

EIGENVALUE ESTIMATES FOR MAGNETIC SCHRÖDINGER OPERATORS IN A WAVEGUIDE

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ABSTRACT. We present an upper estimate for the number of negative eigenvalues below the essential spectrum for the magnetic Schrödinger operator with Aharonov-Bohm magnetic field in a strip.

1. INTRODUCTION

The study of quantum waveguides has attracted much interest in the recent years largely because of their physical importance. They represent many important applications in nanophysical devices as well as flat electromagnetic waveguides (see, e.g., [11] and the references therein). Several interesting results on the spectral properties of these quantities have been obtained, including results on the existence of eigenvalues below the essential spectrum [9, 16, 19]. Some of these results largely depend on the geometry of the waveguide and the conditions imposed at the boundary. Many authors have obtained results on both qualitative and quantitative spectral properties (see, e.g., [9, 12, 13]). The work in this paper is dedicated to the quantitative properties of the discrete spectrum of a Schrödinger operator representing a charged quantum particle whose motion is confined in a straight infinite strip interacting with Aharonov-Bohm magnetic field.

According to the Celebrated Cwikel-Lieb-Rozenblum (CLR) inequality [8, 18, 20], the number of negative eigenvalues of the Schrödinger operator $H_V := -\Delta - V$, $V \geq 0$ on $L^2(\mathbb{R}^n)$, $n \geq 3$ is estimated above by $c \int_{\mathbb{R}^n} V^{n/2} dx$, where c is a constant that depends on the dimension n . For $n = 1$, the analogue of the CLR inequality exists for potentials that are monotone on $\mathbb{R}_+$ or $\mathbb{R}_-$ (see, e.g., [5]), otherwise, the operator H_V has at least one negative eigenvalue, (see, e.g., [15] and the references therein). In the critical case $n = 2$, the CLR inequality fails, because if $\int_{\mathbb{R}^2} V dx > 0$, then the operator $-\Delta - \alpha V$ on $L^2(\mathbb{R}^2)$ has at least one negative eigenvalue for any $\alpha > 0$ (see, e.g., [3, 6, 22]). One other less obvious reason is the failure of the Hardy inequality in dimension two which is one of the important tools in the proof of the CLR inequality. There are several estimates in dimension two of the CLR type and best estimates known so far involve the weighted L^1

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norms and Orlicz norms of the potential (see, e.g. [14, 22]). However, these estimates are not optimal in some sense. When an additional magnetic field is introduced, the situation improves due to the diamagnetic effects, indeed, it is shown in [17] that the magnetic Schrödinger operators do not have weakly coupled eigenvalues. The question of whether it is possible to establish an analogue of the CLR inequality for two-dimensional magnetic Schrödinger operators has been answered in [2] in the case of the Aharonov-Bohm magnetic field and in [16] for a reasonably large class of magnetic fields. In the present paper, we obtain an analogue of the CLR inequality for an Aharonov-Bohm magnetic Schrödinger operator in a two-dimensional strip when the Neumann data is given at the opposite sides of the boundary. Below, is the precise description of the operator studied therein.

Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R}, 0 < x_2 < d\}, d > 0$. Consider the Schrödinger operator H_B in $L^2(\Omega)$ initially defined on the domain $C^\infty(\overline{\Omega})$ given by

$$H_B := \left(\frac{1}{i} \nabla + A \right)^2 \quad (1.1)$$

with homogeneous Neumann boundary conditions maintained at $x_2 = 0$ and $x_2 = d$, where A is a magnetic vector potential for the Aharonov-Bohm magnetic field B , given by

$$A(x_1, x_2) = \Psi \cdot \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right) \quad (1.2)$$

and

$$\Psi := \frac{1}{d} \int_0^d A(x_1, x_2) dx$$

is the magnetic flux. The Magnetic field $B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by

$$B(x_1, x_2) = \nabla \times A = 0.$$

We will obtain an upper estimate for the number of negative eigenvalues of the operator

$$H_{B,V} = H_B - V, V \geq 0 \text{ on } L^2(\Omega), \quad (1.3)$$

where V is an additional electric potential.

2. PRELIMINARIES

Let $\mathcal{H}$ be a complex Hilbert space and q be a Hermitian form with domain $\mathcal{D}(q) \subset \mathcal{H}$. Set

$$Neg(q) := \sup\{\dim \ell \mid q[u] < 0, \forall u \in \ell \setminus \{0\}\}, \quad (2.1)$$

where ℓ is the linear subspace of $\mathcal{D}(q)$ (see, e.g., [4, Theorem 10.2.3]). If q is the quadratic form associated with a self adjoint operator T with no essential spectrum in interval $(-\infty, 0)$, then $Neg(q)$ gives the number of negative eigenvalues of T repeated accordingly to their multiplicities [7, Section 2.1].

Let $A \in L^2_{loc}(\mathbb{R}^2)$ and $V \in L^1_{loc}(\mathbb{R}^2)$. We will understand that the operator $H_B - V$ is self-adjoint whose Friedrichs extension is given by the quadratic form

$$\int_{\Omega} \left(\left| \left(\frac{1}{i} \nabla + A \right) u \right|^2 - V |u|^2 \right) dx, \quad u \in C^\infty(\bar{\Omega}),$$

provided the form is closable and bounded from below.

Let

$$W(x_1) = \|V(x_1, \cdot)\|_{L^\infty(0,d)} = \sup_{0 < x_2 \leq d} |V(x_1)| \quad (2.2)$$

so that

$$\|W\|_{L^1(\mathbb{R}, L^\infty(0,d))} = \|V\|_{L^1(\mathbb{R}, L^\infty(0,d))} < \infty. \quad (2.3)$$

The operator $H_{B,W} = (\frac{1}{i} \nabla + A)^2 - W$, is lower-semi bounded self adjoint operator with essential spectrum $[0, \infty)$ [1, Lemma 5.6.1]. Since $H_{B,V} \geq H_{B,W}$, then

$$Neg(H_{B,V}) \leq Neg(H_{B,W}), \quad (2.4)$$

where $Neg(\cdot)$ is the number of negative eigenvalues of the operator. Therefore to estimate the number of negative eigenvalues of the operator $H_{B,V}$, its enough to estimate the number of negative eigenvalues of the operator $H_{B,W}$.

Now,

$$\begin{aligned} H_{B,W} &= \left(\frac{1}{i} \nabla + A \right)^2 - W \\ &= -\Delta + \frac{2A}{i} \cdot \nabla + |A|^2 - W \\ &= -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \frac{2A}{i} \cdot \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) + |A|^2 - W \\ &= -\frac{\partial^2}{\partial x_1^2} + \frac{2A}{i} \frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2} + \frac{2A}{i} \frac{\partial}{\partial x_2} + |A|^2 - W. \end{aligned}$$

Thus

$$H_{B,W} = -\frac{\partial^2}{\partial x_1^2} + \frac{2A}{i} \frac{\partial}{\partial x_1} + \left(-\frac{\partial^2}{\partial x_2^2} + \frac{2A}{i} \frac{\partial}{\partial x_2} + |A|^2 \right) - W. \quad (2.5)$$

Since $-\frac{\partial^2}{\partial x_2^2} + \frac{2A}{i} \frac{\partial}{\partial x_2} + |A|^2 = \left(\frac{1}{i} \frac{\partial}{\partial x_2} + A \right)^2$, (2.5) can now be written as,

$$H_{B,W} = -\frac{\partial^2}{\partial x_1^2} + \frac{2A}{i} \frac{\partial}{\partial x_1} + \left(\frac{1}{i} \frac{\partial}{\partial x_2} + A \right)^2 - W. \quad (2.6)$$

Define the operator

$$\mathcal{Q} := \frac{1}{i} \frac{\partial}{\partial x_2} + A. \quad (2.7)$$

Let λ be an eigenvalue of $\mathcal{Q}$ and f be the corresponding eigenfunction. Then

$$\mathcal{Q}f = \frac{1}{i} \frac{\partial f}{\partial x_2} + Af = \lambda f, \quad (2.8)$$

so that

$$\frac{1}{i} \frac{df}{dx_2} = (\lambda - A)f.$$

On separating variables and integrating, we obtain

$$\int \frac{1}{f} df = \int i(\lambda - A) dx_2,$$

implying that

$$\ln f = i(\lambda - A)x_2 + c_1, \quad (2.9)$$

where c_1 , is a constant of integration,

$$f(x_2) = ce^{i(\lambda-A)x_2}, c = \ln c_1 \quad (2.10)$$

To determine c , λ and f , we use the following boundary conditions.

$$-f'(0) = f'(d) = 0. \quad (2.11)$$

Differentiating (2.10) with respect to x_2 gives

$$f'(x_2) = i(\lambda - A)ce^{i(\lambda-A)x_2}. \quad (2.12)$$

Now putting the boundary conditions (2.11) into (2.12), we obtain

$$i(A - \lambda)c = 0$$

and

$$i(\lambda - A)ce^{i(\lambda-A)d} = 0.$$

It follows that

$$\lambda = A \text{ or } e^{i(\lambda-A)d} = 0. \quad (2.13)$$

Since $e^{i(\lambda-A)d} \neq 0$, then $\lambda = A$. So from (2.10), we have $f(x_2) = c$. Normalizing $f(x_2) = c$ gives $c = \frac{1}{\sqrt{d}}$. Let $\varphi = \min_{k \in \mathbb{Z}} |\Psi - k|$ and assume without loss of generality that $A = \varphi$, we have

$$f_k(x_2) = \frac{1}{\sqrt{d}} e^{i(\lambda_k - \varphi)x_2}. \quad (2.14)$$

Using the decomposition

$$L^2(\Omega) = L^2(\mathbb{R}, dx_1) \otimes L^2((0, d), dx_2),$$

we have

$$L^2(\Omega) = \bigoplus_{k \in \mathbb{Z}} \left(L^2(\mathbb{R}, dx_1) \otimes \left[\frac{e^{i(\lambda_k - \varphi)x_2}}{\sqrt{d}} \right] \right), \quad (2.15)$$

where $[.]$ denotes the linear span, since $\left\{ e^{i(\lambda_k - \varphi)x_2} \right\}_{k \in \mathbb{Z}}$ is a complete orthonormal system of eigenfunctions in $L^2((0, d), dx_2)$. It follows from (2.6) that

$$H_{B,W} = -\frac{\partial^2}{\partial x_1^2} + \frac{2\varphi}{i} \frac{\partial}{\partial x_1} + (k + \varphi)^2 - W = \bigoplus_{k \in \mathbb{Z}} \left(H_k \otimes 1_k \right), \quad (2.16)$$

where

$$H_k = -\frac{\partial^2}{\partial x_1^2} + \frac{2\varphi}{i} \frac{\partial}{\partial x_1} + (k + \varphi)^2 - W. \quad (2.17)$$

and 1_k is the identity on (2.14).

For any $u \in L^2(0, d)$, u has the representation

$$u(x_1, x_2) = \sum_{k \in \mathbb{Z}} u_k(x_1) f_k(x_2),$$

where

$$u_k(x_1) = \int_0^d u(x_1, x_2) f_k(x_2) dx_2.$$

Let H_A^1 be the completion of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|u\| = \left(\left\| \left(\frac{1}{i} \nabla + A \right) u \right\|^2 + \|u\|^2 \right)^{1/2},$$

where $\|\cdot\|$ denotes the $L^2(\Omega)$ norm. Thus for $u \in H_A^1$, the quadratic form of (2.17) given by

$$\begin{aligned} q_{k,W}[u] := & \sum_{k \in \mathbb{Z}} \left(\langle -u_k''(x_1) f_k(x_2) + \frac{2\varphi}{i} u_k'(x_1) f_k(x_2) + (k + \varphi)^2 u_k(x_1) f_k(x_2) \right. \\ & \left. - W u_k(x_1) f_k(x_2), u_k(x_1) f_k(x_2) \rangle_{L^2(0,d)} \right). \end{aligned}$$

Thus

$$\begin{aligned} q_{k,W}[u] = & \sum_{k \in \mathbb{Z}} \left(- \langle u_k''(x_1) f_k(x_2), u_k(x_1) f_k(x_2) \rangle + \frac{2\varphi}{i} \langle u_k'(x_1) f_k(x_2), u_k(x_1) f_k(x_2) \rangle \right. \\ & \left. + (k + \varphi)^2 \langle u_k(x_1) f_k(x_2), u_k(x_1) f_k(x_2) \rangle - W \langle u_k(x_1) f_k(x_2), u_k(x_1) f_k(x_2) \rangle \right) \\ = & \sum_{k \in \mathbb{Z}} \left(- \int_{\Omega} u_k''(x_1) f_k(x_2) \overline{u_k(x_1) f_k(x_2)} dx_1 dx_2 + \frac{2\varphi}{i} \int_{\Omega} u_k'(x_1) f_k(x_2) \overline{u_k(x_1) f_k(x_2)} dx_1 dx_2 \right. \\ & \left. + (k + \varphi)^2 \int_{\Omega} u_k(x_1) f_k(x_2) \overline{u_k(x_1) f_k(x_2)} dx_1 dx_2 - W \int_{\Omega} u_k(x_1) f_k(x_2) \overline{u_k(x_1) f_k(x_2)} dx_1 dx_2 \right). \end{aligned}$$

It therefore follows that

$$q_{k,W}[u] = \sum_{k \in \mathbb{Z}} \left(\int_{\mathbb{R}} \left(|u_k'(x_1)|^2 + |k + \varphi|^2 |u_k(x_1)|^2 - W |u_k(x_1)|^2 \right) dx_1 \right). \quad (2.18)$$

Lemma 2.1. [21, Theorem 1.1] *Let $Q \in L_{loc}^1(\mathbb{R})$ and consider the following self-adjoint Sturm-Liouville operator*

$$\tau = -\frac{d^2}{dx^2} + \frac{c}{x^2} - Q(x) \text{ on } L^2(\mathbb{R}), \quad (2.19)$$

where c is a constant.

If the operator (2.19) has n negative eigenvalues (counting multiplicities), then

$$n < \frac{1}{\sqrt{4c+1}} \int_{\mathbb{R}} Q(x) dx. \quad (2.20)$$

3. THE MAIN RESULT

Theorem 3.1. *Let $X = L^1(\mathbb{R}, L^\infty(0, d))$, A be the vector potential in (1.2). Then the number of negative eigenvalues of the operator $H_{B,V}$ is estimated above by*

$$Neg(H_{B,V}) \leq \sum'_{k \in \mathbb{Z}} \frac{1}{\sqrt{16|k + \varphi|^2 + 1}} \|V\|_X, \quad (3.1)$$

where $\sum'$ indicates that all summands less than 1 are neglected.

Proof. The operator $H_{B,W}$ is defined via its quadratic form (2.18), that is,

$$q_{k,W}[u] = \sum_{k \in \mathbb{Z}} \left(\int_{\mathbb{R}} \left(|u'_k(x_1)|^2 + (k + \varphi)^2 |u_k(x_1)|^2 - W(x_1) |u_k(x_1)|^2 \right) dx_1 \right). \quad (3.2)$$

By the Hardy inequality [10, theorem 327], we have

$$\begin{aligned} q_{k,W}[u] &\geq \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \left(\frac{1}{4} |u'(x_1)|^2 + \frac{(k + \varphi)^2}{x_1^2} |u(x_1)|^2 - W(x_1) |u_k(x_1)|^2 \right) dx_1 \\ &= \frac{1}{4} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \left(|u'(x_1)|^2 + 4 \frac{(k + \varphi)^2}{x_1^2} |u(x_1)|^2 - 4W(x_1) |u(x_1)|^2 \right) dx_1. \end{aligned} \quad (3.3)$$

The right hand of (3.3) is a quadratic form associated with a family of **self-adjoint** Sturm-Liouville operators of the form

$$\tau(k) := \frac{1}{4} \left(-\frac{d^2}{dx_1^2} + \frac{4(k + \varphi)^2}{x_1^2} - 4W(x_1) \right) \text{ on } L^2(\mathbb{R}). \quad (3.4)$$

Now, it follows from Lemma 2.1 that

$$Neg(\tau(k)) \leq \frac{1}{\sqrt{16(k + \varphi)^2 + 1}} \int_{\mathbb{R}} W(x_1) dx_1 \quad (3.5)$$

Hence, by (2.1) and (3.3), we have

$$Neg(H_{B,W}) \leq \sum'_{k \in \mathbb{Z}} \frac{1}{\sqrt{16|k + \varphi|^2 + 1}} \int_{\mathbb{R}} W(x_1) dx_1. \quad (3.6)$$

Substituting for W (see (2.4)) in (3.6) yields

$$\begin{aligned} Neg(H_{B,V}) &\leq \sum'_{k \in \mathbb{Z}} \frac{1}{\sqrt{16(k + \varphi)^2 + 1}} \int_{\mathbb{R}} \operatorname{ess\,sup}_{0 < x_2 \leq d} V(x_1, x_2) dx_1 \\ &= \sum'_{k \in \mathbb{Z}} \frac{1}{\sqrt{16|k + \varphi|^2 + 1}} \|V\|_X. \end{aligned}$$

□

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