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CENTRALLY-EXTENDED HOMODERIVATIONS ON RINGS

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Abstract. Let R be a ring with center $Z(R)$. A mapping H from R into itself is called a centrally-extended homoderivation on R if for each $x, y \in R$, $H(x+y) - H(x) - H(y) \in Z(R)$ and $H(xy) - H(x)H(y) - H(x)y - xH(y) \in Z(R)$. We present examples of mappings that are centrally-extended homoderivations but not homoderivations. We also study the effect of such mappings on $Z(R)$ and prove some commutativity results.

1. Introduction

Let R be a ring with center $Z(R)$. A mapping $f : R \to R$ is called zero-power valued on R if for each $x \in R$, there exists a positive integer $n(x) > 1$ such that $f^{n(x)}(x) = 0$. A mapping $f : R \to R$ is centralizing on $S \subseteq R$ if $[x, f(x)] \in Z(R)$ for all $x \in S$. A mapping $f : R \to R$ satisfying $[x, y] = [f(x), f(y)]$ for all $x, y \in R$ is called strong commutativity-preserving on R. An additive mapping $d : R \to R$ is a derivation on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

An additive mapping $h : R \to R$ is called a homoderivation on R if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in R$. This notion was introduced by El Sofy [4] by combining the concepts of homomorphisms and derivations on rings. An obvious example of a homoderivation is the additive mapping $h : R \to R$ defined by $h(x) = -x$ for all $x \in R$.

Recently, Bell and Daif [2] defined the concept of a centrally-extended derivation (or CE-derivation) on R as a mapping $D : R \to R$ satisfying $D(x+y) - D(x) - D(y) \in Z(R)$ and $D(xy) - D(x)y - xD(y) \in Z(R)$ for all $x, y \in R$. In any ring R with a derivation d, if we let $D : R \to R$ be a mapping defined by $D(x) = d(x) + f(x)$ where f is a mapping from R into a nonzero central ideal of R, then D is a CE-derivation.

Inspired by the definitions of a homoderivation and a CE-derivation, we introduce a new kind of mapping on rings. Define a mapping $H : R \to R$ to be centrally-extended homoderivation (or CE-homoderivation) on R if for each $x, y \in R$, $H(x+y) - H(x) - H(y) \in Z(R)$ and $H(xy) - H(x)H(y) - H(x)y - xH(y) \in Z(R)$. In this paper, we investigate examples and properties of such

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mappings and prove some commutativity theorems analogous to some of those presented in [2].

2. Examples and Existence Theorems

Since $0 \in Z(R)$, every homoderivation on R is a CE-homoderivation on R. However, the converse is not always true.

Example 2.1. Let R be a ring with a nonzero central ideal I and let f be any mapping from R into I. Let $h : R \to R$ be a homoderivation and define $H : R \to R$ by $H(x) = h(x)+f(x)$ for all $x \in R$. Then, H is a CE-homoderivation since for all $x, y \in R$, we have $H(x + y) - H(x) - H(y) = h(x + y) + f(x + y) - h(x) - f(x) - h(y) - f(y) = f(x + y) - f(x) - f(y) \in I \subseteq Z(R)$, and $H(xy) - H(x)H(y) - H(x)y - xH(y) = h(xy) + f(xy) - h(x)h(y) - h(x)f(y) - f(x)h(y) - f(x)f(y) - h(x)y - f(x)y - xh(y) - xf(y) = f(xy) - h(x)f(y) - f(x)h(y) - f(x)f(y) - f(x)y - xf(y) \in I \subseteq Z(R)$.
Now consider $f : R \to I$ such that $f(x) = c$ where c is a nonzero constant element in I. Then, in this case, H is not a homoderivation.

Example 2.2. Let R_1 be a commutative domain, R_2 a noncommutative prime ring with homoderivation h, and let $R = R_1 \oplus R_2$. Then, R is a semiprime ring. Define a mapping $H : R \to R$ by $H((x, y)) = (g(x), h(y))$ where $g : R_1 \to R_1$ is any mapping which is not a homoderivation. Then, H is a CE-homoderivation since for all $(x_1, y_1), (x_2, y_2) \in R$, we have $H((x_1 + x_2, y_1 + y_2) - H((x_1, y_1)) - H((x_2, y_2)) = (g(x_1 + x_2) - g(x_1) - g(x_2), 0) \in (Z(R_1), Z(R_2)) \subseteq Z(R)$, and $H((x_1x_2, y_1y_2)-H((x_1, y_1))H((x_2, y_2))-H((x_1, y_1))(x_2, y_2)-(x_1, y_1)H((x_2, y_2)) = (g(x_1x_2) - g(x_1)g(x_2) - g(x_1)x_2 - x_1g(x_2), 0) \in (Z(R_1), Z(R_2)) \subseteq Z(R)$.
However, since g is not a homoderivation, H is clearly not a homoderivation. Moreover, $R_1 \oplus \{0\}$ is a central ideal.

A question arises is that under what conditions must a CE-homoderivation on a ring be a homoderivation? We will investigate these conditions and prove our results.

Theorem 2.3. *Let R be any ring with no nonzero central ideals. If $H : R \to R$ is a zero-power valued CE-homoderivation, then H is additive.*

Proof. Let $x, y \in R$ and $z_1 \in Z(R)$ such that

$$H(x + y) = H(x) + H(y) + z_1. \tag{2.1}$$

For arbitrary $r \in R$, there exists $z_2 \in Z(R)$ such that

$$\begin{aligned} H(r(x + y)) &= H(r)H(x + y) + H(r)(x + y) + rH(x + y) + z_2 \\ &= H(r)H(x) + H(r)H(y) + H(r)z_1 + H(r)x + H(r)y + rH(x) \\ &\quad + rH(y) + rz_1 + z_2 \end{aligned} \tag{2.2}$$

On the other hand,

$$\begin{aligned}
H(r(x+y)) &= H(rx+ry) \\
&= H(rx)+H(ry)+z_3 \\
&= H(r)H(x)+H(r)x+rH(x)+z_4+H(r)H(y)+H(r)y+rH(y) \\
&\quad +z_5+z_3
\end{aligned} \tag{2.3}$$

where $z_3, z_4, z_5 \in Z(R)$.
Comparing (2.2) and (2.3) yields $(H(r)+r)z_1+z_2 = z_3+z_4+z_5$. That is,

$$(H(r)+r)z_1 \in Z(R).$$

Since H is zero-power valued on R, we can replace r by $r-H(r)+H^2(r)+\cdots+(-1)^{n(r)-1}H^{n(r)-1}(r)$ to get $rz_1 \in Z(R)$. Since $r \in R$ is arbitrary, $Rz_1 \subseteq Z(R)$. Since R contains no nonzero central ideals, $Rz_1 = \{0\}$. This means that $z_1 \in \mathrm{Ann}(R)$, the two-sided annihilator of R. But $\mathrm{Ann}(R)$ is a central ideal and thus $z_1 = 0$ and by (2.1), H is additive. □

Theorem 2.4. *Let R be a semiprime ring with no nonzero central ideals. If $H : R \to R$ is a zero-power valued CE-homoderivation, then H is a homoderivation.*

Proof. Let $t, x, y \in R$ be arbitrary elements. Observe that

$$H((xy)t)-H(xy)H(t)-H(xy)t-xyH(t) \in Z(R). \tag{2.4}$$

On the other hand,

$$H(x(yt))-H(x)H(yt)-H(x)yt-xH(yt) \in Z(R). \tag{2.5}$$

Subtracting (2.5) from (2.4), we get

$$-H(xy)H(t)-H(xy)t-xyH(t)+H(x)H(yt)+H(x)yt+xH(yt) \in Z(R). \tag{2.6}$$

Let

$$H(xy) = H(x)H(y)+H(x)y+xH(y)+z_1 \tag{2.7}$$

$$H(yt) = H(y)H(t)+H(y)t+yH(t)+z_2 \tag{2.8}$$

where $z_1, z_2 \in Z(R)$. Substituting (2.7) and (2.8) into (2.6) yields

$$-z_1(H(t)+t)+(H(x)+x)z_2 \in Z(R). \tag{2.9}$$

Thus,

$$[-z_1(H(t)+t)+(H(x)+x)z_2, H(t)+t] = 0.$$

This can be reduced to

$$[H(x)+x, H(t)+t]z_2 = 0 \qquad \text{for all } \ x \in R.$$

Since H is zero-power valued on R, we find that

$$[x, H(t)+t]z_2 = 0 \qquad \text{for all } \ x \in R.$$

Replacing x by xr for arbitrary $r \in R$, we get $[x, H(t)+t]rz_2 = 0$. It follows by (2.8) that

$$[x, H(t)+t]R(H(yt)-H(y)H(t)-H(y)t-yH(t)) = 0 \tag{2.10}$$

for all $t, x, y \in R$.
Let P_α be a family of prime ideals of R such that $\bigcap P_\alpha = \{0\}$, and let P denote a typical P_α. Let $\bar{R} = R/P$ and let $\bar{x} = x + P$ be a typical element of $\bar{R}$.
Fix y and t above and let x be arbitrary. Then, from (2.7) and (2.8), z_2 is fixed and z_1 varies with x. From (2.10), we have either $[x, H(t) + t] \in P$ for all $x \in R$ or $z_2 \in P$.

Case1: Assume that $[x, H(t)+t] \in P$ for all $x \in R$. Therefore, $[x, H(t)+t]+P = P$ for all $x \in R$. That is, $\overline{[x, H(t) + t]} = \bar{0}$ for all $x \in R$. Thus, $\overline{H(t) + t} \in Z(\bar{R})$. It follows from (2.9) that $\overline{(H(x) + x)z_2} \in Z(\bar{R})$ for all $x \in R$. Since H is zero-power valued on R, we find that $\overline{xz_2} \in Z(\bar{R})$ for all $x \in R$. That is, $\overline{Rz_2} \subseteq Z(\bar{R})$.

Case2: Assume that $z_2 \in P$. Therefore, $z_2 + P = P$ which implies that $\bar{z_2} = \bar{0}$. Then, $\overline{Rz_2} = \bar{0} \subseteq Z(\bar{R})$.

Both cases imply that $[Rz_2, u] \in P$ for all $u \in R$. Since $\bigcap P_\alpha = \{0\}$, $[Rz_2, u] = 0$ for all $u \in R$. Thus, $Rz_2 \subseteq Z(R)$. Since R contains no nonzero central ideals, $Rz_2 = \{0\}$. This means that $z_2 \in \text{Ann}(R)$, the two-sided annihilator of R. But $\text{Ann}(R)$ is also a central ideal and thus $z_2 = 0$. Hence, by (2.8), $H(yt) = H(y)H(t) + H(y)t + yH(t)$ for all $t, y \in R$. Since, by Theorem 2.3, H is additive, it follows that H is a homoderivation. □

3. Preservation of $Z(R)$

A mapping $f : R \to R$ is said to preserve S if $f(S) \subseteq S$. Bell and Daif [2] proved that if $Z(R)$ contains no nonzero nilpotent elements, then CE-derivations on R preserve $Z(R)$. It is easy to see that zero-power valued homoderivations preserve $Z(R)$. Our purpose in this section is to investigate preservation of $Z(R)$ by CE-homoderivations. The following example shows that CE-homoderivations don't always preserve $Z(R)$.

Example 3.1. Let R_2 be a noncommutative ring with $R_2^2 \subseteq Z(R_2)$. Let R_1 be a zero ring with $(R_1, +) \cong (R_2, +)$ and let $f : (R_1, +) \to (R_2, +)$ be an isomorphism. Define R to be $R_1 \oplus R_2$ and let $H : R \to R$ be given by $H((x, y)) = (0, f(x))$. Then, H is a CE-homoderivation on R since for all $(x_1, y_1), (x_2, y_2) \in R$, it is true that $H((x_1 + x_2, y_1 + y_2) - H((x_1, y_1)) - H((x_2, y_2)) \in (0, 0) \subseteq Z(R)$, and $H((x_1x_2, y_1y_2) - H((x_1, y_1))H((x_2, y_2)) - H((x_1, y_1))(x_2, y_2) - (x_1, y_1)H((x_2, y_2)) \in (0, R_2^2) \subseteq Z(R)$.
Moreover, $R_1 \oplus \{0\} \subseteq Z(R)$. But for $(x, 0) \in R_1 \oplus \{0\}$, $H((x, 0)) = (0, f(x))$ and $f(x) \in R_2$ where R_2 is noncommutative. So, $H(R_1 \oplus \{0\}) \not\subseteq Z(R)$.

Theorem 3.2. *Let R be a ring such that $Z(R)$ contains no nonzero nilpotent elements. If H is a zero-power valued CE-homoderivation on R, then H preserves $Z(R)$.*

Proof. Let $z \in Z(R)$ and $x \in R$. Then, $H(zx) - H(z)H(x) - H(z)x - zH(x) \in Z(R)$ and $H(xz) - H(x)H(z) - H(x)z - xH(z) \in Z(R)$. Subtracting the two expressions yields

$$[H(x), H(z)] + [x, H(z)] \in Z(R) \qquad \text{for all} \quad x \in R$$

which is equivalent to

$$[H(x)+x, H(z)] \in Z(R) \qquad \text{for all } \ x \in R.$$

Since H is zero-power valued on R, we find that

$$[x, H(z)] \in Z(R) \qquad \text{for all } \ x \in R. \tag{3.1}$$

Replacing x by $xH(z)$, we get

$$[x, H(z)]H(z) \in Z(R) \qquad \text{for all } \ x \in R.$$

In particular,

$$[[x, H(z)]H(z), x] = 0 \qquad \text{for all } \ x \in R.$$

Thus, $[x, H(z)][H(z), x] + [[x, H(z)], x]H(z) = 0$ for all $x \in R$. Applying (3.1), we obtain

$$[x, H(z)]^2 = 0 \qquad \text{for all } \ x \in R. \tag{3.2}$$

Since $Z(R)$ contains no nonzero nilpotent elements, it follows from (3.1) and (3.2) that

$$[x, H(z)] = 0 \qquad \text{for all } \ x \in R.$$

Hence, $H(z) \in Z(R)$. That is, H preserves $Z(R)$. □

CE-homoderivations which preserve $Z(R)$ may also preserve subsets of $Z(R)$, in particular the set $K(R) = \{x \in Z(R) | xR \subseteq Z(R)\}$ which is the maximal central ideal.

Theorem 3.3. *If H is a zero-power valued CE-homoderivation on R which preserves $Z(R)$, then H preserves $K(R)$.*

Proof. Let $x \in K(R) \subseteq Z(R)$. Since H preserves $Z(R)$, $H(x) \in Z(R)$. For arbitrary $r \in R$, $H(xr) - H(x)H(r) - H(x)r - xH(r) \in Z(R)$. Since $x \in K(R)$, we find that $H(xr), xH(r) \in Z(R)$. Thus, $H(x)(H(r) + r) \in Z(R)$. Since H is zero-power valued on R, $H(x)r \in Z(R)$. This implies that $H(x)R \subseteq Z(R)$. Hence, $H(x) \in K(R)$. That is, H preserves $K(R)$. □

4. Commutativity Results

In this section, we prove the commutativity of prime and semiprime rings admitting CE-homoderivations satisfying various conditions. We begin with this result.

Theorem 4.1. *Let R be a prime ring and let H be a zero-power valued CE-homoderivation on R. If $H(0) \neq 0$, then R is commutative.*

Proof. Since H is a CE-homoderivation, $H(0+0) - H(0) - H(0) \in Z(R)$. Thus, $H(0) \in Z(R)$. Since $H(0x) - H(0)H(x) - H(0)x - 0H(x) \in Z(R)$ for all $x \in R$, $H(0)(H(x) + x) \in Z(R)$ for all $x \in R$. Since H is zero-power valued on R, we find that $H(0)x \in Z(R)$ for all $x \in R$. Therefore,

$$[H(0)x, r] = 0 \qquad \text{for all } \ r, x \in R.$$

Thus, $H(0)[x, r] + [H(0), r]x = 0$ for all $r, x \in R$. Since $H(0) \in Z(R)$, we obtain

$$H(0)[x, r] = 0 \qquad \text{for all } \ r, x \in R.$$

Replacing x by yx and simplifying, we get

$$H(0)y[x,r]=0 \qquad \text{for all } \; r,x,y\in R.$$

Therefore,

$$H(0)R[x,r]=0 \qquad \text{for all } \; r,x\in R.$$

Since R is prime and $H(0)\neq 0$, it follows that $[x,r]=0$ for all $r,x\in R$. Thus, $x\in Z(R)$ for all $x\in R$. Hence, R is commutative. □

Commutativity of rings and strong commutativity-preserving mappings had been studied by many authors (see [1, 3] for example). Bell and Daif [2] proved the commutativity of semiprime rings admitting a CE-derivation D that satisfies the condition $[x,y]=[D(y),D(x)]$ for all $x,y\in R$ if R contains no nonzero nilpotent elements or D is centralizing on R. With a similar approach, we prove the commutativity of semiprime rings admitting a strong commutativity-preserving CE-homoderivation and satisfying some restrictions.

Theorem 4.2. *Let R be a semiprime ring and let H be a zero-power valued strong commutativity-preserving CE-homoderivation on R. If R contains no nonzero nilpotent elements or H is centralizing on R, then R is commutative.*

Proof. By hypothesis, we have

$$[x,y]=[H(x),H(y)] \qquad \text{for all } \; x,y\in R. \tag{4.1}$$

Replacing x by xy, we get $[xy,y]=[H(xy),H(y)]$. Therefore, there exists $z_1\in Z(R)$ such that

$$\begin{aligned}
[x,y]y = \;& [H(x)H(y)+H(x)y+xH(y)+z_1,H(y)]\\
=\;& [H(x)H(y),H(y)]+[H(x)y,H(y)]+[xH(y),H(y)]+[z_1,H(y)]\\
=\;& [H(x),H(y)]H(y)+H(x)[y,H(y)]+[H(x),H(y)]y+[x,H(y)]H(y)
\end{aligned}$$

Using (4.1), we get

$$[x,y]H(y)+H(x)[y,H(y)]+[x,H(y)]H(y)=0$$

which is equivalent to

$$[x,y+H(y)]H(y)+H(x)[y,H(y)]=0 \qquad \text{for all } \; x,y\in R. \tag{4.2}$$

Similarly, we replace x by yx in (4.1) to get

$$\begin{aligned}
y[x,y] = \;& [H(y)H(x)+H(y)x+yH(x)+z_2,H(y)]\\
=\;& [H(y)H(x),H(y)]+[H(y)x,H(y)]+[yH(x),H(y)]+[z_2,H(y)]\\
=\;& H(y)[H(x),H(y)]+H(y)[x,H(y)]+y[H(x),H(y)]+[y,H(y)]H(x)
\end{aligned}$$

where $z_2\in Z(R)$. Using (4.1), we get

$$H(y)[x,y]+H(y)[x,H(y)]+[y,H(y)]H(x)=0$$

which is equivalent to

$$H(y)[x,y+H(y)]+[y,H(y)]H(x)=0 \qquad \text{for all } \; x,y\in R. \tag{4.3}$$

Taking $x = y + H(y)$ in (4.2) and (4.3) yields

$$\begin{aligned} H(y+H(y))[y,H(y)] &= 0 \qquad \text{and,} \\ [y,H(y)]H(y+H(y)) &= 0 \qquad \text{for all } \ y \in R. \end{aligned} \tag{4.4}$$

Thus,

$$[H(y+H(y)),[y,H(y)]] = 0 \qquad \text{for all } \ y \in R. \tag{4.5}$$

For arbitrary $w \in R$, we replace x by xw in (4.2) to get $[xw, y+H(y)]H(y) + H(xw)[y,H(y)] = 0$. Thus, we have $z_3 \in Z(R)$ such that $x[w, y+H(y)]H(y) + [x, y+H(y)]wH(y) + H(x)H(w)[y,H(y)] + H(x)w[y,H(y)] + xH(w)[y,H(y)] + z_3[y,H(y)] = 0$. Using (4.2), we obtain

$$[x, y+H(y)]wH(y) + H(x)H(w)[y,H(y)] + H(x)w[y,H(y)] + z_3[y,H(y)] = 0. \tag{4.6}$$

Taking $x = y + H(y)$ in (4.6), we get $z_4 \in Z(R)$ such that

$$H(y+H(y))(H(w)+w)[y,H(y)] + z_4[y,H(y)] = 0.$$

Since H is zero-power valued on R, we have for $z_5, z_6 \in Z(R)$,

$$H(y+H(y))(w+z_5)[y,H(y)] + z_6[y,H(y)] = 0. \tag{4.7}$$

Multiplying (4.7) by $[y,H(y)]$ from both sides, individually, and then subtracting the resulting equations yield

$$[H(y+H(y))(w+z_5)[y,H(y)],[y,H(y)]] = 0$$

which is equivalent to

$$[H(y+H(y))(w+z_5),[y,H(y)]][y,H(y)] = 0 \qquad \text{for all } \ y \in R.$$

Applying (4.5), we obtain

$$H(y+H(y))[w+z_5,[y,H(y)]][y,H(y)] = 0 \qquad \text{for all } \ y \in R. \tag{4.8}$$

We can reduce (4.8) to get

$$H(y+H(y))[w,[y,H(y)]][y,H(y)] = 0 \qquad \text{for all } \ w, y \in R.$$

Thus, $H(y+H(y))w[y,H(y)]^2 - H(y+H(y))[y,H(y)]w[y,H(y)] = 0$ for all $w, y \in R$. By (4.4), it follows that

$$H(y+H(y))w[y,H(y)]^2 = 0 \qquad \text{for all } \ w, y \in R.$$

Since $[H(y),H(y)] = 0$ for all $y \in R$,

$$H(y+H(y))w[y+H(y),H(y)]^2 = 0 \qquad \text{for all } \ w, y \in R. \tag{4.9}$$

Using (4.1), we can rewrite (4.9) as

$$H(y+H(y))w[H(y+H(y)),H^2(y)]^2 = 0 \qquad \text{for all } \ w, y \in R. \tag{4.10}$$

Multiplying (4.10) from the left by $H^2(y)$, we get

$$H^2(y)H(y+H(y))w[H(y+H(y)),H^2(y)]^2 = 0 \qquad \text{for all } \ w, y \in R. \tag{4.11}$$

Since $w \in R$ is arbitrary, we can replace w by $H^2(y)w$ in (4.10) to get

$$H(y+H(y))H^2(y)w[H(y+H(y)),H^2(y)]^2 = 0 \qquad \text{for all } \ w, y \in R. \tag{4.12}$$

From (4.11) and (4.12), it follows that

$$[H(y+H(y)),H^2(y)]w[H(y+H(y)),H^2(y)]^2=0 \qquad \text{for all } \ w,y\in R. \tag{4.13}$$

Thus,

$$[H(y+H(y)),H^2(y)]^2w[H(y+H(y)),H^2(y)]^2=0 \qquad \text{for all } \ w,y\in R.$$

Since R is semiprime, we have

$$[H(y+H(y)),H^2(y)]^2=0 \qquad \text{for all } \ y\in R. \tag{4.14}$$

Applying (4.1), we can rewrite (4.14) as $[y+H(y),H(y)]^2=0$, which simplifies to

$$[y,H(y)]^2=0 \qquad \text{for all } \ y\in R. \tag{4.15}$$

If R contains no nonzero nilpotent elements, then $[y,H(y)]=0$ for all $y\in R$. Whereas if H is centralizing on R, then $[y,H(y)]\in Z(R)$ for all $y\in R$. However, since the center of semiprime rings contains no nonzero nilpotent elements, (4.15) implies that $[y,H(y)]=0$ for all $y\in R$.
Hence, in both cases we have

$$[y,H(y)]=0 \qquad \text{for all } \ y\in R. \tag{4.16}$$

It follows from (4.6) and (4.16) that $[x,y+H(y)]wH(y)=0$ for all $w,x,y\in R$. Then, $([x,y]+[x,H(y)])wH(y)=0$ for all $w,x,y\in R$. By (4.1),

$$[H(x)+x,H(y)]wH(y)=0 \qquad \text{for all } \ w,x,y\in R.$$

Since H is zero-power valued on R, we find that

$$[x,H(y)]wH(y)=0 \qquad \text{for all } \ w,x,y\in R.$$

It follows, with a similar approach to proving (4.13), that

$$[x,H(y)]w[x,H(y)]=0 \qquad \text{for all } \ w,x,y\in R.$$

By semiprimeness of R, we have $[x,H(y)]=0$ for all $x,y\in R$. Hence, $H(R)\subseteq Z(R)$ and from (4.1), R is commutative. □

With a similar approach to proving Theorem 4.2, we can prove the following:

Theorem 4.3. *Let R be a semiprime ring and let H be a zero-power valued CE-homoderivation on R such that $[x,y]=[H(y),H(x)]$ for all $x,y\in R$. If R contains no nonzero nilpotent elements or H is centralizing on R, then R is commutative.*

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