Gulf Journal of Mathematics
Vol 11, Issue 1 (2021) 73-82

MAXIMAL REGULARITY FOR NON AUTONOMOUS EVOLUTION EQUATIONS AND BLACK-SCHOLES OPERATOR

CHAKIR HILMI [1] AND AHMED SANI [2*]

Abstract. A sufficient topological condition to ensure maximal regularity is given. The result permits to retrieve classical results established recently on non autonomous evolution equations mainly those concerning well-posedness of the non autonomous Cauchy problem. We illustrate the result in the hilbertian context by an application to Black-Scholes operator.

1. Introduction and preliminaries

1.1. **Introduction:** Recently, A particular interest has been granted to non autonomous evolution equations (see for example [19], [1] or [17]). The most worth reason to mention is the existence of multiple applications in the control theory such as [13] in deterministic framework and [12], [14] in the stochastic one. A central topic concerning general evolution problems is the maximal regularity in different functional spaces. In the latter direction, and even for single operators, the decisive result of Baillon's theorem as revisited in [9] concerning boundedness of operator with maximal regularity on $\mathcal{C}([0,T],E)$(i.e space of all continuous mappings on $[0,T]$ with values in a general Banach space E) stopped researches on this type of spaces. As a natural consequence, an obvious way to encounter this barrier is to investigate the L^p-maximal regularity or $\mathcal{C}^\alpha-$regularity. We focus our attention in the current paper on the first type of regularity. Let us make more precise this notion. Fix a strictly positive real T and let D and X be two Banach spaces such that $D \hookrightarrow X$ which means that D is continuously and densely embedded into X.

Definition 1.1. A single operator $A \in L(D,X)$ has L^p-maximal regularity or is of *category* $\mathcal{MR}$ $(p \in (1,\infty))$ if for every $f \in L^p(0,T;X)$ there exists a unique $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D)$ such that

$$\dot{u} + Au = f \quad \text{a.e. on} \quad (0,T), u(0) = 0 \tag{1.1}$$

For reasons cited above, the property of L^p-maximal regularity has been intensively treated due to its applications to proving existence, uniqueness and regularity of solutions of linear and non linear evolution equations. Note that,

Date: Received: Feb 2, 2021; Accepted: May 25, 2021.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 44B20.
Key words and phrases. Toplogical injections, Non autonomous equations,maximal regularity, interpolation, semigroups.

thanks to Poincaré inequality, for finite horizon T, the maximal regularity of the a single operator A is equivalent to tell that the three terms $\dot{u}$, Au and f belong to the same space $L^p(O,T,X)$. The space $\mathcal{MR} = W^{1,p}(0,T;X) \cap L^p(0,T;D)$ is said of maximal regularity and the Trace space Tr is then $Tr = \{u(0); u \in \mathcal{MR}\}$. Endowed with the norm $\|x\| = \sup \|u(0)\|, u \in \mathcal{MR}$ and $u(0) = x$, Tr is a Banach space isometric to an interpolation space between D and X.

A remarkable result consists on taking the maximal regularity of the inhomogeneous Cauchy problem as a bridge to establish well-posedness of the homogeneous one with start data in a largest subspace of E(*Trace space*) denoted usually, as mentioned above, Tr; see mainly the first part of [5] for a rigorous building of this bridge. Therefore, many attempts to generalize statements established for a single operator such given by (1.1) to non autonomous case, that when the operator A depends merely on time denoted from now on $A(t), t \in [0,T]$. In this way, Lions Theorem was proved otherwise in [21] using frozen coefficients method as initiated in the parabolic case by Acquistapace and Terreni [2] when they tried to construct unified approach to abstract linear non autonomous parabolic equations. More applications may be found in [20] mainly when the ambient space is a Hilbert one. Indeed, if X is Hilbert, then the operators with maximal regularity property are characterized as those who generate holomorphic semigroups. Historically, de Simon, according to [4], was the first to prove the direct implication (maximal regularity $\Longrightarrow$ generation of holomorphic semigroup). The converse is not true in general spaces but holds in Hilbert spaces. A counterexample was given recently by Fackler [11] generalizing by this way the results of Kalton and Lancien [15] concerning the fail of maximal regularity on various classes of Banach spaces including L^p-spaces with $1 < p \neq 2 < +\infty$. For an overview on one parameter semigroups, one may see [6] and for more generalized and recent theory on *biparametrized* semigroups we refer to [3].

The evolutionary case (time dependence $A(t)$) gained recently a great importance mainly in the mixed studies, both deterministic and stochastic ones. One may consult among others [16].

In the current paper, we treat the maximal regularity in terms of injections and space interpolation. A starter point is the fact that trace space of admissible initial data of the problem (1.1) is as known isomorphic to the interpolation space $Tr = (X,D)_{1/q,p}$ where q is the conjugate exponent of p (i.e $1/q+1/p = 1$). On the other hand, the same space may be seen as the set of all traces of the maximal regularity space $\mathcal{MR} = L^p(0,T;X) \cap W^{1,p}(0,T:D)$. In clearer words $Tr = \{u(0), u \in \mathcal{MR}\}$. For arbitrary fixed $t_0 \in [0,T]$ R. Chill proved that also $Tr = \{u(t_0), u \in \mathcal{MR}\}$. So these different but equivalent characterizations of the Trace space prompt us to see the L^p-maximal regularity in a normalizing topological point of view.

Thus, in the first section we recall technical tools required for an autonomous comprehension of our purposes and which are necessary to establish the main results. In the second section, we prove clearly that the Black-Scholes operator has the maximal regularity property in some suitable space. We restrict ourselves to hilbertian case in order to take profit from the famous characterization of maximal regularity in this adequate framework.The third section is devoted to

present a sufficient condition to ensure maximal regularity for a given single operator and the basic definitions and some properties of the relevant spaces we deal with in the paper. Extrapolation to non autonomous Cauchy problem will be exposed. Then at last, we recapitulate and give some ways to improve by weakening hypothesis.

1.2. **Preliminaries.**

1.2.1. *Some topological embeddings.* In general, the compact embeddings allows to restrict questions on existence of solutions to treatment of boundedness in some appropriate subspaces which are assumed to be compactly embedded in the vast ambient space. With usual notations, it is well known that

Proposition 1.2. *The Sobolev space $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n, n \in \mathbb{N}$.*

The generalization of proposition (1.2) to vector-valued functions is easy to state in the same way like the scalar valued functions. The proof is elementary and we refer for example to [8]. In topological words, compacity of embedding means simply that the boundedness respect to the norm of a subspace allows convergent subsequences with respect to the ambient space norm.

When the embedding is just continuous, such conclusions fail in general because the subspace topology is richer than the whole space one. So to warrant convergence of a sequence in the big space needs convergence (and not just boudedness) in the small one.

For the sick of simplicity and brevity, we recall just the following proposition which explains by estimation the combination of both types of embeddings

Proposition 1.3. *Let D, T and X three Banach spaces such that $D \subset T \subset X$ and*

- *D is compactly embedded in T.*
- *T is continuously embedded in X. Then*

$$\forall \epsilon > 0 \ \exists \eta > 0 \ \forall x \in D : \ \|x\|_T \leq \epsilon \|x\|_D + \eta \|x\|_X. \tag{1.2}$$

One may prove proposition (1.3) by contradiction using elementary arguments. The estimate (1.2) may be interpreted as a "quasi" equivalence between the norms $\|.\|_T$ and $\|.\|_X$ modulo a very small $D-$ norm perturbation $(\epsilon\|.\|_D)$.

1.2.2. *Maximal regularity.* The definition (1.1) concerns single operators. In the non autonomous, we adopt the following definition

Definition 1.4. Let $p \in (1, \infty)$. The family $A(t)$ is said to have Lp^-maximal regularity (and we write $A \in \mathcal{MR}p(0,T)$) if for every $f \in L^p(0,T;X)$ there exists a unique $u \in MR(0,T)$ such that

$$\dot{u}(t) + A(t)u(t) = f(t) \quad \text{a.e. on } (0,T), \quad u(0) = 0.$$

It is known that the maximal regularity, as given in definition (1.4) does not hold even if all operators $A(t)$ have this property. Additional hypothesis on

$t \mapsto A(t)$ are required. For example, Prüss and Schnaubelt established the maximal regularity for parabolic equations in [18] under two hypotheses: norm continuity of $t \mapsto A(t)$ and *individual* maximal regularity for each operator $A(t)$. Arendt et al.(see [5]) generalized this result to bounded and measurable functions $t \mapsto A(t)$ invoking a new notion of *relative continuity*. The main idea was a local perturbation by smooth family of operators. As mentioned at introduction, the main interest of maximal regularity consists in integration of the homogeneous Cauchy problem with start data in a the interpolation space Tr. Here, we pursue this vision.
The principal objective of the current paper consists in proving otherwise well-posedness of non autonomous Cauchy problem under the punctual maximal regularity with some additional hypotheses on images of the operators $A(t)$ as stated in the theorem 2.1 below.

2. Main result

Given an index set I and given an operator family $A(t)_{t\in I}$, we denote V_I the space spanned by $\cup_{t\in I} R_t$; $R_t = A(t)D$. Beside the punctual maximal regularity of each operator $A(t)$, we assume that

- H_1**:** *Rank condition*:For every finite subdivision $\lambda_0 = 0 < \lambda_1 < \lambda_2 < ... < \lambda_{n+1} = T, n \in \mathbb{N}$ one has $D \hookrightarrow V_{[\lambda_k,\lambda_{k+1}]}, k = 0, 1, 2..n$.
- H_2**:** *Continuity embedding*:For all $k \in \{0, 1, 2...n\}$, $V_{[\lambda_k,\lambda_{k+1}]}$ is continuously embedded in X.

Under these hypothesis, we prove the following theorem

Theorem 2.1. *Let $A : t \mapsto \mathcal{L}(D, X)$ a family of maximal regular operators satisfying H_1 and H_2 above. For all $f \in L^p(0, T; X)$ the problem*

$$\dot{u} + A(t)u(t) = f \text{ a.e on } [0, T] \; u(0) = x \tag{2.1}$$

has a unique solution for all $x \in Tr$.

It is worth to mention that for the proof, domains $D_t = D(A(t))$ are not necessarily equal, but for convenience, we assume that their intersection $\cap(D_t)_{t\geq 0}$ is topologically large enough (say dense in X). In addition, for a suitable mental representation of the space Tr, we recall that Tr is isometric to the interpolation space $(X, D_{\frac{1}{2};2})$ when we consider $p = \frac{1}{2}$. In fact, as *maximal regularity-property* does not depend on p, the choice of this parameter may be arbitrary in $(1, \infty)$. Although, it is necessary to mention that the strength and geometry of Tr may vary with p. We will explore this aspect in the future research.

Proof. Consider now a finite subdivision $\lambda_0 = 0 < \lambda_1 < \lambda_2 < ... < \lambda_{n+1} = T, n \in \mathbb{N}$ such that the common domain D is compactly embedded in the image V_k for all $k \in \{0, 1, 2 \dots n\}$ and V_k is continuously embedded in the ambient space X. For a frozen interval $I_k = [\lambda_k, \lambda_{k+1}[$, by hypotheses H_1 and H_2, we have for all (s, r) in I_k^2:

$$\|A(s)x - A(r)x\| \leq 2\epsilon\|x\|_D + 2\eta\|x\|_X.$$

This latter inequality is a simple reformulation of (1.2) in Proposition 1.2. In particular, for $s = \lambda_k$, we obtain

$$\|A(\lambda_k)x - A(r)x\| \leq 2\epsilon\|x\|_D + 2\eta\|x\|_X \quad \text{for all} \quad r \in I_k. \tag{2.2}$$

Hence the initial problem 2.1 is divided on subproblems $(\mathcal{P}_k)$ on each cell $I_k = [\lambda_k, \lambda_{k+1}[$ as a simple perturbation problem

$$(\mathcal{P}_k): \quad \dot{u_k}(r)+A(\lambda_k)u_k(r)+A(r)u(r) = f(r) \text{ a.e on } [\lambda_k, \lambda_{k+1}[\ u_k(\lambda_k) = u_{\lambda_{k-1}}(\lambda_k),$$

which is well-posed by robustness of maximal regularity. In fact, it is easy to prove well-posedness in the same way as done in [5] using the classical characterization of maximal regularity by the sum of two convenient operators and a usual perturbation result. The main tool to do so is the following classical result (see [5, Proposition 1.3])

Proposition 2.2. *Let A be of category $\mathcal{MR}$ and $B: t \mapsto B(t)$ a non autonomous perturbation measurable of A. Suppose $B(t)$ satisfy hypotheses H_1 and H_2. Then the problem*

$$\dot{u} + Au + B(t)u = f \quad a.e. \quad on \quad (a,b), \quad u(a) = x$$

is well-posed for all $x \in Tr$.

To complete the claim, it suffices to consider the subproblem $\mathcal{P}_k$ as a perturbed problem of the the singular (autonomous) operator A_{λ_k} by the smooth mapping (which plays the same role as B in Proposition 2.2 above) $A: r \in I_k = [\lambda_k, \lambda_{k+1}[\mapsto A(r)$.

□

3. Black-Scholes operator

Here we introduce and study the maximal regularity of Black-Scholes operator. We cite, among others, the interesting paper [7] as a rigorous reference for the bridge between forms and operators associated with those in Hilbertian structure (the ambient X is a Hilbert space, denoted henceforth H and the form is defined on a dense subspace V). Of course, results on operators which arise from elliptic forms are assumed known. If it is not the case, one may consult fruitfully [21] or [20].
Let us recall necessary ingredients for such study mainly the form associated with the operator and its domain.

3.1. **Study in $L^2(0,+\infty)$.** Let $H = L^2(0,+\infty)$. Consider for given constants $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$ the problem:

$$Au = ax^2D^2u + bxDu - bu$$

with natural domain $H^2(0,+\infty)$. We give in details Einemann method to determine precisely the form associated with the operator A acting on L^2.

Henceforth, we consider the following form:

$$\mathfrak{a}(u,v) = (xDu, xDv) + (2a-b)(xDu, v) + b(u,v)$$

with domain

$$V = \{u \in W^{1,1}_{loc}((0,+\infty),\ xDu \in H\}.$$

In fact V is endowed with hilbertian structure as states the following proposition

Proposition 3.1. *The domain V of the form $\mathfrak{a}$ is a Hilbert space with inner product :*

$$(u,v)_V = (u,v) + (xDu, xDv).$$

In the Hilbertian framework, the hilbertian structure of V is important for our purposes since the norm arising from this structure allows to embed V continuously and densely in H.

Proof. [of the proposition 3.1 above] We recall the product of two separable Hilbert spaces remains separable. Consider the isomorphism

$$\phi :\ V \rightarrowtail H \times H$$

,

$$u \mapsto (u, xDu)$$

which is in fact an isometric isomorphism. To prove the claim, it suffices to prove that ϕ is closed.

For a sequence $(u_n)_n$ of elements of V such that $(\phi(u_n)) \longrightarrow (u,v)$. Then $\lim_n u_n = u$ in H and $\lim xDu_n = v$ in H. The closure of ϕ will be established when we prove that $u \in W^{1,1}(0,+\infty)$ and that $xDu = v$.
For all function θ with compact support in $(0,+\infty)(\theta \in D(0,+\infty))$, we have:

$$(u, D\theta) = \lim_n (u_n, D\theta) = -\lim_n (Du_n, \theta) = -\lim_n (xDu_n, \frac{1}{x}\theta) = -(\frac{1}{x}v, \theta).$$

This yields $u \in W^{1,1}(0,+\infty)$ and the derivative in the sense of distributions is: $Du = \frac{1}{x}v$. Since the space $(V, \|.\|_V)$ is isometric to the Banach space $H \times H$ it is complete itself. □

Remark 3.2. The norm defined on V satisfies clearly $\|.\|_H \leq \|.\|_V$ which means functionally speaking that $V \underset{d}{\hookrightarrow} H$, ($V$ is continuously embedded in H.)

Here we pursue our investigation on V in order to describe suitably its topological structure and some of its remarkable subsets. In particular, we have:

Proposition 3.3. *The set $D((0,+\infty))$ is dense in V when this latter is endowed with $\|.\|_V$-topology.*

Proof. Let us denote H^0 the closure of $D(0,+\infty))$ in V, i.e $H^0 = \overline{D(0,+\infty)}^{\|.\|_V}$. It is enough to prove that $H^0 = V$.

To do so, we remark that any $u \in V \cap C_c(0,+\infty)$ is a limit of a sequence elements of de K. To see it, one may consider a *mollifier* (i.e A sequence $(\rho_n)_n$ with support in $[-\frac{1}{n}, \frac{1}{n}]$ such that $\int_{\mathbb{R}} \rho_n(x)dx = 1$). A well known result is that $u_n = \rho_n \star u$ belongs to $D(0,+\infty)$ and satisfies: $u_n \longrightarrow u$ and $Du_n = \rho_n \star Du \longrightarrow Du$ in H. This yields:

$$\|u_n - u\|_V \leq \|u_n - u\|_H + \|xD(u_n - u)\|_H \leq \|u_n - u\|_H + \|Du_n - Du\|_H.$$

Similarly, one may show that for every $u \in V$, there exists a sequence in $V \cap C_c(0,+\infty)$ which approximates u in V. In the sequel, we give an approximating sequence, without calculation details. For this, we consider the function $\psi_n \in W^{1,1} \cap C_c(0,+\infty)$ defined for all integer n by :

$$\psi_n(x) = \begin{cases} 0; & \text{if } x \in [0;\frac{1}{n}] \cup [n+1;\infty[\\ \text{nx-1}; & \text{if } x \in [\frac{1}{n};\frac{2}{n}] \\ 1; & \text{if } [\frac{2}{n};n] \\ \text{-x+n+1}; & \text{if } x \in [n;n+1]. \end{cases}$$

A direct calculation shows that the sequence $(u_n)_n$ in H^0 given by $(u_n)_n = (u\psi_n)_n$ converges in V to u; in other words $H^0 = V$. □

More description of V is given in the following. Besides the continuous embedding in H, it was proved that $V \subset C(0,+\infty)$ [10]. But by the following example, it is impossible to establish the embedding of V (see always [10]) in $L^\infty(0;+\infty)$:

Example 3.4. Let $g(x) = \min(\ln x; 0)$. Then g belongs to V. Furthermore $\|g\|_V^2 = 3$.

The example (3.4) proves that an estimation such $\|.\|_\infty \leq C\|.\|_V$ for a constant C is impossible.

A fundamental example for our topic is $S \mapsto (K-S)_+$ which expresses in Black-Scholes model that the option price V or V_t can't exceed the final value K fixed in advance at the horizon τ.

Example 3.5. Let $h(x) = (K-x)_+ = \max(K-x;0)$. Then h belongs to V. moreover $\|h\|_V^2 = \frac{2}{3}K^3$.

3.2. **Come back to the form.** Let us now retreat the form $\mathfrak{a}$:

$$\mathfrak{a}(u,v) = (xDu, xDv) + (2a-b)(xDu, v) + b(u,v)$$

with domain

$$V = \{u \in W^{1,1}_{loc}((0,+\infty),\ xDu \in H\}.$$

In the previous section , we have proved that V is dens H. In the sequel, we examine some analytic properties of $\mathfrak{a}$ mainly the coercitivity and ellipticity

Proposition 3.6. *The form $\mathfrak{a}$ is continuous, densely defined and elliptic (on H). Moreover, if $b \geq \frac{2}{3}a$ then $\mathfrak{a}$ is coercive.*

Proof. Since $D(\mathbb{R}+)$ is dense in V and V is densely embedded in H, then the form $\mathfrak{a}$ is well and densely defined on H. Its continuity is an immediate consequence of:

$$\mathfrak{a}(u,v) \leq (a + |2a-b| + |b|)\|u\|\|v\| \quad \text{for all} \quad (u,v) \in V^2.$$

Let us now show that $\mathfrak{a}$ is $H-$elliptic. For a given $(u,v) \in V^2$, the density of $D(\mathbb{R}^+)$ in V ensures the existence of a sequence ϕ_n of elements of $D(\mathbb{R}^+)$ which converges to u in V. Taking in mind that $x\phi_n$ remains in $D(\mathbb{R}^+)$ for all $n \in \mathbb{N}$, the following calculation makes sense:

$$\begin{aligned} 2(xDu,u) &= (xDu,u)_H + (u,xDu)_H \\ &= \lim_{n\to\infty}(xDu,\phi_n) + (u,xD\phi_n) \\ &= \lim_{n\to\infty}(u,-Dx\phi_n + xD\phi_n) \\ &= \lim_{n\to\infty}(u,\phi_n) \\ &= -\|u\|_H^2. \end{aligned}$$

it yields that

$$\mathfrak{a}(u,u) = \mathfrak{a}(u) = a\|xDu\|^2 + (b - \frac{1}{2}(2a-b))\|u\|_H^2 = a\|xDu\|_H^2 + (\frac{3}{2}b - a)\|u\|_H^2.$$

The latter estimate expresses that $\mathfrak{a}$ is elliptic. Furthermore, with the choice $\frac{3}{2}b - a \geq 0$, the form $\mathfrak{a}$ becomes positive. □

Naturally, the form $\mathfrak{a}$ is not coercive for arbitrarily chosen values of de a et b. It remains to establish that the operator A is associated to the form $\mathfrak{a}$ on H. It is enough to write for every $u \in D(A) \cap W^{2,1}_{loc}(\mathbb{R}^+)$:

$$\begin{aligned} (Au,v)_H &= \mathfrak{a}(u,v) \\ &= a(xDu,Dv) + (2a-b)(xDu,v) + b(u,v) \\ &= a(Du,x^2Dv + 2xv) - b(xDu - u,v) \\ &= a(Du,Dx^2v) - b(xDu - u,v) \\ &= a(D^2u,x^2v) - b(xDu - u,v) \\ &= -(ax^2D^2u + bxDu - xu,v)_H \end{aligned}$$

Since $\mathfrak{a}$ satisfies all required properties of regularity (dense domain, ellipticity...), the associated operator $-A$ is a generator of an analytic semigroup on H. Since the framework is hilbertian, the operator of Black-Scholes is then of category $\mathcal{MR}$.

Remark 3.7. When the concern is just to approximate regular solutions in space like $C^2(\mathbb{R})$, it is useful to avoid the term of degeneracy x^2 which is adjacent to the second differential term D^2 by change of variable in some usual situations. For example, to discard the degeneracy term in the the following problem

$$\begin{cases} \frac{\partial D}{\partial t}u + \frac{1}{2}\sigma^2S^2\frac{\partial D^2}{\partial S^2}u + rS\frac{\partial D}{\partial S}u - ru = 0 \\ u(x,0) = \phi(x), \end{cases}$$

it is known and easily retrievable that, thanks to change:

$$\begin{cases} S = Ke^x \\ z(x,t) = u(S,t), \end{cases}$$

the new variable z satisfies the more *accessible* equation:

$$\frac{\partial D}{\partial t}z + \frac{1}{2}\sigma^2\frac{\partial D^2}{\partial x^2}z + (r - \frac{1}{2}\sigma^2)\frac{\partial D}{\partial x}z - rz = 0$$

With the backward $s = (T-t)\sigma^2$ and the rescale $q = \frac{2r}{\sigma^2}$, we improve the equation as follows

$$\frac{\partial D}{\partial s}z + \frac{1}{2}\frac{\partial D^2}{\partial x^2}z + \frac{1}{2}(q-1)\frac{\partial D}{\partial x}z - \frac{1}{2}qz = 0.$$

Finally, for a classical European option, one may resolve the problem after the last change: $z(x,t) = \exp(\gamma x + \omega s)y(x,s)$. A development of this elementary calculus will appear in our future works in order to improve some classical schemes of approximation.

Acknowledgement. Our prior acknowledgement and thanks to the referee(s) and to our colleagues at LAMA Laboratory for fruitful discussions.

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[1]Université Ibn Zohr, Faculté des Sciences, Département de Mathématiques, Agadir, Morocco
Email address: ahmedsani82@gmail.com

[2]Université Ibn Zohr, Faculté des Sciences, Département de Mathématiques, Agadir, Morocco
Email address: chakir.hilmi@gmail.com