are torsions (see Proposition 3.1) and the restriction homomorphisms $Res^G_{G_p} : H^k(G,\mathbb{Z})_p \longrightarrow H^k(G_p,\mathbb{Z})^G$ for all $k \geq 1$ are isomorphisms between the p-primary parts of $H^k(G,\mathbb{Z})$ and $H^k(G_p,\mathbb{Z})^G$ the invariant elements of the p-Sylow subgroups G_p of G, where p runs over all prime numbers divinding the order of G (see Proposition 3.3).
For our group we will be interested in two cases :
If p/m : G_p is normal and $H^k(G,\mathbb{Z})_p \cong H^k(G_p,\mathbb{Z})^{G/G_p}$.
If p/n : G_p is abelian and $H^k(G,\mathbb{Z})_p \cong H^k(G_p,\mathbb{Z})^{N_p}$.
Hence we get the group isomorphism Proposition 3.9 : For every $k \geq 1$,

$$H^k(G,\mathbb{Z}) \longrightarrow \prod_{p/m} H^k(G_p,\mathbb{Z})^{G/G_p} \times H^k(\langle B\rangle,\mathbb{Z})$$

In order to describe the cohomology ring $H^*(G;\mathbb{Z})$ by means of Chern classes of

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representations of G_p we establish the propositions below :

Proposition 4.3 : For every $k \geq 1$, $H^{2k}(\langle B\rangle, \mathbb{Z}) = \frac{\mathbb{Z}}{n\mathbb{Z}}(c_1(\theta))^k$

$$\begin{array}{rccc} \theta\colon & \langle B\rangle & \longrightarrow & \mathbb{C}^* \\ & B & \longmapsto & e^{\frac{2i\pi}{n}} \end{array}$$

where $c_1(\theta)$ is the first Chern class of θ.

Proposition 4.4 :For every $k \geq 1$,

$$H^{2k}(G_p, \mathbb{Z})^{G/G_p} = \begin{cases} 0 & if \quad k \ncong 0(d_p) \\ \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}(c_1(\theta_p))^k & if \quad k = qd_p \end{cases}$$

where

$$\begin{array}{rccc} \theta_p\colon & \langle A_p\rangle & \longrightarrow & \mathbb{C}^* \\ & A_p & \longmapsto & e^{\frac{2i\pi}{p^\alpha}} \end{array}$$

Which implies that G has a periodic cohomology (see Proposition 4.8). Then by induction methods we obtain the two results stated as follows:

Proposition 4.9 : Let p be a prime divisor of m. There exists a representation ζ_p of G of degree d_p such that :

$$\zeta_p(A) = \begin{bmatrix} e^{\frac{2i\pi}{m}} & 0 & \dots & 0 \\ 0 & e^{\frac{2i\pi r}{m}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{\frac{2i\pi r^{d_p-1}}{m}} \end{bmatrix} \; and \; \zeta_p(B) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ e^{\frac{2i\pi d_p}{n}} & 0 & \dots & \dots & 0 \end{bmatrix}$$

where $Res^G_{G_p}(c_{d_p}(\zeta_p))$ is a generator of $H^{2d_p}(G_p, \mathbb{Z})^{G/G_p}$.

Proposition 4.12: There exists $\zeta \in R(G)$ where $\zeta(A)$ is a unitary matrix and $\zeta(B) = e^{\frac{2i\pi}{n}}$ such that $Res^G_{\langle B\rangle}(c_1(\zeta))$ generates $H^2(\langle B\rangle, \mathbb{Z})$.

The arguments that have been carried out lead to Theorem 2.1 :

There exists a graded ring isomorphism

$$H^*(G, \mathbb{Z}) = \mathbb{Z} \oplus \frac{\mathbb{Z}}{n\mathbb{Z}}[c_1(\zeta)]^+ \times \prod_{p/m} \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}[c_{d_p}(\zeta_p)]^+$$

such that $c_1(\zeta)$ is the first Chern class of the representation ζ of G where $\zeta(A)$ is a unitary matrix, $\zeta(B) = e^{\frac{2i\pi}{n}}$ and $c_{d_p}(\zeta_p)$ are the d_p Chern classes of representations ζ_p of G given by:

$$\zeta_p(A) = \begin{bmatrix} e^{\frac{2i\pi}{m}} & 0 & \dots & 0 \\ 0 & e^{\frac{2i\pi r}{m}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{\frac{2i\pi r^{d_p-1}}{m}} \end{bmatrix}$$

$$\zeta_p(B) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ e^{\frac{2i\pi d_p}{n}} & 0 & \dots & \dots & 0 \end{bmatrix}$$

where d_p is the order of r in $(\frac{\mathbb{Z}}{p^\alpha \mathbb{Z}})^*$ and p^α stands for the highest power of p dividing m.

2. Statement of the main result and preliminaries

Let G be the finite split metacyclic group of presentation
$G = \langle A, B / A^m = B^n = e, BAB^{-1} = A^r, r^n \equiv 1(m), (n(r-1), m) = 1\rangle$.

Theorem 2.1. *There exists a graded ring isomorphism*

$$H^*(G,\mathbb{Z}) = \mathbb{Z} \oplus \frac{\mathbb{Z}}{n\mathbb{Z}}[c_1(\zeta)]^+ \times \prod_{p/m} \frac{\mathbb{Z}}{p^\alpha \mathbb{Z}}[c_{d_p}(\zeta_p)]^+ \tag{2.1}$$

such that $c_1(\zeta)$ is the first Chern class of the representation ζ of G where $\zeta(A)$ is a unitary matrix, $\zeta(B) = e^{\frac{2i\pi}{n}}$ and $c_{d_p}(\zeta_p)$ are the d_p Chern classes of representations ζ_p of G given by:

$$\zeta_p(A) = \begin{bmatrix} e^{\frac{2i\pi}{m}} & 0 & \dots & 0 \\ 0 & e^{\frac{2i\pi r}{m}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{\frac{2i\pi r^{d_p-1}}{m}} \end{bmatrix}$$

$$\zeta_p(B) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ e^{\frac{2i\pi d_p}{n}} & 0 & \dots & \dots & 0 \end{bmatrix}$$

where d_p is the order of r in $(\frac{\mathbb{Z}}{p^\alpha\mathbb{Z}})^*$ and p^α stands for the highest power of p dividing m.

This section is devoted to the description of the group we are dealing with. Several notations and reminders will be introduced in order to make the paper self contained.
Let G be the finite group of presentation:
$G = \langle A, B / A^m = B^n = e, BAB^{-1} = A^r, r^n \equiv 1(m), (n(r-1), m) = 1\rangle$.
Thus G is a group of which all Sylow subgroups are cyclic. Moreover, G is split and metacyclic. Its commutator $[G, G]$ is isomorphic to $\langle A\rangle \cong \frac{\mathbb{Z}}{m\mathbb{Z}}$ and its derived group $G/[G, G]$ is isomorphic to $\langle B\rangle \cong \frac{\mathbb{Z}}{m\mathbb{Z}}$. Furthermore, $G \cong \frac{\mathbb{Z}}{m\mathbb{Z}} \rtimes_\rho \frac{\mathbb{Z}}{n\mathbb{Z}}$ is the semi-direct product where ρ: $\frac{\mathbb{Z}}{n\mathbb{Z}} \longrightarrow Aut(\frac{\mathbb{Z}}{m\mathbb{Z}})$ denotes the conjugation morphism of elements of $\frac{\mathbb{Z}}{m\mathbb{Z}}$ by elements of $\frac{\mathbb{Z}}{n\mathbb{Z}}$.
In addition, G is generated by the elements A and B of coprime order m and n, therefore $|G| = mn$. An element x of G is written in the form A^tB^s with $0 \le t \le m-1$ and $0 \le s \le n-1$. We also have $B^sA^t = A^{tr^s}B^s$ for all $t \in \mathbb{N}$ and $s \in \mathbb{N}$.

Let now p be a prime number dividing $|G|$ and G_p a p-Sylow subgroup of G.

(1) If p/m, then G_p is isomorphic to the cyclic subgroup $\langle A_p\rangle$ of G generated by $A_p = A^{m/p^\alpha}$. We have G_p is normal in G, so its normalizer $N_p = G$ and its centralizer C_p is isomorphic to $\langle A, B^{d_p}\rangle$ where d_p is the smallest divisor of n satisfying $r^n \equiv 1(p^\alpha)$. The action of G on G_p boils down to the action of $G/G_p \cong \langle \overline{B}^{\bar{i}}\rangle$ where $\bar{i}$ is the class of i modulo d_p. We point out that p is different from 2 because $(m, r(r-1)) = 1$.

(2) If p/n, then G_p is isomorphic to the cyclic subgroup $\langle B_p\rangle$ of G generated by $B_p = B^{n/p^\alpha}$. In this case the normalizer of G_p coincides with the centralizer $N_p = C_p$. Thus the action of N_p on G_p is trivial.

Recall that for each prime number p the p -Sylow subgroups of G are conjugates. We choose a representative G_p in this conjugation class. The collection $\{G_p;\ p/[G:1]\}$ is called a representative family of p-Sylow subgroups of G.

3. Computation of $H^*(G, \mathbb{Z})$

Let G be a finite group and $\mathbb{Z}$ the trivial $\mathbb{Z}[G]$-module. First of all, the cohomology of finite groups is torsion.

Proposition 3.1. [3] *For every $k \geq 1$,*

$$[G:1]H^k(G, \mathbb{Z}) = 0. \tag{3.1}$$

Hence, if we want to determine the cohomology of a finite group G, we only need to compute the p-primary parts for all primes p dividing $|G|$.

Corollary 3.2. *For every $k \geq 1$,*

$$H^k(G, \mathbb{Z}) = \prod_{p/[G:1]} H^k(G, \mathbb{Z})_p. \tag{3.2}$$

where $H^k(G, \mathbb{Z})_p = \left\{z \in H^k(G, \mathbb{Z})/\exists n \in \mathbb{N}\ p^n.z = 0\right\}$ is the p-primary part of $H^k(G, \mathbb{Z})$.

Proposition 3.3. *There exists a graded ring isomorphism*

$$H^*(G, \mathbb{Z}) \cong \mathbb{Z} \oplus \prod_{p/|G|} H^*(G, \mathbb{Z})_p \tag{3.3}$$

where $H^*(G, \mathbb{Z})_p = \oplus_{k\geq 1} H^k(G, \mathbb{Z})_p$.

Proof. Since the set of prime divisors p of $|G|$ is finite, the projections $H^*(G, \mathbb{Z}) \longrightarrow H^*(G, \mathbb{Z})_p$ yield a graded abelian group isomorphism

$$H^*(G, \mathbb{Z}) \longrightarrow \mathbb{Z} \oplus \oplus_{p/|G|} H^*(G, \mathbb{Z})_p \longrightarrow \mathbb{Z} \oplus \prod_{p/|G|} H^*(G, \mathbb{Z})_p$$

Moreover, these projections are ring homomorphisms because $H^*(G, \mathbb{Z})_p$ is a graded ideal of $H^*(G, \mathbb{Z})$ then $\oplus_{q/|G|, q\neq p} H^*(G, \mathbb{Z})_q$ is also a graded ideal of $H^*(G, \mathbb{Z})$. □

In order to assemble the cohomology of the finite group G, we study the action of G on the cohomology of some of its subgroups.
Let H be a subgroup of G and g an element of G, we denote by ${}^gH = gHg^{-1}$.
The cochains complex map

$$\begin{array}{rccc} c_g\colon & Hom_{\mathbb{Z}[H]}(P_*,\mathbb{Z}) & \longrightarrow & Hom_{\mathbb{Z}[{}^gH]}(P_*,\mathbb{Z}) \\ & f & \longmapsto & c_g(f)\colon p \mapsto g.f(g^{-1}.p) \end{array}$$

induces the isomorphisms for all $k \in \mathbb{N}$

$$c_g\colon H^k(H,\mathbb{Z}) \longrightarrow H^k({}^gH,\mathbb{Z})$$

where P_* is a projective resolution of $\mathbb{Z}$.

Definition 3.4. Let $k \geq 1$. An element $\alpha \in H^k(H,\mathbb{Z})$ is said invariant if $Res^H_{H\cap {}^gH}(\alpha) = Res^{{}^gH}_{H\cap {}^gH}(c_g(\alpha))$ for all $g \in G$.

The following proposition connects the p-primary parts of $H^*(G,\mathbb{Z})$ to the cohomology of p-Sylow subgroups of G.

Proposition 3.5. [8]*Let* $\{G_p;\ p/[G:1]\}$ *be a representative family of p-Sylow subgroups of* G *then for all* $k \geq 1$ *:*

$$Res^G_{G_p}\colon H^k(G,\mathbb{Z})_p \longrightarrow H^k(G_p,\mathbb{Z})^G \tag{3.4}$$

is an isomorphism, where $H^k(G_p,\mathbb{Z})^G$ *are the invariant elements of* $H^k(G_p,\mathbb{Z})$*.*

Proposition 3.6. *There exists a graded ring isomorphism*

$$H^*(G,\mathbb{Z}) \cong \mathbb{Z} \oplus \prod_{p/|G|} H^*(G_p,\mathbb{Z})^G \tag{3.5}$$

where $H^*(G_p,\mathbb{Z})^G = \oplus_{k\geq 1} H^k(G_p,\mathbb{Z})^G$*.*

Proof. We have for all $p/|G|$, a graded abelian group isomorphism $H^*(G,\mathbb{Z})_p \longrightarrow H^*(G_p,\mathbb{Z})^G$ induced by the ring homomorphism $Res^G_{G_p}$. Hence $H^*(G,\mathbb{Z})_p \longrightarrow H^*(G_p,\mathbb{Z})^G$ is a ring isomorphism.

□

Proposition 3.7. [8] *If* G_p *is normal then for all* $k \geq 1$,

$$Res^G_{G_p}\colon H^k(G,\mathbb{Z})_p \longrightarrow H^k(G_p,\mathbb{Z})^{G/G_p} \tag{3.6}$$

is an isomorphism.

Proposition 3.8. [8]*If* G_p *is abelian then for all* $k \geq 1$,

$$Res^G_{G_p}\colon H^k(G,\mathbb{Z})_p \longrightarrow H^k(G_p,\mathbb{Z})^{N_p} \tag{3.7}$$

is an isomorphism.

For our group G, as enounciated before, we will discuss the two particular cases:

1) If p/m then G_p is normal hence $H^k(G,\mathbb{Z})_p \cong H^k(G_p,\mathbb{Z})^{G/G_p}$.
2) If p/n, consequently G_p is abelian then $H^k(G,\mathbb{Z})_p \cong H^k(G_p,\mathbb{Z})^{N_p}$.

So we have :

Proposition 3.9. *For every $k \geq 1$,*

$$H^k(G,\mathbb{Z}) \longrightarrow \prod_{p/m} H^k(G_p,\mathbb{Z})^{G/G_p} \times H^k(\langle B\rangle,\mathbb{Z}) \tag{3.8}$$

is a group isomorphism.

Proof. We know that $H^k(G,\mathbb{Z}) \cong \prod_{p/|G|} H^k(G,\mathbb{Z})_p$ is an isomorphism .

- When p/n: G_p is abelian then $H^k(G,\mathbb{Z})_p \cong H^k(G_p,\mathbb{Z})^{N_p}$. And since the action of N_p on G_p is trivial, it follows that $H^k(G_p,\mathbb{Z})^{N_p} \cong H^k(G_p,\mathbb{Z})$. We also have $H^k(\langle B\rangle,\mathbb{Z})_p \cong H^k(G_p,\mathbb{Z})$, hence the following triangle is commutative :

$$\begin{array}{ccc} H^k(G,\mathbb{Z})_p & \xrightarrow{\quad Res^G_{G_p} \quad} & H^k(G_p,\mathbb{Z}) \\ & {\scriptstyle Res^G_{\langle B\rangle}} \searrow \qquad \nearrow {\scriptstyle Res^{\langle B\rangle}_{G_p}} & \\ & H^k(\langle B\rangle,\mathbb{Z})_p & \end{array}$$

  We deduce that $\prod_{p/n} H^k(G,\mathbb{Z})_p \cong H^k(\dfrac{\mathbb{Z}}{n\mathbb{Z}},\mathbb{Z})$ is a group isomorphism.

- When p/m: $G_p = \langle A_p\rangle$ is a normal subgroup of G, therefore $H^k(G,\mathbb{Z})_p \cong H^k(G_p,\mathbb{Z})^{G/G_p}$ is an isomorphism. We deduce that $\prod_{p/m} H^k(G,\mathbb{Z})_p \cong \prod_{p/m} H^k(G_p,\mathbb{Z})^{G/G_p}$ is an isomorphism.

□

As a result:

Corollary 3.10. *For every $k \in \mathbb{N}$,*

$$H^{2k+1}(G,\mathbb{Z}) = 0 \tag{3.9}$$

.

Proof. Let $k \in \mathbb{N}$. For every cyclic group H, we have $H^{2k+1}(H,\mathbb{Z}) = 0$ then $H^{2k+1}(G,\mathbb{Z}) = 0$. □

4. Chern classes of groups representations

Definition 4.1. [8] Let $\rho\colon G \to GL_n(\mathbb{C})$ be a representation of a discrete group G of degree n and $p\colon EG \to BG$ the associated universal G-bundle .
The k-th Chern class $c_k(\rho)$ is the k-th Chern class of the vector bundle $EG \times_\rho \mathbb{C}^n$ of dimension n on BG defined by :

$$c_k(\rho) = c_k(EG \times_\rho \mathbb{C}^n) \in H^{2k}(BG,\mathbb{Z}) = H^{2k}(G,\mathbb{Z}), \quad \text{k=0,1,...,n.} \tag{4.1}$$

Note that G acts on the right on EG by Deck transformations and on the left on $\mathbb{C}^n$ by ρ.

These classes verify :

Proposition 4.2. [8]

(1) $c_k(\rho) = 0$ *if* $k > n$.

(2) *If* $f\colon H \to G$ *is a group homomorphism then* $c_k(\rho \circ f) = f^*(c_k(\rho))$.
(3) *If* $C(\rho) = 1 + c_1(\rho) + ... + c_n(\rho)$ *is the total Chern class then* $C(\rho_1 \oplus \rho_2) = C(\rho_1)C(\rho_2)$.
(4) $c_1\colon Hom(G, \mathbb{C}^*) \xrightarrow{\simeq} H^2(G, \mathbb{Z})$ *is an isomorphism .*

Proposition 4.3. *For every* $k \geq 1$,

$$H^{2k}(\langle B\rangle, \mathbb{Z}) = \frac{\mathbb{Z}}{n\mathbb{Z}}(c_1(\theta))^k \tag{4.2}$$

$$\begin{array}{rccc} \theta\colon & \langle B\rangle & \longrightarrow & \mathbb{C}^* \\ & B & \longmapsto & e^{\frac{2i\pi}{n}} \end{array}$$

where $c_1(\theta)$ *is the first Chern class of* θ.

Proof. We know that $Hom(\langle B\rangle, \mathbb{C}^*) \xrightarrow{c_1} H^2(\langle B\rangle, \mathbb{Z})$ is an isomorphism, as θ generates $Hom(\langle B\rangle, \mathbb{C}^*)$ then $c_1(\theta)$ generates $H^2(\langle B\rangle, \mathbb{Z})$.
The fact that $\langle B\rangle$ is of period 2 implies that $c_1(\theta)^k$ is the generator in dimension $2k$. □

Proposition 4.4. *For every* $k \geq 1$,

$$H^{2k}(G_p, \mathbb{Z})^{G/G_p} = \begin{cases} 0 & if \quad k \not\equiv 0(d_p) \\ \dfrac{\mathbb{Z}}{p^\alpha\mathbb{Z}}(c_1(\theta_p))^k & if \quad k = qd_p \end{cases} \tag{4.3}$$

where

$$\begin{array}{rccc} \theta_p\colon & \langle A_p\rangle & \longrightarrow & \mathbb{C}^* \\ & A_p & \longmapsto & e^{\frac{2i\pi}{p^\alpha}} \end{array}$$

The proof of the Proposition 4.4 requires some technical lemmas:

Lemma 4.5. *The generator* $\overline{B}$ *of* G/G_p *acts on* θ_p *sending it to* θ_p^r.

Proof. If $\overline{B}\theta_p$ denotes this action we have $(\overline{B}\theta_p)(A_p) = \theta(A_p^r)$.
Remember that B acts on G_p by sending A_p to A_p^r , by definition we get $(B(\theta_p)(A_p) = (\theta_p(A_p^r)) = \theta_p^r(A_p)$. □

Lemma 4.6. *The generator* $\overline{B}$ *of* G/G_p *acts on* $H^{2k}(G_p, \mathbb{Z})$ *by sending* $\lambda(c_1(\theta_p))^k$ *to* $\lambda r^k(c_1(\theta_p))^k$ *where* $\lambda \in \dfrac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$.

Proof. Indeed, let $x \in H^{2k}(\dfrac{\mathbb{Z}}{p^\alpha\mathbb{Z}}, \mathbb{Z})$ and $\overline{B}x$ the action $\overline{B}$ on x.

$$\overline{B}x = \overline{B}(\lambda c_1(\theta_p)^k) = \lambda\overline{B}(c_1(\theta_p)^k) \quad (4.4)$$
$$= \lambda(\overline{B}(c_1(\theta_p))^k \quad (4.5)$$
$$= \lambda(c_1(\overline{B}\theta_p))^k \quad (4.6)$$
$$= \lambda(c_1(\theta_p^r))^k \quad (4.7)$$
$$= \lambda(rc_1(\theta_p))^k \quad (4.8)$$
$$= \lambda r^k c_1(\theta_p))^k. \quad (4.9)$$

□

Proof of Proposition 4.4
Let $x = \lambda(c_1(\theta_p))^k \in H^{2k}(G_p, \mathbb{Z})^{G/G_p}$.
Hence $x \in H^{2k}(G_p, \mathbb{Z})^{G/G_p} \Leftrightarrow \lambda(c_1(\theta_p))^k = \lambda r^k(c_1(\theta_p))^k \Leftrightarrow \lambda(r^k - 1) \equiv 0(p^\alpha)$.
We distinguish three cases:
First case : $(p, r^k - 1) = 1$ which means that $\lambda = 0(p^\alpha)$.
Second case : p divides $r^k - 1$.
If $r^k - 1 = p^{\alpha_1}p_1$ with $(p_1, p) = 1$ and $0 < \alpha_1 < \alpha$ then we have $\lambda p^{\alpha_1}p_1 \equiv 0(p^\alpha)$, therefore $\lambda p_1 \equiv p^{\alpha-\alpha_1})$, which implies $\lambda \equiv 0(p)$.
Consider $x = (1+r^k+...+r^{k(d_p-1)})(c_1(\theta_p))^k$. Let us show that $x \in H^{2k}(G_p, \mathbb{Z})^{G/G_p}$.
$\overline{B}x = (1+r^k+...+r^{k(d_p-1)})r^k(c_1(\theta_p))^k = x$ because $r^{d_p} \equiv 1(p^\alpha)$ with d_p the order of r in $(\frac{\mathbb{Z}}{p^\alpha\mathbb{Z}})^*$. So $1+r^k+...+r^{k(d_p-1)} \equiv 0(p)$. On the other hand $r^k \equiv 1(p)$ which implies that $d_p \equiv 1 + r^k + ... + r^{k(d_p-1)}(p)$. Hence $d_p \equiv 0(p)$ which is impossible because d_p/n and $(n, m) = 1$.
Third case :If $r^k - 1 = p^{\alpha_1}p_1$ with $(p_1, p) = 1$ and $\alpha_1 \geq \alpha$.
Therefore $r^k \equiv 1(p^\alpha)$ hence $k = qd_p$.
For any $\lambda \in \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}$ we have $\lambda(r^k - 1) \equiv 0(p^\alpha)$
then $H^{2k}(G_p, \mathbb{Z})^{G/G_p} = \frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}(c_1(\theta_p))^q$.

Definition 4.7. [7]Let G be a finite group of order N and let k be an integer ≥ 1. We say that G has a periodic cohomology with period k if there exists an element $u \in H^k(G, \mathbb{Z})$ of order equal to N.

Proposition 4.8. *The cohomological period of G is $2.lcm_{p/m}(d_p)$, where $lcm_{p/m}(d_p)$ is the smallest common multiple of d_p for p/m.*

Proof. Let $|H^{2d}(G, \mathbb{Z})|$ be the order of $H^{2d}(G, \mathbb{Z})$ where $d = lcm_{p/m}(d_p)$.
$|H^{2d}(G, \mathbb{Z})| = |H^{2d}(\langle B\rangle, \mathbb{Z})| \times \prod_{p/m} |H^{2d}(G_p, \mathbb{Z})^{G/G_p}|$ since it is a direct product of groups. Since $|H^{2d}(\langle B\rangle, \mathbb{Z})| = |\frac{\mathbb{Z}}{n\mathbb{Z}}| = n$ due to the fact that 2 is the period of $\langle B\rangle$, and $|H^{2d}(G_p, \mathbb{Z})^{G/G_p}| = |\frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}| = p^\alpha$ because d is a multiple of d_p. Thus $|H^{2d}(G, \mathbb{Z})| = m \times n = |G|$. So $2d$ is a cohomological period of G. Furthemore this is the smallest period $|H^{2k}(G, \mathbb{Z})| = |G| \Leftrightarrow |H^{2k}(G_p, \mathbb{Z})^{G/G_p}| = p^\alpha \Leftrightarrow$ k is a multiple of d_p for p/m, which shows that $2d$ is the smallest period. □

By induction methods we get:

Proposition 4.9. *Let p be a prime divisor of m. There exists ζ_p a representation of G of degree d_p such that :*

$$\zeta_p(A) = \begin{bmatrix} e^{\frac{2i\pi}{m}} & 0 & \dots & 0 \\ 0 & e^{\frac{2i\pi r}{m}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{\frac{2i\pi r^{d_p-1}}{m}} \end{bmatrix} \tag{4.10}$$

$$\zeta_p(B) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ e^{\frac{2i\pi d_p}{n}} & 0 & \dots & \dots & 0 \end{bmatrix} \tag{4.11}$$

where $Res^G_{G_p}(c_{d_p}(\zeta_p))$ is a generator of $H^{2d_p}(G_p, \mathbb{Z})^{G/G_p}$.

To establish the lattest proposition, we need the following lemmas:

Lemma 4.10. *Let C_p be the centralizer of $G_p = \langle A_p \rangle$ in G, and $\hat{\theta}_p$ the extension of θ_p to C_p.*
If $\zeta_p = Ind^G_{C_p}(\hat{\theta}_p)$ is the induced representation from $\hat{\theta}_p$ then ζ_p has degree d_p where

$$\zeta_p(A) = \begin{bmatrix} e^{\frac{2i\pi}{m}} & 0 & \dots & 0 \\ 0 & e^{\frac{2i\pi r}{m}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{\frac{2i\pi r^{d_p-1}}{m}} \end{bmatrix}$$

and

$$\zeta_p(B) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ e^{\frac{2i\pi d_p}{n}} & 0 & \dots & \dots & 0 \end{bmatrix}$$

Proof. The representation θ_p has an extension $\hat{\theta}_p$ to C_p defined by $\hat{\theta}_p(B^{d_p}) = e^{\frac{2i\pi d_p}{n}}$ and $\hat{\theta}_p(A) = e^{\frac{2i\pi}{m}}$.

$G/C_p = \{B^{1-i}\}_{1\leq i\leq d_p} = \left\{1, B^{-1}, B^{-2}, ..., B^{-d_p+1}\right\}$. Therefore by setting $\zeta_p = ind_{G_p}^{G}(\hat{\theta}_p)$ then, ζ_p is of degree d_p and $\zeta_p(x) = (\hat{\theta}_p(B^{i-1}xB^{1-j}))_{\substack{1\leq i\leq d_p\\ 1\leq j\leq d_p}}$
Using the equality of matrices in the basis $\underbrace{\mathbb{C}\oplus\mathbb{C}\oplus ... \oplus\mathbb{C}}_{d_p\, fois}$, with the convention $\hat{\theta}_p(y) = 0$ if $y \notin C_p$.

$$\zeta_p(A) = (\hat{\theta}_p(B^{i-1}AB^{1-j}))_{\substack{1\leq i\leq d_p\\ 1\leq j\leq d_p}} = (\hat{\theta}_p(A^{r^{i-1}}B^{i-j}))_{\substack{1\leq i\leq d_p\\ 1\leq j\leq d_p}} \tag{4.12}$$

Hence

$$\zeta_p(A) = \begin{bmatrix} e^{\frac{2i\pi}{m}} & 0 & \cdots & 0 \\ 0 & e^{\frac{2i\pi r}{m}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\frac{2i\pi r^{d_p-1}}{m}} \end{bmatrix}$$

$$\zeta_p(B) = (\hat{\theta}_p(B^{i-1}BB^{1-j}))_{\substack{1\leq i\leq d_p\\ 1\leq j\leq d_p}} = (\hat{\theta}_p(B^{1+i-j}))_{\substack{1\leq i\leq d_p\\ 1\leq j\leq d_p}} \tag{4.13}$$

Hence

$$\zeta_p(B) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ e^{\frac{2i\pi d_p}{n}} & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Because $B^{1+i-j} \in C_p \Leftrightarrow 1 + i - j \equiv 0(d_p)$.
As $2 - d_p \leq 1 + i - j \leq d_p$, two cases arrise : $1 + i - j = 0$ or $1 + i - j = d_p$
If $1 + i - j = 0$, so $j = i + 1$ and $\hat{\theta}_p(1) = 1$
If $1 + i - j = d_p$, from $2 \leq 1 + i \leq 1 + d_p$ and $1 + d_p \leq j + d_p \leq 2d_p$ we have $i + 1 = j + d_p = 1 + d_p$ hence, $i = d_p$ and $j = 1$. So $\hat{\theta}_p(B^{d_p}) = e^{\frac{2i\pi d_p}{n}}$. □

Lemma 4.11. *Let $\eta_p = Res_{G_p}^{G}(\zeta_p)$ be the restriction of ζ_p to G_p. So*

$$\begin{cases} \eta_p = \theta_p \oplus \theta_p^r \oplus ... \oplus \theta_p^{r^{d_p-1}} \\ c_{d_p}(\eta_p) = 1 \times r \times r^2 \times ... \times r^{d_p-1}(c_1(\theta_p))^{d_p} \end{cases} \tag{4.14}$$

Proof. We deduce that $\eta_p = \theta_p \oplus \theta_p^r \oplus ... \oplus \theta_p^{r^{d_p-1}}$ from the equality

$$\zeta_p(A_p) = \begin{bmatrix} e^{\frac{2i\pi}{m}} & 0 & \dots & 0 \\ 0 & e^{\frac{2i\pi r}{p^\alpha}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{\frac{2i\pi r^{d_p-1}}{p^\alpha}} \end{bmatrix}$$

Hence $c(\eta_p) = c(\theta_p)c(\theta_p^r)...c(\theta_p^{r^{d_p-1}})$. In dimension $2d_p$ we have :
$c_{d_p}(\eta_p) = c_1(\theta_p)c_1(\theta_p^r)...c_1(\theta_p^{r^{d_p-1}}) = 1 \times r \times ...r^{d_p-1}(c_1(\theta_p))^{d_p}$. □

Proof of Proposition 4.9. As $1 \times r \times ...r^{d_p-1}$ is coprime with p, $c_{d_p}(\eta_p)$ generates $H^{2d_p}(G_p, \mathbb{Z})^{G/G_p}$.

Proposition 4.12. *There exists $\zeta \in R(G)$ defined by $\zeta(A)$ a unitary matrix and $\zeta(B) = e^{\frac{2i\pi}{n}}$ such that $Res^G_{\langle B\rangle}(c_1(\zeta))$ generates $H^2(\langle B\rangle, \mathbb{Z})$.*

Proof. Consider $\zeta\colon G \xrightarrow{p} \langle B\rangle \xrightarrow{\theta} \mathbb{C}^*$ such that $G \xrightarrow{p} \langle B\rangle$ is the canonical projection and $\theta\colon \langle B\rangle \to \mathbb{C}^*$ denotes the representation defined by $\theta(B) = e^{2i\pi/n}$.

Thus ζ is a representation of G with $\zeta(A)$ is a unitary matrix and $\zeta(B) = e^{\frac{2i\pi}{n}}$.
Consequently, $Res^G_{\langle B\rangle}(c_1(\zeta)) = Res^G_{\langle B\rangle}(c_1(\theta \circ p)) = Res^G_{\langle B\rangle}(p^* c_1(\theta)) = c_1(\theta)$ which is a generator of $H^2(\langle B\rangle, \mathbb{Z})$. □

5. Proof of main result

Theorem 5.1. *There exists a graded ring isomorphism*

$$H^*(G, \mathbb{Z}) = \mathbb{Z} \oplus \frac{\mathbb{Z}}{n\mathbb{Z}}[c_1(\zeta)]^+ \times \prod_{p/m} \frac{\mathbb{Z}}{p^\alpha \mathbb{Z}}[c_{d_p}(\zeta_p)]^+$$

such that $c_1(\zeta)$ is the first Chern class of the representation ζ of G where $\zeta(A)$ is a unitary matrix, $\zeta(B) = e^{\frac{2i\pi}{n}}$ and $c_{d_p}(\zeta_p)$ are the d_p Chern classes of representations ζ_p of G given by:

$$\zeta_p(A) = \begin{bmatrix} e^{\frac{2i\pi}{m}} & 0 & \dots & 0 \\ 0 & e^{\frac{2i\pi r}{m}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{\frac{2i\pi r^{d_p-1}}{m}} \end{bmatrix}$$

$$\zeta_p(B) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ e^{\frac{2i\pi d_p}{n}} & 0 & \dots & \dots & 0 \end{bmatrix}$$

where d_p is the order of r in $(\frac{\mathbb{Z}}{p^\alpha\mathbb{Z}})^*$ and p^α stands for the highest power of p dividing m.

Proof. Step 1: We have the graded rings isomorphism

$$H^*(G, \mathbb{Z}) = \mathbb{Z} \oplus H^*(G, \mathbb{Z})^+$$

Step 2: According to Proposition 3.9

$$H^*(G, \mathbb{Z})^+ \cong H^*(\frac{\mathbb{Z}}{n\mathbb{Z}}, \mathbb{Z})^+ \times \prod_{p/m} H^*(G_p, \mathbb{Z})^{+^{G/G_p}}$$

Step 3: It follows from Proposition 4.12

$$\frac{\mathbb{Z}}{n\mathbb{Z}}[c_1(\zeta)]^+ \cong H^*(\langle B\rangle, \mathbb{Z})^+$$

Step 4: Proposition 4.9 yields to

$$\frac{\mathbb{Z}}{p^\alpha\mathbb{Z}}[c_p(\zeta_p)]^+ \cong H^*(G_p, \mathbb{Z})^{+^{G/G_p}}$$

Hence the Theorem 2.1. □

Remark 5.2. Let BG be the classifying space of G. The previous computations of the cohomology ring $H^*(G;\mathbb{Z})$ allow the authors to describe some invariants of BG such as its topological and connective K-theory rings (see [4]). In [1] Azi and Hamraoui computed the k- representation ring of the special group $SL_d(k)$, where k a commutative field and they deduced in [2] a computation of the topological K-theory ring of its classifying space.

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