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ON SOME INDEPENDENT EQUITABLE DOMINATION OF GRAPHS

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Abstract. One of the most interesting facts about the study of connectivity and working is in the area of domination. By this, we determine the independent equitable domination number of some graphs and some results. Moreover, we investigate graphs with the existence of independent equitable dominating set.

1. Introduction

Let G be a connected simple graph. Suppose $v \in V$, the *neighborhood* of v is the set $N_G(v) = \{u \in V(G) : \ uv \in E(G)\}$. Given $D \subseteq V$, the set $N_G(D) = N(D) = \bigcup_{v \in D} N_G(v)$ and the set $N_G[D] = N[D] = D \cup N(D)$ are the *open neighborhood* and the *closed neighborhood* of D, respectively. In this paper, we denote $\Delta(G)$ and $\delta(G)$ to be the minimum and maximum degree of G, respectively. That is,

$$\Delta(G) = \max_{v \in V} \deg(v)$$

and

$$\delta(G) = \min_{v \in V} \deg(v),$$

respectively.

The concept of domination has been so popular in Graph Theory and its applications anchored to some applications on computer engineering, finance and other allied sciences in mathematics. In fact, there are a number of papers who have started paving ways on introducing types and studies of domination. Nayaka, Alwardi and Puttaswamy calculated the transversal domination number for some familises of graphs and obtained some bounds and relations with other domination parameter. Chavalurajaru and Chaitra [2] computed the sign domination number of an airthmetic graph and its efficient number domination. To be specific more on what could domination influence to graphs, Purnima, et. al. [8] studied the 2-point set domination of a cactus and provided alternate proof for solving 2-point set domination of a unicyclic graphs. Aside from imparting different

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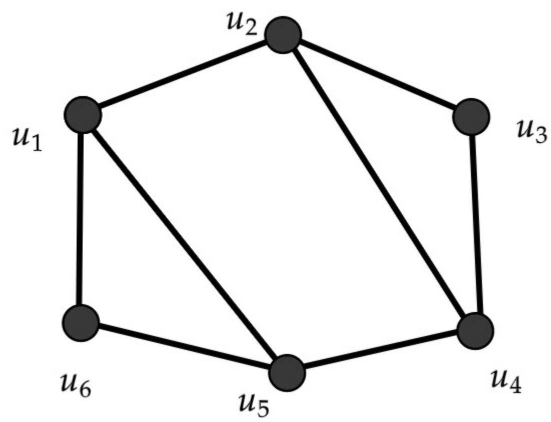
application, it has gained a lot of developed ideas from different areas in mathematics. Visweswaran and Vadhel [9] considered the dominating sets complement of the annihilating ideal graph and studied the influence of the dominating sets of such annihilating ideals on the ring structure and vice-versa. We say that D is the *dominating set* of G if for every $v \in V(G) \backslash D$, there exists $u \in D$ such that $uv \in E(G)$, that is, u is said to dominate v. Thus, $N[D] = V$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. On a similar, the concept of domination is not just limited to the work on the vertices a graph. In fact, there are some edge-domination which is analogous to the domination working on the graph as elements. Alqesmah, Alwardi and Rangarajan [1] introduced the concept of edge injective domination, injective independence edge number and edge injective domatic number of a given graph and established exact values for some standard graphs, bounds and some interesting results.

One of the concepts of independent set was used by DeMaio and Jacobson [3], in which they used the independent set to show the fibonacci and lucas sequences relationship in a Tadpole Graph. A set $I \subseteq V(G)$ is an *independent set* of a vertices if no two vertices in I are adjacent. The concept of the independent domination was introduced by Allan and Laskar in [4]. A subset $D_i(G) \subseteq V(G)$ is an independent dominating set of G if $D_i(G)$ is both an independent and dominating set. The minimum cardinality of the independent dominating set is called *independent domination number* of G, and is denoted by $\gamma_i(G)$.

The concept of Equitable Domination was introduced by Deepak [5] and stated the idea of equitable bondage number of a graph and provided sharp bounds and exact values for some standard graphs. Similarly, Caay and Arugay in [6], investigated the perfect equitable dominations in graphs and provided values for some graphs. A subset $D_e(G)$ of $V(G)$ is called an *equitable dominating set* if for every $v \in V(G) \backslash D_e(G)$ there exists a vertex $u \in D_e(G)$ such that $uv \in E(G)$ and $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of such dominating set is called *equitable domination number* and is denoted by $\gamma_e(G)$.

In this paper, we introduced the concept of independent equitable domination in a graph G. The *independent equitable dominating set* in G is the set $D_{ie}(G) \subseteq V(G)$ which is both independent set and equitable dominating set of G. The minimum cardinality of the independent equitable dominating set is called *independent equitable domination* number of G and is denoted by $\gamma_{ie}(G)$. Moreover, in this study, if the graph has an independent equitable dominating set, then the graph is said to be γ_{ie}-graph, otherwise, γ_{ie}-graph. Such set is γ_{ie}-set, otherwise, non-γ_{ie}-set.

Consider the graph below:



Note that u_1 can dominate u_2, u_5 and u_6, and u_4 can dominate u_2, u_3 and u_5. Thus the set $\{u_1, u_4\}$ is a dominating set. Note that $\{u_2, u_5\}, \{u_1, u_2\}$ are also dominating set. If we list all the possible dominating set, the minimal set has the cardinality of 2. Hence, without loss of generality, we take $\{u_1, u_4\}$. Since $u_1u_4 \notin E(G)$, it follows that $\{u_1, u_4\}$ is an independent dominating set. Observe also that

$$\begin{aligned}
|\deg(u_1) - \deg(u_2)| &= 0 \leq 1\\
|\deg(u_1) - \deg(u_5)| &= 0 \leq 1\\
|\deg(u_1) - \deg(u_6)| &= 1 \leq 1\\
|\deg(u_4) - \deg(u_2)| &= 0 \leq 1\\
|\deg(u_4) - \deg(u_3)| &= 1 \leq 1\\
|\deg(u_4) - \deg(u_5)| &= 0 \leq 1
\end{aligned}$$

Thus, $\{u_1, u_4\}$ is also equitable dominating set. Thus, the set is independent equitable dominating set. Taking the cardinality, we have $\gamma_{ie}(G) = 2$.

Following the definition of the independent equitable domination (IED) in graphs, we have immediate observations:

Remark 1.1. This is in reference of the definition of the independent equitable domination of graphs.

(i) All independent equitable dominating sets are equitable dominating sets.
(ii) If $a, b \in D_{ie}(G)$, then $ab \notin E(G)$.
(iii) A singleton set, say $\{a\}$ is an independent set.

2. IED of Some Graphs

The following are the results of the independent equitable domination of graphs.

Proposition 2.1. *Given a complete graph K_n, $\gamma_{ie}(K_n) = 1$.*

Proof. Let $a \in V(Kn)$. Then for all $b_i \in V(Kn), i = \{1, 2, \cdots, n-1\}, ab_i \in E(Kn)$. Thus, a dominates b_i $i = \{1, 2, \cdots, n-1\}$. Since K_n is a complete graph, all vertices have the same degree. Hence, $D_{ie}(K_n)\{a\}$ and so $\gamma_{ie}(K_n) = 1$. □

Theorem 2.2. *Given any graph G with $\Delta(G) = |V(G)| - 1$, then $\gamma_{ie}(G) = 1$ if $\Delta(G) - \delta(G) = 1$.*

Proof. We will consider the following cases:

- Case 1: If $n \equiv 0 \mod 3$.
  We divide the vertices in a group of 3. Without loss of generality, suppose the first group of three vertices are labeled in consecutive manner as v_1, v_2, v_3. If v_1 dominates v_2, v_3 is not dominated. If v_3 dominates v_2, v_1 is not dominated. If v_2 dominates v_1, it also dominates v_3. Simlarly, for the second group of vertices v_4, v_5, v_6, we have v_5 that dominates v_4 and v_6. Continuing the process, we obtain vertices place in the middle of every group that can dominate the rest. Since 3 divides n and the number of vertices are divided by 3. We have $\left\lceil \dfrac{n}{3} \right\rceil$.
- Case 2: If $n \equiv 1 \mod 3$.
  Dividing the vertices in a group of 3 vertices per group, we have one group of one vertex. Without a loss of generality, we label such vertex as v_1 and the vertices next to it are v_2, v_3, v_4. Further suppose v_1 dominates v_2. Then we may assume v_4 dominates v_3. Suppose that the next group of vertices are labeled v_5, v_6, v_7. Then v_4 from the previous group dominate v_5 and so we may assume v_7 dominates v_6. Continuing this process, we can find out that the dominating vertices are the third vertex in each group. There are $\dfrac{n-1}{3}$ groups of three vertices. Since v_1 is also a dominating vertex, it follows that the dominating vertices are $\dfrac{n-1}{3} + 1$ or in this case is $\left\lceil \dfrac{n}{3} \right\rceil$.
- Case 3: If $n \equiv 2 \mod 3$.
  Dividing the vertices in a group of 3 vertices per group, we have one group of one vertex. Without a loss of generality, we label such verteices as v_1 and v_2, and the vertices next to it are v_3, v_4, v_5. Further suppose v_2 dominates v_1. Then following the same processwith case 2, we arrive at the same answer as $\dfrac{n-1}{3} + 1$ or in this case is $\left\lceil \dfrac{n}{3} \right\rceil$.

□

Proposition 2.3. *Given path Pn and cycle Cn, $n \geq 3$, $\gamma_{ie}(P_n) = \gamma_{ie}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.*

Proof. Note that a vertex can dominate another vertex when they are adjacent to each other. Thus, the maximum number of vertices dominated by one vertex is 2 of P_n and C_n and the minimum $N_P(v_i) = 1$ and $N_C(v_j) = 2$ where $v_i \in V(P_n)$ and $v_j \in V(C_n)$, $i, j = 1, 2, \cdots, n$. So to minimize the number of vertices selected for dominating set is to partition the vertices of P_n and C_n into 3 vertices that includes one of the dominating set. Thus for $|V(P_n)| = |V(C_n)| = n$, the partition can be $\frac{n}{3}$ if 3 divides n and $\frac{n-1}{3} + a$ if 3 does not divide n where $a \leq 2$. Note that for a remaining vertices, there exists a vertex which is also a dominating set which means for a case that 3 does not divide n, there is one additional vertex which is in $D_{ie}(G)$ added to $n-1$ before it is divided by 3. Thus in any case, the partition is $\frac{n}{3}$. Similarly, suppose $A = a, b, c$ be one of the partitions of $V(P_n)$ or $V(C_n)$. Without loss of generality, suppose $a \in D_{ie}(P_n \text{ or } C_n)$ and $b, c \in V(P_n \text{ or } C_n) \backslash D_{ie}(P_n \text{ or } C_n)$. Then $|\deg(a) - \deg(b)| \leq 1$ and $|\deg(a) - \deg(c)| \leq 1$. This proves the fact that $\gamma_{ie}(P_n) = \gamma_{ie}(C_n) = \dfrac{n}{3}$. □

Proposition 2.4. *The following graphs are non-γ_{ie}-graph:*

- *A star graph, S_n, $n \geq 4$*
- *A wheel graph, W_n, $n \geq 6$*
- *A fan graph, F_n, $n \geq 5$*

The following is another result showing a graph which is non-γ_{ie}-graph:

Theorem 2.5. *Given a tree T with the largest induced subgraph as path Pn, T is non-γ_{ie}-graph if there are more than one pendant vertices incident to any vertex in the path.*

The proof is easy, we only see that if we select $a \in V(P_n)$, then its degree is at least 2 (for the adjacent vertices in path) plus 2 (for at least number of pendant vertices incident to a) but one of the pendant has only one adjacent vertex. This shows that T does not have equitable dominating set, and so T is non-γ_{ie}-graph.

3. IED of the Corona and Join of Graphs

The following are the results of the independent equitable domination of graphs formed by corona and join operations of graphs.

Definition 3.1. The **join** $G+H$ of two graphs G and H is the graph with vertex set

$$V(G+H) = V(G) \cup V(H)$$

and the edge set

$$E(G+H) = E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}.$$

Definition 3.2. The **corona** $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H

Theorem 3.3. *Any non-complete graph G, $G_k + H_k$ is a non-γ_{ie}-graph if H_k is a complete graph with $k \geq 5$.*

Proof. Suppose $G_k + H_k$ is a γ_{ie}-graph. Without loss of generality, consider $G_k = P_k$. Then let $a \in V(H_k)$, then a dominates all other $2k-1$ vertices in $G_k + H_k$. So, $a \in D_{ie}$. Observe that a dominates $b_i \in V(P_k), i = 1, 2, \cdots, k$ such that b_i is adjacent to at most 2 other vertices in P_k and k vertices in H_k which implies that $\deg(b_i) = k+2$. So,

$$\begin{aligned} |\deg(a) - \deg(b_i)| &= (2k-1) - (k+2) \\ &= k-3 \\ &\geq 2, \quad \text{since } k \geq 5. \end{aligned}$$

This is a contradiction. Therefore, $G_k + H_k$ is non-γ_{ie}-graph. □

Proposition 3.4. *$C_n + K_p$ is non-γ_{ie}-graph if $|n-p| > 1$.*

Proposition 3.5. *$C_n + K_p$ is γ_{ie}-graph if $p - n \leq 1$ and $n \leq 5$, that is, $\gamma_{ie}(C_n + (K_p) = 1$.*

Proof. Let $a \in V(K_p)$. Then a is adjacent to $b_i \in V(K_p), i = 1, 2, \cdots, p-1$ and is also adjacent to $c_j \in V(C_n), j = 1, 2, \cdots, n$. Thus, $a \in D_{ie}(C_n + K_p)$. Since $\{a\}$ is a singleton set, by Remark 1.1, $D_{ie}(C_n + K_p)$ is an independent dominating set.

Observe that a is adjacent to $p-1$ vertices of K_p and to n vertices of C_n. So, $\deg(a) = n + p - 1$. Moreover, since $b_i \in V(K_p)$, $i = 1, 2, \cdots, p-1$, it follows that the $\deg(b_i) = n + p - 1$. Thus, $|\deg(a) - \deg(b_i)| = 0 \leq 1$. Similarly, $c_j \in V(C_n)$, $\quad j = 1, 2, \cdots, n$. This means that $\deg(c_j) = p + 2$. Thus, $|\deg(a) - \deg(c_j)| = (p - 1 + n) - (p - 2) = n - 3 \leq 1$ since $n \leq 4$. Therefore, $C_n + K_p$ is γ_{ie}-graph and $\gamma_{ie}(C_n + (K_p) = 1$ if $p - n \leq 1$. □ □

Proposition 3.6. *$Pn \circ C_m$ is non-γ_{ie}-graph if $m \geq 5$.*

Proof. Let $a \in V(P_n)$ and $b \in V(C_m)$. Note that a dominates all m vertices of C_m. Also for $b_i \in V(C_m), i = 1, 2, \cdots, m$, we have $\deg(b_i) = 3$. So if $m \geq 5$, then $|\deg(a) - \deg(b_i)| \geq 2$. Thus, it violates the definition of the equitable domination that $|\deg(a) - \deg(b_i)| \leq 1$. Therefore, $Pn \circ C_m$ is non-γ_{ie}-graph. □

Proposition 3.7. *$P_n \circ C_m$ is γ_{ie}-graph if $m \leq 4$ and $n \leq 4$. Moreover,*

$$\gamma_{ie}(Pn \circ C_m) = \left\lceil \frac{m}{3} \right\rceil \left(n - \left\lceil \frac{n}{2} \right\rceil \right) + \left\lceil \frac{n}{2} \right\rceil.$$

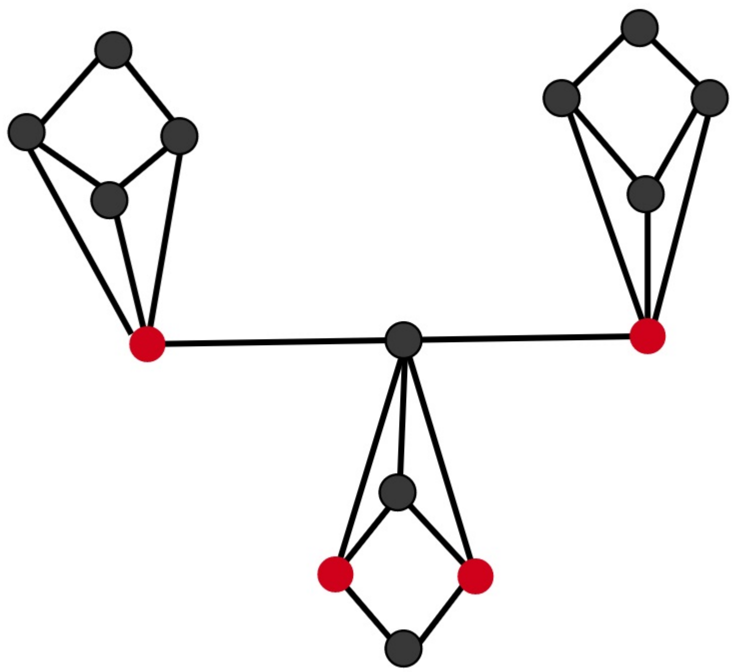
Proof. Without loss of generality, we first select vertices in path. Then have $\left\lceil \frac{n}{2} \right\rceil$ number of vertices we select that dominate the other vertices of P_n and some vertices of C_m adjacent to these vertices we have selected. Let $a \in V(P_n)$ be the first vertex of P_n arranged in numerical order (could be starting from left to right or right to left). We select the first vertex of Pn so that its degree is at most 5 (with atmost 4 vertices from C_m and 1 adjacent vertex on P_n) so that from $m_i \in V(C_m) i = 1, 2, \cdots, m$, $|\deg(a) - \deg(m_i)| \leq 1$ and the other selected vertex from P_n is the last vertex. Note that there are $n - \left\lceil \frac{n}{1} \right\rceil$ C'_ms whose vertices are not dominated. If we choose the vertices in the path to dominate the vertices of $n - \left\lceil \frac{n}{1} \right\rceil$ C'_ms, we cannot formulate the independent set. Thus, we select vertices from $n - \left\lceil \frac{n}{1} \right\rceil$ C'_ms and by Proposition 2.3, we can select $\left\lceil \frac{m}{3} \right\rceil$ from $n - \left\lceil \frac{n}{1} \right\rceil$ C'_ms. Thus, adding the results we have $\left\lceil \frac{m}{3} \right\rceil \left(n - \left\lceil \frac{n}{2} \right\rceil \right) + \left\lceil \frac{n}{2} \right\rceil$. □

Consider the example below

Example 3.8. The $\gamma_{ie}(P_3 \circ C_4) = 4$.

Solution Using Proposition 3.7, we have

$$\begin{aligned}\gamma_{ie}(P_3 \circ C_4) &= \left\lceil \frac{m}{3} \right\rceil \left(n - \left\lceil \frac{n}{2} \right\rceil \right) + \left\lceil \frac{n}{2} \right\rceil \\ &= \left\lceil \frac{4}{3} \right\rceil \left(3 - \left\lceil \frac{3}{2} \right\rceil \right) + \left\lceil \frac{3}{2} \right\rceil \\ &= 2(3-2)+2 \\ &= 4.\end{aligned}$$



Proposition 3.9. *$P_n \circ K_m$ is non-γ_{ie}-graph if $n \geq 3$.*

Proof. Let $a \in V(P_n)$. Then a dominates all m vertices in K_m and a also dominates the adjacent vertices of Pn at most 2. Thus, $\deg(a) \leq m+2$. If $b \in V(K_m)$ then b dominates all other $m-1$ vertices of K_m and one of the vertices of P_n. Thus, $\deg(b) = (m-1)+1 = m$. So, $|\deg(a) - \deg(b)| = 2 > 1$. Therefore, $P_n \circ K_m$ is non-γ_{ie}-graph if $n \geq 3$. □

Corollary 3.10. *$P_n \circ K_m$ is γ_{ie}-graph if $n \leq 2$. Moreover, $\gamma_{ie}(P_n \circ K_m) = n$.*

4. Conclusion

In this paper, we have evaluated the parameters of independent equitable dominations to some graphs and some graphs resulting to binary operations such as join and corona. Further, we examine some graphs and operations with the existence of independent equitable domination because not all graphs have this set. The study is beyond helpful in the application of real-life scenario. This is in fact used as an application in making decision to relief distribution and aid from the government to the people especially this time of pandemic.

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