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THE UNION OF TWO GRAPHS ASSOCIATED WITH THE SET OF ALL NON-ZERO ANNIHILATING IDEALS OF A COMMUTATIVE RING

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Abstract. The rings considered in this article are commutative with identity which are not integral domains. Let R be a ring. Let $\mathbb{A}(R)$ denote the set of all annihilating ideals of R and let us denote $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. With R, in this article we associate an undirected graph denoted by $G(R)$ whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in $G(R)$ if and only if either $IJ \neq (0)$ or $I + J \notin \mathbb{A}(R)$. If R is reduced, then in Section 2, we prove that $G(R)$ is connected and determine the diameter and radius of $G(R)$. If R is not reduced, then in Section 3, we answer when $G(R)$ is connected and determine the diameter and radius of $G(R)$ when $G(R)$ is connected.

1. Introduction and preliminaries

The rings considered in this article are commutative with identity which are not integral domains. Motivated by the research work of I. Beck in [11], several algebraists associated graphs with algebraic structures and investigated the interplay between the algebraic properties of the algebraic objects and the graph-theoretic properties of the graphs associated with them. The graphs associated with commutative rings that motivate the research work presented in this article are mainly the zero-divisor graphs of commutative rings and the annihilating-ideal graphs of commutative rings. Let R be a ring. We denote the set of all zero-divisors of R by $Z(R)$ and the set $Z(R)\backslash\{0\}$ by $Z(R)^*$. Recall from [8] that the *zero-divisor graph of* R, denoted by $\Gamma(R)$ is an undirected graph whose vertex set is $Z(R)^*$ and distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $xy = 0$. For an excellent and inspiring survey article on the research work done in the area of zero-divisor graphs of commutative rings, one can refer [4].

Let R be a ring. Recall from [12] that an ideal I of R is said to be an *annihilating ideal* if there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. As in [12], we denote the set of all annihilating ideals of R by $\mathbb{A}(R)$ and the set $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. Let R be a ring such that R is not an integral domain. Recall from [12] that the *annihilating-ideal graph of* R, denoted by $\mathbb{AG}(R)$ is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in $\mathbb{AG}(R)$ if and

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only if $IJ = (0)$. The concept of the annihilating-ideal graph of a ring was introduced by M. Behboodi and Z. Rakeei in [12] and several inspiring results on annihilating-ideal graphs of commutative rings were proved in [12, 13]. There are several research articles in the literature which studied the interplay between the ring-theoretic properties of a commutative ring R and the graph-theoretic properties of $\mathbb{AG}(R)$ (see for example, [1, 2, 18, 28]).

The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a simple graph. Recall from [10, Definition 1.1.13] that the *complement of* G, denoted by G^c is defined as a graph whose vertex set is V and distinct $a, b \in V$ are joined by an edge in G^c if and only if there is no edge in G joining a and b. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$.

Let R be a ring which is not an integral domain. In [5], D.F. Anderson and A. Badawi introduced the concept of the *total graph of* R, denoted by $T(\Gamma(R))$ is an undirected graph with $V(T(\Gamma(R))) = Z(R)^*$ and distinct vertices x and y are adjacent in $T(\Gamma(R))$ if and only if $x + y \in Z(R)$. Several interesting theorems on $T(\Gamma(R))$ were proved in [5, 6]. Moreover, $T(\Gamma(R))$ was generalized and investigated in [7]. Furthermore, the total graph of a commutative ring was also studied in [3].

Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. Motivated by the research work done on the total graph of a commutative ring in [3, 5], we introduced and investigated an undirected graph, denoted by $\Omega(R)$ in [25], whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in $\Omega(R)$ if and only if $I + J \in \mathbb{A}(R)$. The results proved on the graph considered in the present article is also motivated by the interesting results proved by A. Duric, S. Jevdenic, P. Oblak, and N. Stopar in [15] on the total zero-divisor graph of commutative rings. Let R be a ring such that $Z(R)^* \neq \emptyset$. Recall from [15] that the *total zero-divisor graph of* R, denoted by $ZT(R)$ is an undirected graph whose vertex set is $Z(R)^*$ and distinct vertices x and y are adjacent in $ZT(R)$ if and only if $xy = 0$ and $x + y \in Z(R)$.

Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. Motivated by the above mentioned research work done by several mathematicians, in this article, with R, we associate an undirected graph denoted by $G(R)$ such that $V(G(R)) = \mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in $G(R)$ if and only if either $IJ \neq (0)$ or $I + J \notin \mathbb{A}(R)$. Let $i \in \{1, 2\}$. Let $G_i = (V_i, E_i)$ be an undirected graph. Then their *union*, denoted by $G_1 \cup G_2$ is an undirected graph with $V(G_1 \cup G_2) = V_1 \cup V_2$ and $E(G_1 \cup G_2) = E_1 \cup E_2$ [14, page 26]. Hence, it follows that $G(R) = (\mathbb{AG}(R))^c \cup (\Omega(R))^c$. For a ring with $\mathbb{A}(R)^* \neq \emptyset$, the graph $(\mathbb{AG}(R))^c$ was studied in [26] and the graph $(\Omega(R))^c$ was studied in [27]. Recall from [10] that a subgraph H of a graph G is said to be a *spanning subgraph of* G if $V(H) = V(G)$. Thus both $(\mathbb{AG}(R))^c$ and $(\Omega(R))^c$ are spanning subgraphs of $G(R)$.

Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. The aim of this article is to discuss some results regarding the connectedness of $G(R)$ and determine the diameter of $G(R)$ and the radius of $G(R)$, whenever $G(R)$ is connected.

Let us recall the following definitions, notations, and results from commutative ring theory before we give a brief summary of results that are proved in this article. Let R be a ring. We denote the set of all prime ideals of R by $Spec(R)$, the set of all maximal ideals of R by $Max(R)$, and the set of all minimal prime ideals of R by $Min(R)$. We denote the set of all units of R by $U(R)$. R is said to be *quasilocal* if R has only one maximal ideal. A Noetherian quasilocal ring is referred to as a *local ring*. We denote the nilradical of R by $nil(R)$ and R is said to be *reduced* if $nil(R) = (0)$. The Krull dimension of R is simply referred to as dimension of R and is denoted by $dim R$. Whenever a set A is a subset of a set B and $A \neq B$, we denote it symbolically by $A \subset B$. The cardinality of a set A is denoted by $|A|$. Let $n \in \mathbb{N}$ with $n \geq 2$. We denote the ring of integers modulo n by $\mathbb{Z}_n$.

Let I be an ideal of a ring R with $I \neq R$. Let $\mathfrak{p} \in Spec(R)$. Recall from [20] that $\mathfrak{p}$ is said to be a *maximal N-prime of I* if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z_R(\frac{R}{I}) = \{r \in R \mid rx \in I \text{ for some } x \in R \backslash I\}$. Hence, $\mathfrak{p} \in Spec(R)$ is a maximal N-prime of (0) if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z(R)$. It is clear that $S = R \backslash Z(R)$ is a multiplicatively closed subset of R. Let $x \in Z(R)$. Then $Rx \cap S = \emptyset$. Hence, it follows from Zorn's lemma and [21, Theorem 1] that there exists a maximal N-prime $\mathfrak{p}$ of (0) in R such that $x \in \mathfrak{p}$. Therefore, if $\{\mathfrak{p}_\alpha\}_{\alpha \in \Lambda}$ is the set of all maximal N-primes of (0) in R, then it follows that $Z(R) = \bigcup_{\alpha \in \Lambda} \mathfrak{p}_\alpha$.

Let I be an ideal of a ring R with $I \neq R$. Let $\mathfrak{p} \in Spec(R)$. Recall from [19] that $\mathfrak{p}$ is said to be an *associated prime of I in the sense of Bourbaki* if $\mathfrak{p} = (I :_R x)$ for some $x \in R$. In this case, we say that $\mathfrak{p}$ is a B-prime of I. Let $\mathfrak{p}$ be a maximal N-prime of (0) in R. It is easy to verify that $\mathfrak{p}$ is a B-prime of (0) in R if and only if $\mathfrak{p} \in \mathbb{A}(R)$.

Let $G = (V, E)$ be a graph. G is said to be *connected* if for each pair of distinct vertices $a, b \in V$, there exists at least one path in G between a and b. Let G be connected. Recall from [10] that for distinct vertices a, b of G, the *distance between a and b* denoted by $d(a, b)$ is defined as the length of a shortest path in G between a and b. We define $d(a, a) = 0$. We define the *diameter of G*, denoted by $diam(G)$ as $diam(G) = sup\{d(a, b) \mid a, b \in V\}$.

Let $G = (V, E)$ be a connected graph. Let $a \in V$. Recall from [10] that the *eccentricity of a*, denoted by $e(a)$ is defined as $e(a) = sup\{d(a, b) \mid b \in V\}$. The *radius of G*, denoted by $r(G)$ is defined as $r(G) = min\{e(a) \mid a \in V\}$. A simple graph G is said to be *complete* if every pair of distinct vertices of G are adjacent in G. Let $n \in \mathbb{N}$. A complete graph with n vertices is denoted by K_n [10, Definition 1.1.11].

This article consists of three sections. Section 1 is on introduction. Unless otherwise specified, the rings considered in this article are commutative with identity which are not integral domains. Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. In Section 2 of this article, we focus on reduced rings. Let R be a reduced ring which is not an integral domain. It is proved in Corollary 2.2 that $G(R)$ is connected and $diam(G(R)) \leq 2$. It is shown in Theorem 2.5 that $G(R)$ is complete if and only if $|Min(R)| = 2$. For a von Neumann regular ring R which is not a field, it is deduced in Corollary 2.6 that $G(R)$ is complete if and only if $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. Let T be a ring which can possibly be non-reduced. Suppose that $G(T)$ is connected. It is proved in Theorem 2.9 that

$r(G(T)) = 1$ if and only if there exists $\mathfrak{p} \in Min(T) \cap \mathbb{A}(T)^*$ such that $e(\mathfrak{p}) = 1$ in $G(T)$. For a reduced ring R, it is deduced in Corollary 2.10 that the statements (1) $r(G(R)) = 1$; (2) There exists $\mathfrak{p} \in Min(R) \cap \mathbb{A}(R)^*$ such that $e(\mathfrak{p}) = 1$ in $G(R)$; (3) $Min(R) \cap \mathbb{A}(R)^* \neq \emptyset$ are equivalent. In Example 2.13, a von Neumann regular ring R is provided such that $diam(G(R)) = r(G(R)) = 2$. In Example 2.14, a reduced ring R from [17, page 16] is mentioned and the reduced ring R is such that $Min(R)$ is infinite, each $\mathfrak{p} \in Min(R)$ is an annihilating ideal of R, $diam(G(R)) = 2$, and $r(G(R)) = 1$.

The rings considered in Section 3 are non-reduced. Let R be a non-reduced ring with $|\mathbb{A}(R)^*| \geq 2$. It is proved in Lemma 3.1 that there exist distinct $I, J \in \mathbb{A}(R)^*$ such that I and J are not adjacent in $G(R)$. Let R be a non-reduced ring which admits $\mathfrak{p}$ as its unique maximal N-prime of (0). It is shown in Proposition 3.7 that $G(R)$ is connected if and only if $\mathfrak{p}$ is not a B-prime of (0) in R. Moreover, in the case when $G(R)$ is connected, it is proved that $diam(G(R)) = 2$. Furthermore, if $nil(R) \notin \mathbb{A}(R)$, then it is verified that $r(G(R)) = 2$. If $\mathfrak{p} \notin \mathbb{A}(R)$ and if given $I \in \mathbb{A}(R)^*$, there exists a non-zero $x \in nil(R)$ such that $Ix = (0)$, then it is verified that $r(G(R)) = 2$. Let R be a non-reduced ring such that R admits $\{\mathfrak{p}_i \mid i \in \{1, 2\}\}$ as its set of all maximal N-primes of (0). It is observed in Proposition 3.9 that $G(R)$ is connected and $2 \leq diam(G(R)) \leq 3$. If $\mathfrak{p}_i$ is not a B-prime of (0) in R for at least one $i \in \{1, 2\}$, then it is noted in Proposition 3.11 that $diam(G(R)) = 2$. If $\mathfrak{p}_1 = ((0) :_R u)$ and $\mathfrak{p}_2 = ((0) :_R v)$ for some $u, v \in \bigcap_{i=1}^{2} \mathfrak{p}_i$, then it is proved in Proposition 3.13 that $diam(G(R)) = 3$ and $r(G(R)) = 2$. If $\mathfrak{p}_1 = ((0) :_R u)$ for some $u \in \bigcap_{i=1}^{2} \mathfrak{p}_i$ and $\mathfrak{p}_2 I \neq (0)$ for any non-zero ideal I of R with $I \subseteq \bigcap_{i=1}^{2} \mathfrak{p}_i$, then it is shown in Proposition 3.14 that $diam(G(R)) = 2$ and $r(G(R)) = 1$. For a non-reduced ring R which admits at least three maximal N-primes of (0), it is observed in Proposition 3.16 that $G(R)$ is connected and $diam(G(R)) = 2$. Several examples are given to illustrate the results proved in this section (see Examples 3.8, 3.12, 3.15, and 3.17).

2. Results

Let R be a reduced ring such that R is not an integral domain. We prove in this section that $G(R)$ is connected with $diam(G(R)) \leq 2$. We characterize R such that $G(R)$ is complete. The aim of this section is also to characterize R such that $r(G(R)) = 1$.

Lemma 2.1. *Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. Let $I, J \in \mathbb{A}(R)^*$ be such that $I \neq J$, $I^2 \neq (0)$, and $J^2 \neq (0)$. Then in $G(R)$, there exists a path of length at most two between I and J.*

Proof. We can assume that I and J are not adjacent in $G(R)$. Then $IJ = (0)$ and $I + J \in \mathbb{A}(R)$. Since $I^2 \neq (0)$ and $J^2 \neq (0)$, it follows that $I(I + J) = I^2 \neq (0)$ and $J(I + J) = J^2 \neq (0)$. Hence, I and $I + J$ (respectively, J and $I + J$) are adjacent in $G(R)$. Therefore, with $A = I + J$, $I - A - J$ is a path of length two between I and J in $G(R)$. $\square$

Corollary 2.2. *Let R be a reduced ring which is not an integral domain. Then $G(R)$ is connected and $diam(G(R)) \leq 2$.*

Proof. Let $I, J \in \mathbb{A}(R)^*$ be such that $I \neq J$. Since R is reduced, it follows that $I^2 \neq (0)$ and $J^2 \neq (0)$. We know from Lemma 2.1 that in $G(R)$, there exists a path of length at most two between I and J. Therefore, we obtain that $G(R)$ is connected and $diam(G(R)) \leq 2$. $\square$

Lemma 2.3. *Let R be a reduced ring which is not an integral domain. If $Min(R) \not\subseteq \mathbb{A}(R)$, then $diam(G(R)) = 2$.*

Proof. We know from Corollary 2.2 that $G(R)$ is connected and $diam(G(R)) \leq 2$. Since R is not an integral domain, there exist $x, y \in R\backslash\{0\}$ such that $xy = 0$. We are assuming that $Min(R) \not\subseteq \mathbb{A}(R)$. Let $\mathfrak{p} \in Min(R)$ be such that $\mathfrak{p} \notin \mathbb{A}(R)$. Then $\mathfrak{p}x \neq (0)$ and $\mathfrak{p}y \neq (0)$. Hence, $\mathfrak{p} \not\subseteq ((0) :_R x) \cup ((0) :_R y)$. Therefore, there exists $p \in \mathfrak{p}$ such that $px \neq 0$ and $py \neq 0$. Let us denote Rpx by I and Rpy by J. From $xy = 0$, it follows that $IJ = (0)$ and so, $I, J \in \mathbb{A}(R)$. As R is reduced, $I^2 \neq (0)$ and $J^2 \neq (0)$. Therefore, from $IJ = (0)$, we get that $I \neq J$. We know from [21, Theorem 84] that $\mathfrak{p} \subseteq Z(R)$. Hence, $p \in Z(R)$ and so, $Rp \in \mathbb{A}(R)$. Observe that $I + J \subseteq Rp$. Hence, $I + J \in \mathbb{A}(R)$. From $IJ = (0)$ and $I + J \in \mathbb{A}(R)$, it follows that I and J are not adjacent in $G(R)$. Hence, $d(I, J) \geq 2$ in $G(R)$ and therefore, $diam(G(R)) \geq 2$. This proves that $diam(G(R)) = 2$. $\square$

Lemma 2.4. *Let R be a reduced ring which is not an integral domain. If $G(R)$ is complete, then $|Min(R)| = 2$.*

Proof. It is known that each prime ideal of a ring T (reduced or not) contains a minimal prime ideal of T [21, Theorem 10]. Since R is reduced, $nil(R) = (0)$. Hence, it follows from [9, Proposition 1.8] and [21, Theorem 10] that $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$. From R is not an integral domain, we obtain that $|Min(R)| \geq 2$.

We are assuming that $G(R)$ is complete. Hence, we obtain from Lemma 2.3 that $Min(R) \subseteq \mathbb{A}(R)$. Suppose that $|Min(R)| \geq 3$. Let $\{\mathfrak{p}_i \mid i \in \{1, 2, 3\}\} \subseteq Min(R)$. Now for each $i \in \{1, 2, 3\}$, there exists $x_i \in R\backslash\{0\}$ such that $\mathfrak{p}_i x_i = (0)$. Let $i \in \{1, 2, 3\}$. From $x_i^2 \neq 0$, it follows that $x_i \notin \mathfrak{p}_i$. Since distinct minimal prime ideals are not comparable under the inclusion relation, it follows from $\mathfrak{p}_i x_i = (0)$ that $x_i \in \mathfrak{p}$ for any $\mathfrak{p} \in Min(R)\backslash\{\mathfrak{p}_i\}$. In particular, $x_i \in \mathfrak{p}_j$ for all $j \in \{1, 2, 3\}\backslash\{i\}$. Hence, $x_i x_j = 0$ for all distinct $i, j \in \{1, 2, 3\}$. Let $I_1 = Rx_1$ and $I_2 = Rx_2$. Observe that $I_1, I_2 \in \mathbb{A}(R)^*$ with $I_1 \neq I_2$. From $x_1 x_2 = 0$ and $(x_1 + x_2)x_3 = 0$, we get that $I_1 I_2 = (0)$ and $I_1 + I_2 \in \mathbb{A}(R)$. Therefore, I_1 and I_2 are not adjacent in $G(R)$. This is in contradiction to the assumption $G(R)$ is complete. Hence, we obtain that $|Min(R)| \leq 2$ and so, $|Min(R)| = 2$. $\square$

Theorem 2.5. *Let R be a reduced ring which is not an integral domain. The following statements are equivalent:*
(1) *$G(R)$ is complete.*
(2) *$|Min(R)| = 2$.*

Proof. (1) $\Rightarrow$ (2) We are assuming that $G(R)$ is complete. Therefore, we obtain from Lemma 2.4 that $|Min(R)| = 2$.
(2) $\Rightarrow$ (1) Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Since R is reduced, it follows that $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$ and hence, $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$. It is clear that any annihilating ideal of a ring (reduced or not) is contained in its set of all zero-divisors. From $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$, it follows that if $A \in \mathbb{A}(R)^*$, then $A \subseteq \mathfrak{p}_i$ for some $i \in \{1, 2\}$ and such an i is necessarily unique, since $\bigcap_{j=1}^{2} \mathfrak{p}_j = (0)$. Let $I, J \in \mathbb{A}(R)^*$ with $I \neq J$. If both I and J are contained in $\mathfrak{p}_1$, then $I \not\subseteq \mathfrak{p}_2$, $J \not\subseteq \mathfrak{p}_2$ and so, $IJ \not\subseteq \mathfrak{p}_2$. Hence, $IJ \neq (0)$ and therefore, I and J are adjacent in $G(R)$. Similarly, if $I \cup J \subseteq \mathfrak{p}_2$, then neither I nor J is contained in $\mathfrak{p}_1$ and so, $IJ \not\subseteq \mathfrak{p}_1$. Hence, $IJ \neq (0)$ and therefore, I and J are adjacent in $G(R)$. Suppose that $I \cup J \not\subseteq \mathfrak{p}_i$ for each $i \in \{1, 2\}$. Then $I + J \not\subseteq \bigcup_{i=1}^{2} \mathfrak{p}_i = Z(R)$. Therefore, $I + J \notin \mathbb{A}(R)$. Hence, I and J are adjacent in $G(R)$. This shows that $G(R)$ is complete. $\square$

Recall that a ring R is said to be *von Neumann regular* if given $a \in R$, there exists $b \in R$ such that $a = a^2 b$ [16, Exercise 16, page 111]. Let R be a von Neumann regular ring which is not a field. Let a be a non-zero non-unit of R. Now, there exists $b \in R$ such that $a = a^2 b$. This implies that $ab = a^2 b^2 = (ab)^2$. Let $e = ab$. Then e is an idempotent element of R with $e \notin \{0, 1\}$. We know from $(a) \Leftrightarrow (d)$ of [16, Exercise 16, page 111] that a ring R is von Neumann regular if and only if R is reduced and $dim R = 0$. For a ring R, we denote the set of all proper ideals of R by $\mathbb{I}(R)$ and we denote $\mathbb{I}(R) \backslash \{(0)\}$ by $\mathbb{I}(R)^*$.

Corollary 2.6. *Let R be a von Neumann regular ring which is not a field. The following statements are equivalent:*
(1) *$G(R)$ is complete.*
(2) *$R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$.*

Proof. By hypothesis, R is von Neumann regular which is not a field. Notice that R is reduced and $dim R = 0$.
(1) $\Rightarrow$ (2) We are assuming that $G(R)$ is complete. Hence, we obtain from (1) $\Rightarrow$ (2) of Theorem 2.5 that $|Min(R)| = 2$. Since $dim R = 0$, we get that $Min(R) = Max(R)$. Let $Min(R) = Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. From R is reduced, it follows that $\bigcap_{i=1}^{2} \mathfrak{m}_i = (0)$. Hence, by [9, Proposition 1.10(ii) and (iii)] that the mapping $f : R \to \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2}$ given by $f(r) = (r + \mathfrak{m}_1, r + \mathfrak{m}_2)$ is an isomorphism of rings. Let $i \in \{1, 2\}$. With $F_i = \frac{R}{\mathfrak{m}_i}$, we obtain that F_i is a field and $R \cong F_1 \times F_2$ as rings.
(2) $\Rightarrow$ (1) We are assuming that $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. Let us denote the ring $F_1 \times F_2$ by T. Observe that $\mathbb{I}(T)^* = \mathbb{A}(T)^* = \{(0) \times F_2, F_1 \times (0)\}$. It is clear that $((0) \times F_2) + (F_1 \times (0)) = F_1 \times F_2 \notin \mathbb{A}(T)$.

Hence, $(0) \times F_2$ and $F_1 \times (0)$ are adjacent in $G(T)$ and so, $G(T)$ is complete. From $R \cong T$ as rings, we obtain that $G(R)$ is complete. $\square$

Let R be a reduced ring which is not an integral domain. We prove in Theorem 2.9 that $r(G(R)) = 1$ if and only if there exists at least one $\mathfrak{p} \in Min(R)$ such that $\mathfrak{p} \in \mathbb{A}(R)^*$. We use Lemma 2.7 and Proposition 2.8 in the proof of Theorem 2.9.

Lemma 2.7. *Let R be a reduced ring which is not an integral domain. Let $J \in \mathbb{A}(R)^*$. The following statements are equivalent:*
(1) *J and I are adjacent in $G(R)$ for any $I \in \mathbb{A}(R)^*$ with $I \neq J$.*
(2) *For any $a \in Z(R)^* \backslash J$, either $Ja \neq (0)$ or $J + Ra \notin \mathbb{A}(R)$.*

Proof. We know from Corollary 2.2 that $G(R)$ is connected and $diam(G(R)) \leq 2$.
$(1) \Rightarrow (2)$ For this part of the proof, we do not need the assumption that R is reduced. Let $a \in Z(R)^*$ be such that $a \notin J$. It is clear that $Ra \in \mathbb{A}(R)^*$ and $Ra \neq J$. Since J and Ra are adjacent in $G(R)$, it follows that either $Ja \neq (0)$ or $J + Ra \notin \mathbb{A}(R)$.
$(2) \Rightarrow (1)$ Let $I \in \mathbb{A}(R)^*$ be such that $I \neq J$. Suppose that $JI = (0)$. Since R is reduced, $I^2 \neq (0)$. From $JI = (0)$, it follows that $I \not\subseteq J$. Let $a \in I \backslash J$. Since $I \subseteq Z(R)$, we get that $a \in Z(R)^*$. As $Ja = (0)$, it follows that $J + Ra \notin \mathbb{A}(R)$. From $J + Ra \subseteq J + I$, we obtain that $J + I \notin \mathbb{A}(R)$. This shows that J and I are adjacent in $G(R)$. $\square$

Let $G = (V, E)$ be a simple graph with $|V| \geq 2$. If there exists $v \in V$ such that v is adjacent to any $w \in V$ in G with $w \neq v$, then it is well-known that G is connected and $diam(G) \leq 2$. We include a proof of it for the sake of completeness. Let $v_1, v_2 \in V$ be such that $v_1 \neq v_2$. Suppose that v_1 and v_2 are not adjacent in G. As v is adjacent to any $w \in V$ in G with $w \neq v$, we get that $v_1 - v - v_2$ is a path of length two between v_1 and v_2 in G. Hence, G is connected and $diam(G) \leq 2$.

Proposition 2.8. *Let R be a ring such that $|V(G(R)) = \mathbb{A}(R)^*| \geq 2$. Suppose that $G(R)$ is connected. Let $J \in \mathbb{A}(R)^*$ be such that $e(J) = 1$ in $G(R)$. Then there exists $\mathfrak{p} \in Min(R)$ such that $J \subseteq \mathfrak{p}$, $\mathfrak{p} \in \mathbb{A}(R)^*$, and $e(\mathfrak{p}) = 1$ in $G(R)$.*

Proof. We know from [12, Theorem 2.1] that $\mathbb{AG}(R)$ is connected. By hypothesis, $|\mathbb{A}(R)^*| \geq 2$. Hence, there exists $I \in \mathbb{A}(R)^*$ with $I \neq J$ such that $JI = (0)$. From $e(J) = 1$ in $G(R)$, we get that $J + I \notin \mathbb{A}(R)$. Therefore, it follows from [13, Lemma 1.5] that J is not nilpotent.

Let $\mathcal{C} = \{A \in \mathbb{A}(R)^* \mid J \subseteq A\}$. As $J \in \mathbb{A}(R)^*$, there exists $r \in R \backslash \{0\}$ such that $Jr = (0)$. Let $A \in \mathcal{C}$. We claim that $Ar = (0)$. Suppose that $Ar \neq (0)$. As $A \in \mathbb{A}(R)^*$, it is clear that $Ar \in \mathbb{A}(R)^*$. Observe that $J \neq Ar$. For if $J = Ar$, then $J^2 = J(Ar) = (0)$. This is in contradiction to the fact that J is not nilpotent. Therefore, $J \neq Ar$. It is clear that $J(Ar) = (0)$ and $J + Ar \subseteq A$ and so, $J + Ar \in \mathbb{A}(R)$. This implies that J and Ar are not adjacent in $G(R)$. This is in contradiction to the hypothesis that $e(J) = 1$ in $G(R)$. Therefore, for each $A \in \mathcal{C}$, $Ar = (0)$. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a chain in $\mathcal{C}$. Let $A = \bigcup_{\alpha \in \Lambda} A_\alpha$. As for any

$\alpha, \beta \in \Lambda$, either $A_\alpha \subseteq A_\beta$ or $A_\beta \subseteq A_\alpha$, it follows that A is an ideal of R. From $A_\alpha r = (0)$ for each $\alpha \in \Lambda$, we get that $Ar = (0)$. As $J \subseteq A_\alpha$ for each $\alpha \in \Lambda$, it follows that $J \subseteq A$. Indeed, $J \subseteq \bigcap\limits_{\alpha \in \Lambda} A_\alpha$. Hence, $A \in \mathcal{C}$ and as $A_\alpha \subseteq A$ for each $\alpha \in \Lambda$, we obtain that A is an upper bound for the chain $\{A_\alpha\}_{\alpha \in \Lambda}$. This shows that any chain in $\mathcal{C}$ has an upper bound in $\mathcal{C}$. Therefore, we obtain from Zorn's lemma that $\mathcal{C}$ admits a maximal element. Let $\mathfrak{p} \in \mathcal{C}$ be a maximal element of $\mathcal{C}$. From $\mathfrak{p} \in \mathcal{C}$, it follows that $J \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \mathbb{A}(R)^*$. Observe that $\mathfrak{p}r = (0)$. It is clear that $\mathfrak{p} \neq R$. Let $a, b \in R$ be such that $ab \in \mathfrak{p}$. Suppose that $a \notin \mathfrak{p}$. Now, $J \subseteq \mathfrak{p} \subset \mathfrak{p} + Ra$. Since $\mathfrak{p}$ is a maximal element of $\mathcal{C}$, it follows that $\mathfrak{p} + Ra \notin \mathcal{C}$. Therefore, $\mathfrak{p} + Ra \notin \mathbb{A}(R)$. From $\mathfrak{p}r = (0)$ and $ab \in \mathfrak{p}$, we obtain that $abr = 0$. Hence, $(\mathfrak{p} + Ra)br = (0)$. It follows from $\mathfrak{p} + Ra \notin \mathbb{A}(R)$ that $br = 0$. Therefore, $(\mathfrak{p} + Rb)r = (0)$ and so, $\mathfrak{p} + Rb \in \mathbb{A}(R)^*$. Now, $J \subseteq \mathfrak{p} \subseteq \mathfrak{p} + Rb$. Since $\mathfrak{p}$ is a maximal element of $\mathcal{C}$, we get that $\mathfrak{p} = \mathfrak{p} + Rb$. Therefore, $b \in \mathfrak{p}$. This proves that $\mathfrak{p} \in Spec(R)$. From $Jr = (0)$ and J is not nilpotent, it follows that $J \neq Rr$. As $e(J) = 1$ in $G(R)$, $Rr \in \mathbb{A}(R)^*$ with $Jr = (0)$, we get that $J + Rr \notin \mathbb{A}(R)$. Hence, we obtain from [13, Lemma 1.5] that $r \notin nil(R)$. Since $nil(R) = \bigcap\limits_{\mathfrak{q} \in Min(R)} \mathfrak{q}$, it follows that $r \notin \mathfrak{q}$ for some $q \in Min(R)$. Now, $\mathfrak{p}r = (0) \subseteq \mathfrak{q}$. As $r \notin \mathfrak{q}$, we obtain that $\mathfrak{p} \subseteq \mathfrak{q}$ and so, $\mathfrak{p} = \mathfrak{q} \in Min(R)$. We next verify that $e(\mathfrak{p}) = 1$ in $G(R)$. Let $I \in \mathbb{A}(R)^*$ be such that $I \neq \mathfrak{p}$. If $\mathfrak{p}I \neq (0)$, then $\mathfrak{p}$ and I are adjacent in $G(R)$. Suppose that $\mathfrak{p}I = (0)$. As $J \subseteq \mathfrak{p}$, it follows that $JI = (0)$. From J is not nilpotent, it follows that $I \neq J$. As $e(J) = 1$ in $G(R)$, we get that $J + I \notin \mathbb{A}(R)$. From $J + I \subseteq \mathfrak{p} + I$, we obtain that $\mathfrak{p} + I \notin \mathbb{A}(R)$. Hence, $\mathfrak{p}$ and I are adjacent in $G(R)$. This proves that $e(\mathfrak{p}) = 1$ in $G(R)$. □

Theorem 2.9. *Let R be a ring such that $|\mathbb{A}(R)^*| \geq 2$. Suppose that $G(R)$ is connected. The following statements are equivalent:*
(1) $r(G(R)) = 1$.
(2) *There exists $\mathfrak{p} \in Min(R)$ such that $\mathfrak{p} \in \mathbb{A}(R)^*$ and $e(\mathfrak{p}) = 1$ in $G(R)$.*

Proof. (1) $\Rightarrow$ (2) We are assuming that $r(G(R)) = 1$. Hence, there exists $J \in \mathbb{A}(R)^*$ such that $e(J) = 1$ in $G(R)$. We know from Proposition 2.8 that there exists $\mathfrak{p} \in Min(R)$ such that $J \subseteq \mathfrak{p}$, $\mathfrak{p} \in \mathbb{A}(R)^*$, and $e(\mathfrak{p}) = 1$ in $G(R)$.
(2) $\Rightarrow$ (1) This is clear. □

Corollary 2.10. *Let R be a reduced ring which is not an integral domain. The following statements are equivalent:*
(1) $r(G(R)) = 1$.
(2) *There exists $\mathfrak{p} \in Min(R)$ such that $\mathfrak{p} \in \mathbb{A}(R)^*$ with $e(\mathfrak{p}) = 1$ in $G(R)$.*
(3) *There exists $\mathfrak{p} \in Min(R)$ such that $\mathfrak{p} \in \mathbb{A}(R)^*$.*
In particular, if $Min(R)$ is finite, then $r(G(R)) = 1$.

Proof. Since R is not an integral domain, there exist $x, y \in R \backslash \{0\}$ such that $xy = 0$. Let $I = Rx$ and let $J = Ry$. It is clear that $I, J \in \mathbb{A}(R)^*$. As R is reduced, $x^2 \neq 0$ and so, from $xy = 0$, it follows that $I \neq J$. Hence, $|\mathbb{A}(R)^*| \geq 2$. We know from Corollary 2.2 that $G(R)$ is connected and $diam(G(R)) \leq 2$.

First, we verify that if $I \in \mathbb{A}(R)$, then $I \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in Min(R)$. Now, $Ir = (0)$ for some $r \in R\backslash\{0\}$. Since R is reduced, $\bigcap_{\mathfrak{q} \in Min(R)} \mathfrak{q} = (0)$. From $r \neq 0$, we get that $r \notin \mathfrak{q}$ for some $\mathfrak{q} \in Min(R)$. As $Ir = (0) \subseteq \mathfrak{q}$ and from $r \notin \mathfrak{q}$, we obtain that $I \subseteq \mathfrak{q}$.

(1) $\Rightarrow$ (2) This follows from (1) $\Rightarrow$ (2) of Theorem 2.9.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) We are assuming that there exists $\mathfrak{p} \in Min(R)$ such that $\mathfrak{p} \in \mathbb{A}(R)^*$. We verify that $e(\mathfrak{p}) = 1$ in $G(R)$. Let $a \in Z(R)^*$ be such that $a \notin \mathfrak{p}$. We claim that $\mathfrak{p} + Ra \notin \mathbb{A}(R)$. For if $\mathfrak{p} + Ra \in \mathbb{A}(R)$, then there exists $\mathfrak{q} \in Min(R)$ such that $\mathfrak{p} + Ra \subseteq \mathfrak{q}$. This implies that $\mathfrak{p} \subseteq \mathfrak{q}$ and so, $\mathfrak{p} = \mathfrak{q}$. Hence, we arrive at $Ra \subseteq \mathfrak{p}$ and this is in contradiction to the assumption that $a \notin \mathfrak{p}$. Therefore, $\mathfrak{p} + Ra \notin \mathbb{A}(R)$ and so, it follows from (2) $\Rightarrow$ (1) of Lemma 2.7 that $e(\mathfrak{p}) = 1$ in $G(R)$. Hence, $r(G(R)) = 1$.

We next verify the in particular part of this corollary. We are assuming that $Min(R)$ is finite. Let $|Min(R)| = n$. Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, \ldots, n\}\}$. Since $\bigcap_{i=1}^{n} \mathfrak{p}_i = (0)$ and R is not an integral domain, we get that $n \geq 2$. Let $i \in \{1, 2, \ldots, n\}$. Since distinct minimal prime ideals are not comparable under the inclusion relation, it follows from [9, Proposition 1.11(ii)] that there exists $x_i \in (\bigcap_{j \in \{1,2,\ldots,n\}\backslash\{i\}} \mathfrak{p}_j)\backslash\mathfrak{p}_i$. Observe that $x_i \neq 0$ and $\mathfrak{p}_i x_i = (0)$. This proves that $\mathfrak{p}_i \in \mathbb{A}(R)^*$. Hence, we obtain from (3) $\Rightarrow$ (1) of this corollary that $r(G(R)) = 1$. $\square$

Corollary 2.11. *Let T be a ring. Let $n \geq 2$. Let $\{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$ be a set of n incomparable prime ideals of T. Let $I = \bigcap_{i=1}^{n} \mathfrak{p}_i$. Let $R = \frac{T}{I}$. Then R is reduced and $|Min(R)| = n$. Moreover, $G(R)$ is complete if $n = 2$ and if $n \geq 3$, then $diam(G(R)) = 2$ and $r(G(R)) = 1$.*

Proof. It is clear that $R = \frac{T}{I}$ is reduced and that for each $i \in \{1, 2, \ldots, n\}$, $\mathfrak{P}_i = \frac{\mathfrak{p}_i}{I} \in Spec(R)$. Since by hypothesis, $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are not comparable under the inclusion relation for all distinct $i, j \in \{1, 2, \ldots, n\}$, it follows that $\mathfrak{P}_i$ and $\mathfrak{P}_j$ are not comparable under the inclusion relation. Let $\mathfrak{P} \in Spec(R)$. Then $\mathfrak{P} = \frac{\mathfrak{p}}{I}$ for some $\mathfrak{p} \in Spec(T)$ with $\mathfrak{p} \supseteq I$. It follows from [9, Proposition 1.11(ii)] that $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some $i \in \{1, 2, \ldots, n\}$ and so, $\mathfrak{P} \supseteq \frac{\mathfrak{p}_i}{I}$. It now follows that $Min(R) = \{\mathfrak{P}_i \mid i \in \{1, 2, \ldots, n\}\}$. Hence, $|Min(R)| = n$. If $n = 2$, then we know from (2) $\Rightarrow$ (1) of Theorem 2.5 that $G(R)$ is complete. If $n \geq 3$, then it follows from Corollary 2.2 and (1) $\Rightarrow$ (2) of Theorem 2.5 that $diam(G(R)) = 2$. Since $Min(R)$ is finite, we obtain from the in particular part of Corollary 2.10 that $r(G(R)) = 1$. $\square$

Let $T = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K. We know from [9, Corollary 7.6] that T is Noetherian and we know from [23, Theorem 3, page 281] that $dimT = 2$. As T is a unique factorization domain (UFD), it follows that each height one prime ideal of T is principal. Observe that we obtain from [21, Theorem 144] that T has an infinite number of non-associate prime elements. Let $n \geq 2$. Let $p_1, p_2, \ldots, p_n$ be non-associate prime elements of T. Let $I = \bigcap_{i=1}^{n} Tp_i$ and let $R = \frac{T}{I}$. Then it follows from Corollary 2.11 that R is reduced, $G(R)$ is complete if $n = 2$, and if $n \geq 3$, then $diam(G(R)) = 2$ and $r(G(R)) = 1$.

In Example 2.13, we provide a von Neumann regular ring R such that $Min(R) \cap \mathbb{A}(R)^* = \emptyset$ and $G(R)$ is connected with $diam(G(R)) = r(G(R)) = 2$. We use Lemma 2.12 in the verification of Example 2.13.

Lemma 2.12. *Let R be a von Neumann regular ring which is not a field. Let $\mathfrak{m} \in Max(R)$. Then the following statements are equivalent:*
(1) $\mathfrak{m} \in \mathbb{A}(R)^*$.
(2) $\mathfrak{m}$ *is principal.*

Proof. Let a be any element of R. Since R is von Neumann regular, we obtain from (1) $\Rightarrow$ (3) of [16, Exercise 29, page 113] that $a = ue$, where $u \in U(R)$ and e is an idempotent element of R.
(1) $\Rightarrow$ (2) We are assuming that $\mathfrak{m} \in \mathbb{A}(R)^*$. Hence, there exists $r \in R\backslash\{0\}$ such that $\mathfrak{m}r = (0)$. Notice that $r = ue$ for some $u \in U(R)$ and $e = e^2$. From $r \neq 0$, we get that $e \neq 0$. Since $r \notin U(R)$, it follows that $e \neq 1$. Thus $e \notin \{0, 1\}$. Now, $\mathfrak{m}e = (0)$. It follows from $e \neq 0$ that $e \notin \mathfrak{m}$. Therefore, $1 - e \in \mathfrak{m}$ and so, $R(1 - e) \subseteq \mathfrak{m}$. Let $m \in \mathfrak{m}$. Then $m = m1 = m(e + 1 - e) = m(1 - e) \in R(1 - e)$. This implies that $\mathfrak{m} \subseteq R(1 - e)$ and so, $\mathfrak{m} = R(1 - e)$ is principal.
(2) $\Rightarrow$ (1) We are assuming that $\mathfrak{m}$ is principal. Let $m \in \mathfrak{m}$ be such that $\mathfrak{m} = Rm$. Since $dim R = 0$, we know from [21, Theorem 84] that $\mathfrak{m} \subseteq Z(R)$. Hence, there exists $r \in R\backslash\{0\}$ such that $mr = 0$ and so, $\mathfrak{m}r = (0)$. Therefore, $\mathfrak{m} \in \mathbb{A}(R)^*$. One can also apply the following argument to show that $\mathfrak{m} \in \mathbb{A}(R)^*$. Since R is von Neumann regular, $m = ue$, where $u \in U(R)$ and e is an idempotent element of R with $e \notin \{0, 1\}$. Now, $1 - e \neq 0$ and $m(1 - e) = 0$. Hence, $\mathfrak{m}(1 - e) = (0)$. This shows that $\mathfrak{m} \in \mathbb{A}(R)^*$. ☐

Example 2.13. Let L be the field of algebraic numbers (that is, L is the algebraic closure of $\mathbb{Q}$) and let A be the ring of all algebraic integers (that is, A is the integral closure of $\mathbb{Z}$ in L). Let $R = \frac{A}{\sqrt{2A}}$. Then R is a von Neumann regular ring, $Min(R) \cap \mathbb{A}(R)^* = \emptyset$, and $G(R)$ is connected with $diam(G(R)) = r(G(R)) = 2$.

Proof. We know from [16, Proposition 42.8] that A is a one-dimensional Prüfer domain with the following properties: (1) Each non-unit of A belongs to uncountably many maximal ideals of A and (2) If I is any ideal of A with $I = \sqrt{I}$, then I is idempotent (that is, $I = I^2$). As 2 is not a unit in A, it follows that 2 belongs to uncountably many maximal ideals of A. Therefore, the ring $R = \frac{A}{\sqrt{2A}}$ is such that $dim R = 0$ and $Max(R)$ is uncountable. From $dim R = 0$, it follows that $Max(R) = Min(R)$. We know from [9, Exercise 1.13(ii), page 9] that $\sqrt{\sqrt{2A}} = \sqrt{2A}$. Hence, we get that R is reduced. Thus $dim R = 0$ and R is reduced. Therefore, R is a von Neumann regular ring. We next verify that $Min(R) \cap \mathbb{A}(R)^* = \emptyset$. Let $\mathfrak{p} \in Min(R) = Max(R)$. Then there exists

$\mathfrak{m} \in Max(A)$ such that $2 \in \mathfrak{m}$ and $\mathfrak{p} = \frac{\mathfrak{m}}{\sqrt{2A}}$. We claim that $\mathfrak{p} \notin \mathbb{A}(R)^*$. For if $\mathfrak{p} \in \mathbb{A}(R)^*$, then we know from (1) $\Rightarrow$ (2) of Lemma 2.12 that $\mathfrak{p}$ is principal. Let $m \in \mathfrak{m}$ be such that $\mathfrak{p} = R(m + \sqrt{2A})$. This implies that $\mathfrak{m} = \sqrt{mA + 2A}$. We know from [22, page 86] that A is a Bezout domain. Therefore, there exists $c \in mA + 2A$ such that $mA + 2A = cA$ and so, we get that $\mathfrak{m} = \sqrt{cA}$. This is impossible, since we know from [16, Proposition 42.8(1)] that c belongs to uncountably many maximal ideals of A. Hence, we obtain that $Min(R) \cap \mathbb{A}(R)^* = \emptyset$. As R is reduced, we know from Corollary 2.2 that $G(R)$ is connected and $diam(G(R)) \leq 2$. Since R has an uncountable number of minimal prime ideals, it follows from (1) $\Rightarrow$ (2) of Theorem 2.5 that $diam(G(R)) \geq 2$ and so, $diam(G(R)) = 2$. Since no minimal prime ideal of R is an annihilating ideal of R, it follows from (1) $\Rightarrow$ (3) of Corollary 2.10 that $r(G(R)) \geq 2$ and so, $r(G(R)) = 2$. Therefore, $diam(G(R)) = r(G(R)) = 2$. □

The reduced ring R given in Example 2.14 below is due to R. Gilmer and W. Heinzer and it appeared in [17, page 16]. The reduced ring R is such that R has a unique maximal N-prime of its zero ideal, R has an infinite number of minimal prime ideals and each one of them is an annihilating ideal of R, $diam(G(R)) = 2$, and $r(G(R)) = 1$.

Example 2.14. Let $\{X_i\}_{i=1}^{\infty}$ be a set of indeterminates over a field K. Let $D = \bigcup_{n=1}^{\infty} K[[X_1, \ldots, X_n]]$, where for each $n \geq 1$, $K[[X_1, \ldots, X_n]]$ is the power series ring in $X_1, \ldots, X_n$ over K. Let I be the ideal of D generated by $\{X_iX_j \mid i, j \in \mathbb{N}, i \neq j\}$. Let $R = \frac{D}{I}$. Then R is a reduced ring, $Z(R)$ is an ideal of R, R has an infinite number of minimal prime ideals and each one of them is an annihilating ideal of R, $diam(G(R)) = 2$, and $r(G(R)) = 1$.

Proof. For each $i \in \mathbb{N}$, it is convenient to denote $X_i + I$ by x_i. It was already observed in [17, Example, page 16] that R is a reduced ring, $dimR = 1$, R is quasilocal with $\mathfrak{m} = \sum_{n=1}^{\infty} Rx_n$ as its unique maximal ideal, $\{\mathfrak{p}_i\}_{i=1}^{\infty}$ is the set of all minimal prime ideals of R, where for each $i \in \mathbb{N}$, $\mathfrak{p}_i$ is the ideal of R generated by $\{x_j \mid j \in \mathbb{N}\backslash\{i\}\}$. It was also remarked in [17, page 16] that $Z(R) = \mathfrak{m}$. Observe that for each $i \in \mathbb{N}$, $\mathfrak{p}_i x_i = (0 + I)$. As $x_i \neq 0 + I$, it follows that $\mathfrak{p}_i \in \mathbb{A}(R)^*$ for each $i \in \mathbb{N}$. It follows from Corollary 2.2 and (1) $\Rightarrow$ (2) of Theorem 2.5 that $diam(G(R)) = 2$. We know from (3) $\Rightarrow$ (1) of Corollary 2.10 that $r(G(R)) = 1$. □

3. Some more results

Let R be a ring which is not reduced. The aim of this section is to discuss some results regarding the connectedness of $G(R)$.

Lemma 3.1. *Let R be a ring. If R is not reduced and if $|\mathbb{A}(R)^*| \geq 2$, then there exist $I, J \in \mathbb{A}(R)^*$ with $I \neq J$ such that I and J are not adjacent in $G(R)$.*

Proof. By hypothesis, R is not reduced. Hence, it is possible to find $x \in R\backslash\{0\}$ such that $x^2 = 0$. Let us denote Rx by I. It is clear that $I \in \mathbb{A}(R)^*$. By hypothesis, $|\mathbb{A}(R)^*| \geq 2$. We know from [12, Theorem 2.1] that $\mathbb{AG}(R)$ is connected. Therefore, there exists $J \in \mathbb{A}(R)^*$ with $J \neq I$ such that $IJ = (0)$. Since I is a nilpotent ideal of R (indeed, $I^2 = (0)$) and $J \in \mathbb{A}(R)^*$, we obtain from [13, Lemma 1.5] that $I + J \in \mathbb{A}(R)$. From $IJ = (0)$ and $I + J \in \mathbb{A}(R)$, we obtain that I and J are not adjacent in $G(R)$. □

Lemma 3.2. *Let R be a ring which is not reduced. Suppose that $|\mathbb{A}(R)^*| \geq 2$. If there exists a maximal N-prime $\mathfrak{p}$ of (0) in R such that $\mathfrak{p}$ is not a B-prime of (0) in R, then $G(R)$ is connected and $diam(G(R)) = 2$.*

Proof. We know from [26, Lemma 2.2] that $(\mathbb{AG}(R))^c$ is connected and $diam((\mathbb{AG}(R))^c) \leq 2$. Since $(\mathbb{AG}(R))^c$ is a spanning subgraph of $G(R)$, it follows that $G(R)$ is connected and $diam(G(R)) \leq 2$. As R is not reduced and $|\mathbb{A}(R)^*| \geq 2$, we obtain from Lemma 3.1 that there exist $I, J \in \mathbb{A}(R)^*$ such that $d(I, J) \geq 2$ in $G(R)$. Therefore, $diam(G(R)) = 2$. □

Lemma 3.3. *Let R be a ring which is not reduced. If $nil(R) \notin \mathbb{A}(R)$, then $G(R)$ is connected and $diam(G(R)) = r(G(R)) = 2$.*

Proof. We use f.g. for finitely generated. If I is a f.g. ideal of R with $I \subseteq nil(R)$, then I is nilpotent and hence, $I \in \mathbb{A}(R)$. By hypothesis, $nil(R) \notin \mathbb{A}(R)$. Therefore, $nil(R)$ is not f.g. and so, R contains an infinite number of annihilating ideals.

Let $\{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda}$ be the set of all maximal N-primes of (0) in R. As $nil(R) \subseteq \mathfrak{p}$ for any $\mathfrak{p} \in Spec(R)$ and $nil(R) \notin \mathbb{A}(R)$, it follows that $\mathfrak{p}_\alpha \notin \mathbb{A}(R)$ for each $\alpha \in \Lambda$. Therefore, $\mathfrak{p}_\alpha$ is not a B-prime of (0) in R for each $\alpha \in \Lambda$. It follows from Lemma 3.2 that $G(R)$ is connected and $diam(G(R)) = 2$. We now show that $r(G(R)) = 2$. Since $diam(G(R)) = 2$, we get that $r(G(R)) \leq 2$. Let $I \in \mathbb{A}(R)^*$. From the assumption that $nil(R) \notin \mathbb{A}(R)$, it follows that $\mathfrak{p} \notin \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. Hence, we obtain from Proposition 2.8 that $e(I) \geq 2$ in $G(R)$. We also provide a direct argument for the fact that $e(I) \geq 2$ in $G(R)$. Now, there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. Since $nil(R) \notin \mathbb{A}(R)$, we obtain that $nil(R) \not\subseteq ((0) :_R r) \cup ((0) :_R I)$. Therefore, it is possible to find $x \in nil(R)$ such that $rx \neq 0$ and $Ix \neq (0)$. Let $n \geq 2$ be least with the property that $rx^n = 0$. Then $rx^{n-1} \neq 0$. Let $J = R(rx^{n-1})$. Then $J \in \mathbb{A}(R)^*$ and $J^2 = (0)$. We claim that $I \neq J$. For if $I = J$, then $Ix = Jx = (0)$. This is in contradiction to the fact that $Ix \neq (0)$. Therefore, $I \neq J$. Observe that $IJ = (0)$, $(I + J)J = (0)$, and so, $I + J \in \mathbb{A}(R)$. This implies that I and J are not adjacent in $G(R)$. Hence, $d(I, J) \geq 2$ in $G(R)$ and so, $e(I) \geq 2$ in $G(R)$. Therefore, $e(I) = 2$ in $G(R)$ for each $I \in \mathbb{A}(R)^*$. Hence, $r(G(R)) = 2$. □

In Example 3.12(3), we provide a non-reduced ring T_1 such that $nil(T_1) \in \mathbb{A}(T_1)$, $G(T_1)$ is connected with $diam(G(T_1)) = 2$ but $r(G(T_1)) = 1$.

Lemma 3.4. *Let R be a ring which is not reduced. Suppose that $|\mathbb{A}(R)^*| \geq 2$ and given $I \in \mathbb{A}(R)^*$, there exists a non-zero $x \in nil(R)$ such that $Ix = (0)$. Then* *given $I \in \mathbb{A}(R)^*$, there exists $A \in \mathbb{A}(R)^*$ with $A \neq I$ such that I and A are not adjacent in $G(R)$.*

Proof. Let $I \in \mathbb{A}(R)^*$. As $|\mathbb{A}(R)^*| \geq 2$ and $\mathbb{AG}(R)$ is connected by [12, Theorem 2.1], we obtain that there exists $J \in \mathbb{A}(R)^*$ with $J \neq I$ such that $IJ = (0)$. If I is nilpotent, then we know from [13, Lemma 1.5] that $I + J \in \mathbb{A}(R)$. Hence, with $A = J$, we obtain that I and A are not adjacent in $G(R)$. Suppose that I is not nilpotent. By hypothesis, there exists a non-zero $x \in nil(R)$ such that $Ix = (0)$. Let $A = Rx$. As A is nilpotent, from the assumption that I is not nilpotent, it follows that $A \neq I$. Now, $IA = (0)$ and $I + A \in \mathbb{A}(R)$. Hence, I and A are not adjacent in $G(R)$. □

Corollary 3.5. *Let R be a ring which is not reduced. Suppose that R has at least one maximal N-prime $\mathfrak{p}$ of (0) such that $\mathfrak{p}$ is not a B-prime of (0) in R and given $I \in \mathbb{A}(R)^*$, there exists a non-zero $x \in nil(R)$ such that $Ix = (0)$. Then $G(R)$ is connected with $diam(G(R)) = r(G(R)) = 2$.*

Proof. Let $x \in nil(R)$, $x \neq 0$. Since $\mathfrak{p}$ is not a B-prime of (0) in R, it follows that $\mathfrak{p}x \neq (0)$. Hence, there exists $y \in \mathfrak{p}$ such that $yx \neq 0$. As $y \in Z(R)^*$ and $yx \neq 0$, it follows that $Ry, Ryx \in \mathbb{A}(R)^*$. We assert that $Ry \neq Ryx$. For if $Ry = Ryx$, then $y = ryx$ for some $r \in R$. This implies that $y(1 - rx) = 0$. As $x \in nil(R)$, it follows from [9, Exercise 1, page 10] that $1 - rx \in U(R)$. From $y \neq 0$, we get that $y(1 - rx) \neq 0$. This is a contradiction and so, $Ry \neq Ryx$. Therefore, $|\mathbb{A}(R)^*| \geq 2$.

Since $\mathfrak{p}$ is not a B-prime of (0) in R, we obtain from Lemma 3.2 that $G(R)$ is connected and $diam(G(R)) = 2$. We next verify that $r(G(R)) = 2$. Let $I \in \mathbb{A}(R)^*$. We claim that $e(I) \geq 2$ in $G(R)$. We know from Lemma 3.4 that there exists $A \in \mathbb{A}(R)^*$ with $A \neq I$ such that $d(I, A) \geq 2$ in $G(R)$. This shows that $e(I) \geq 2$ in $G(R)$ for each $I \in \mathbb{A}(R)^*$. Therefore, $e(I) = 2$ in $G(R)$ for each $I \in \mathbb{A}(R)^*$ and so, $r(G(R)) = 2$. □

Lemma 3.6. *Let R be a ring which is not reduced. Let $\mathfrak{p}$ be a maximal N-prime of (0) in R such that $\mathfrak{p} \in \mathbb{A}(R)$. If $\mathfrak{p}I \neq (0)$ for each non-zero ideal I of R with $I \subseteq nil(R)$, then $G(R)$ is connected with $diam(G(R)) = 2$ and $r(G(R)) = 1$.*

Proof. Let $x \in nil(R)$ with $x \neq 0$. By hypothesis, $\mathfrak{p}x \neq (0)$. Hence, there exists $p \in \mathfrak{p}$ such that $px \neq 0$. Since $\mathfrak{p} \subseteq Z(R)$, we obtain that $Rp, Rpx \in \mathbb{A}(R)^*$. It follows as in the proof of Corollary 3.5 that $Rp \neq Rpx$ and so, $|\mathbb{A}(R)^*| \geq 2$.

Let $I \in \mathbb{A}(R)^*$ be such that $I \neq \mathfrak{p}$. We assert that $\mathfrak{p}$ and I are adjacent in $G(R)$. This is clear if $\mathfrak{p}I \neq (0)$. Suppose that $\mathfrak{p}I = (0)$. Then it follows from the hypothesis that $I \not\subseteq nil(R)$. We claim that $I \not\subseteq \mathfrak{p}$. For if $I \subseteq \mathfrak{p}$, then $I^2 \subseteq \mathfrak{p}I = (0)$ and so, $I^2 = (0)$. This is impossible, since $I \not\subseteq nil(R)$. Therefore, $I \not\subseteq \mathfrak{p}$. We verify that $\mathfrak{p} + I \notin \mathbb{A}(R)$. For if $\mathfrak{p} + I \in \mathbb{A}(R)$, then $\mathfrak{p} + I \subseteq Z(R)$. This implies that $(\mathfrak{p} + I) \cap (R \backslash Z(R)) = \emptyset$. Since $R\backslash Z(R)$ is a multiplicatively closed subset of R, it follows from Zorn's lemma and [21, Theorem 1] that there exists a maximal N-prime $\mathfrak{P}$ of (0) in R such that $\mathfrak{p} + I \subseteq \mathfrak{P}$. Hence, $\mathfrak{p} \subseteq \mathfrak{P}$ and $I \subseteq \mathfrak{P}$. From $\mathfrak{p} \subseteq \mathfrak{P}$, we get that $\mathfrak{p} = \mathfrak{P}$ and so, $I \subseteq \mathfrak{p}$. This is in contradiction to the fact that $I \not\subseteq \mathfrak{p}$. Therefore, $\mathfrak{p} + I \notin \mathbb{A}(R)$. Hence, $\mathfrak{p}$ and I are adjacent in $G(R)$. Thus for any $I \in \mathbb{A}(R)^*$ with $I \neq \mathfrak{p}$, $\mathfrak{p}$ and I are adjacent in $G(R)$.

Let $I_1, I_2 \in \mathbb{A}(R)^*$ be such that $I_1 \neq I_2$. Suppose that I_1 and I_2 are not adjacent in $G(R)$. It is clear that $I_1 - \mathfrak{p} - I_2$ is a path of length two between I_1 and I_2 in $G(R)$. This proves that $G(R)$ is connected and $diam(G(R)) \leq 2$. Since R is not reduced, we obtain from Lemma 3.1 that there exist distinct $I, J \in \mathbb{A}(R)^*$ such that $d(I, J) \geq 2$ in $G(R)$. Therefore, $diam(G(R)) \geq 2$ and so, $diam(G(R)) = 2$. As $\mathfrak{p}$ is adjacent to any $I \in \mathbb{A}(R)^*$ in $G(R)$ with $I \neq \mathfrak{p}$, we get that $e(\mathfrak{p}) = 1$ in $G(R)$. Therefore, $r(G(R)) = 1$. □

Example 3.15(2) provided in this article illustrates Lemma 3.6.

Proposition 3.7. *Let R be a ring which is not reduced. Suppose that R has $\mathfrak{p}$ as its unique maximal N-prime of (0) and $|\mathbb{A}(R)^*| \geq 2$. The following statements are equivalent:*
(1) *$G(R)$ is connected.*
(2) *$\mathfrak{p}$ is not a B-prime of (0) in R.*

Moreover, if either (1) or (2) holds (and hence, both (1) and (2) hold), then $diam(G(R)) = 2$. If $nil(R) \notin \mathbb{A}(R)$, then (2) holds and $r(G(R)) = 2$. Furthermore, if (2) holds and if given $I \in \mathbb{A}(R)^*$, there exists a non-zero $x \in nil(R)$ such that $Ix = (0)$, then $r(G(R)) = 2$.

Proof. (1) ⇒ (2) We are assuming that $G(R)$ is connected. Suppose that $\mathfrak{p}$ is a B-prime of (0) in R. Then there exists $x \in \mathfrak{p}$, $x \neq 0$ such that $\mathfrak{p} = ((0) :_R x)$. Let us denote the ideal Rx by I. It is clear that $Rx \in \mathbb{A}(R)^*$. Let $J \in \mathbb{A}(R)^*$ be such that $J \neq I$. Observe that $J \subseteq Z(R) = \mathfrak{p}$. Hence, $IJ = (0)$, $I + J \subseteq \mathfrak{p}$, and so, $I + J \in \mathbb{A}(R)$. This implies that I is an isolated vertex of $G(R)$. This is impossible, since we are assuming that $G(R)$ is connected and by hypothesis, $|V(G(R)) = \mathbb{A}(R)^*| \geq 2$. Therefore, $\mathfrak{p}$ is not a B-prime of (0) in R.

(2) ⇒ (1) We are assuming that $\mathfrak{p}$ is not a B-prime of (0) in R. Hence, we obtain from Lemma 3.2 that $G(R)$ is connected and $diam(G(R)) = 2$.

Suppose that $nil(R) \notin \mathbb{A}(R)$. As $nil(R) \subseteq \mathfrak{p}$, we obtain that $\mathfrak{p} \notin \mathbb{A}(R)$ and so, $\mathfrak{p}$ is not a B-prime of (0) in R. Therefore, (2) holds. We know from Lemma 3.3 that $diam(G(R)) = r(G(R)) = 2$.

We next verify the furthermore part. We are assuming that (2) holds and given $I \in \mathbb{A}(R)^*$, there exists a non-zero $x \in nil(R)$ such that $Ix = (0)$. We know from Corollary 3.5 that $diam(G(R)) = r(G(R)) = 2$. □

We provide Example 3.8 to illustrate Corollary 3.5 and Proposition 3.7. Let M be a unitary module over a ring R. The direct product of abelian groups R and M, that is, $R \times M = \{(r, m) \mid r \in R, m \in M\}$ can be made into a ring by defining the multiplication as follows: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ for any $(r_1, m_1), (r_2, m_2) \in R \times M$. The ring obtained in this way is called the ring obtained by using *Nagata's principle of idealization* and is denoted by $R(+)M$.

Example 3.8. (1) Let $S = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K. Let $\mathfrak{m} = SX + SY$. Let $T = S_{\mathfrak{m}}$. Let $\mathcal{P}$ denote the set of all pairwise non-associate prime elements of T. Let $W = \bigoplus_{p \in \mathcal{P}} \frac{T}{Tp}$ be the direct sum of T-modules, $\frac{T}{Tp}$, where p varies over $\mathcal{P}$. Let $R = T(+)W$ be the ring obtained

by using Nagata's principle of idealization. Then $\mathfrak{p} = \mathfrak{m}T(+)W$ is the unique maximal N-prime of the zero ideal in R and $\mathfrak{p}$ is not a B-prime of the zero ideal in R. Moreover, $G(R)$ is connected with $diam(G(R)) = r(G(R)) = 2$.
(2) Let V be a rank one valuation domain which is not discrete. (One can take $V = A_{\mathfrak{p}}$, where A is as in Example 2.13 and $\mathfrak{p}$ is any maximal ideal of A.) Let $\mathfrak{m}$ denote the unique maximal ideal of V. Let $m \in \mathfrak{m}$ with $m \neq 0$. Let $R = \frac{V}{Vm}$. Then R has $\frac{m}{Vm}$ as its unique maximal N-prime of $(0+Vm)$ in R, $\frac{\mathfrak{m}}{Vm}$ is not a B-prime of $(0+Vm)$ in R, and $G(R)$ is connected with $diam(G(R)) = r(G(R)) = 2$.

Proof. Since $S = K[X,Y]$ is a unique factorization domain (UFD), it follows from [21, Theorem 5] and [9, Proposition 3.11(iv)] that $T = S_{\mathfrak{m}}$ is a UFD. We know from [9, Corollary 7.6] and [9, Proposition 7.3] that T is Noetherian. We know from [23, Theorem 3, page 281] that $height\mathfrak{m} = 2$ and hence, we obtain from [9, Proposition 3.11(iv)] that $height\mathfrak{m}T = 2$. Therefore, it follows from [21, Theorem 144] that T admits an infinite number of non-associate prime elements. It was already verified in the proof of [25, Example 2.8] that $\mathfrak{p} = \mathfrak{m}T(+)W$ is the unique maximal N-prime of the zero ideal in $R = T(+)W$ and $\mathfrak{p}$ is not a B-prime of the zero ideal in R. It is clear that $((0)(+)W)^2$ equals the zero ideal of R. Let $I \in \mathbb{A}(R)^*$. We claim that there exists a non-zero $x \in nil(R)$ such that Ix is the zero ideal of R. If $I \subseteq (0)(+)W$, then $I((0)(+)W)$ equals the zero ideal of R. Suppose that $I \not\subseteq (0)(+)W$. As $I \in \mathbb{A}(R)^*$, there exists a non-zero element (b,w) $(b \in T, w \in W)$ of R such that $I(b,w)$ equals the zero ideal of R. From $I \not\subseteq (0)(+)W$, it follows that there exist $a \in T\backslash\{0\}$ and $u \in W$ such that $(a,u) \in I$. Let 0_W denote the zero element of W. From $(a,u)(b,w) = (0,0_W)$, we get that $ab = 0$ and so, $b = 0$. Hence, $w \neq 0_W$. The element $x = (0,w) \in nil(R)$ is such that $x \neq (0,0_W)$ and Ix is the zero ideal of R. Thus given $I \in \mathbb{A}(R)^*$, there exists a non-zero $x \in nil(R)$ such that Ix equals the zero ideal of R. Hence, we obtain from Corollary 3.5 that $G(R)$ is connected and $diam(G(R)) = r(G(R)) = 2$.
(2) Let A be the ring of all algebraic integers in L, where L is the algebraic closure of the field $\mathbb{Q}$. Let $\mathfrak{p} \in Max(A)$. We know from [16, Proposition 42.8] that A is a one-dimensional Prüfer domain. Hence, we obtain that $A_{\mathfrak{p}}$ is a valuation domain and $dimA_{\mathfrak{p}} = 1$. We know from [16, Proposition 42.8(2)] that $\mathfrak{p} = \mathfrak{p}^2$ and so, $\mathfrak{p}A_{\mathfrak{p}} = (\mathfrak{p}A_{\mathfrak{p}})^2$. Hence, $\mathfrak{p}A_{\mathfrak{p}}$ is not principal. Therefore, $A_{\mathfrak{p}}$ is a rank one valuation domain and is not discrete.

Let V be any rank one valuation domain which is not discrete. We denote the unique maximal ideal of V by $\mathfrak{m}$. Now, $R = \frac{V}{Vm}$, where m is any non-zero element of $\mathfrak{m}$. It was already verified in [24, Example 3.1] that R has $\frac{\mathfrak{m}}{Vm}$ as its unique maximal N-prime of $(0 + Vm)$ and $\frac{\mathfrak{m}}{Vm}$ is not a B-prime of $(0 + Vm)$ in R. From $Spec(V) = \{(0), \mathfrak{m}\}$, it follows that $Spec(R) = \{\frac{\mathfrak{m}}{Vm}\}$. Therefore, we obtain from [9, Proposition 1.8] that $nil(R) = \frac{\mathfrak{m}}{Vm}$. Thus $nil(R) \notin \mathbb{A}(R)$. Hence, we obtain from the moreover part of Proposition 3.7 that $G(R)$ is connected and $diam(G(R)) = r(G(R)) = 2$. □

Let R be a non-reduced ring which admits exactly two maximal N-primes of (0). We prove in Proposition 3.9 that $G(R)$ is connected with $2 \leq diam(G(R)) \leq 3$.

Proposition 3.9. *Let R be a ring which is not reduced. Let $\{\mathfrak{p}_i \mid i \in \{1,2\}\}$ denote the set of all maximal N-primes of* (0) *in* R*. Then* $G(R)$ *is connected and* $2 \leq diam(G(R)) \leq 3$.

Proof. By hypothesis, R is not reduced. Hence, $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$. We know from [26, Lemma 4.4] that $(\mathbb{AG}(R))^c$ is connected and $2 \leq diam((\mathbb{AG}(R))^c) \leq 3$. Since $(\mathbb{AG}(R))^c$ is a spanning subgraph of $G(R)$, we obtain that $G(R)$ is connected and $diam(G(R)) \leq 3$. As R is not reduced, it follows from Lemma 3.1 that $2 \leq diam(G(R))$. Therefore, $2 \leq diam(G(R)) \leq 3$. □

Lemma 3.10. *Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the statement of Proposition 3.9. Then* $r(G(R)) \leq 2$.

Proof. We know from Proposition 3.9 that $G(R)$ is connected and
$2 \leq diam(G(R)) \leq 3$. As R is not reduced by hypothesis, it follows that $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$. It can be shown as in the proof of [26, Lemma 4.4] that there exist $a \in \mathfrak{p}_1 \backslash \mathfrak{p}_2$, $b \in \mathfrak{p}_2 \backslash \mathfrak{p}_1$ such that $ab \neq 0$. Observe that $a + b \notin \bigcup_{i=1}^{2} \mathfrak{p}_i = Z(R)$. We claim that $e(Ra) \leq 2$ in $G(R)$. Let $I \in \mathbb{A}(R)^*$ be such that $I \neq Ra$. Suppose that Ra and I are not adjacent in $G(R)$. Then $I(Ra) = (0)$ and $I + Ra \in \mathbb{A}(R)$. From $a + b \notin Z(R)$, $I \neq (0)$, and $Ia = (0)$, we get that $Ib = I(a + b) \neq (0)$. Now, $Ra - Rb - I$ is a path of length two between Ra and I in $(\mathbb{AG}(R))^c$ and hence, in $G(R)$. This proves that $d(Ra, I) \leq 2$ in $G(R)$. Therefore, $e(Ra) \leq 2$ in $G(R)$ and so, $r(G(R)) \leq 2$. □

Proposition 3.11. *Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the statement of Proposition 3.9. If $\mathfrak{p}_i$ is not a B-prime of* (0) *in* R *for some* $i \in \{1,2\}$*, then* $diam(G(R)) = 2$.

Proof. We know from Proposition 3.9 that $G(R)$ is connected and
$2 \leq diam(G(R)) \leq 3$. We can assume without loss of generality that $\mathfrak{p}_1$ is not a B-prime of (0) in R. It follows from Lemma 3.2 that $diam(G(R)) = 2$. □

We provide Example 3.12 to illustrate Proposition 3.11.

Example 3.12. (1) Let R be as in the statement of Example 3.8(2). Let $T = R \times R$. Then T is not reduced, T has exactly two maximal N-primes $\mathfrak{p}_1, \mathfrak{p}_2$ of its zero ideal such that $\mathfrak{p}_i$ is not a B-prime of its zero ideal for each $i \in \{1,2\}$, and $G(T)$ is connected with $diam(G(T)) = r(G(T)) = 2$.
(2) Let R be as in the statement of Example 3.8(2). Let $S = R \times \mathbb{Z}_4$. Then S is not reduced, S has exactly two maximal N-primes of its zero ideal such that one between them is not a B-prime of its zero ideal, and $G(S)$ is connected with $diam(G(S)) = r(G(S)) = 2$.
(3) Let R be as in Example 2.14. Let $T_1 = R \times \mathbb{Z}_4$. Then T_1 is not reduced, T_1 has exactly two maximal N-primes of its zero ideal such that one between them is not a B-prime of the zero ideal in T_1, and $G(T_1)$ is connected with $diam(G(T_1)) = 2$ and $r(G(T_1)) = 1$.

Proof. (1) It is already noted in the proof of Example 3.8(2) that $R = \frac{V}{Vm}$ has $\frac{\mathfrak{m}}{Vm}$ as its unique maximal N-prime of $(0+Vm)$, $\frac{\mathfrak{m}}{Vm}$ is not a B-prime of $(0+Vm)$, and $nil(R) = \frac{\mathfrak{m}}{Vm}$. As R is not reduced, we get that $T = R \times R$ is not reduced. Observe that $Z(T) = (Z(R) \times R) \cup (R \times Z(R)) = (\frac{\mathfrak{m}}{Vm} \times R) \cup (R \times \frac{\mathfrak{m}}{Vm})$. Let $\mathfrak{p}_1 = \frac{\mathfrak{m}}{Vm} \times R$ and let $\mathfrak{p}_2 = R \times \frac{\mathfrak{m}}{Vm}$. It is clear that $Max(T) = \{\mathfrak{p}_i \mid i \in \{1,2\}\}$. From $Z(T) = \bigcup_{i=1}^{2} \mathfrak{p}_i$, it follows that $\{\mathfrak{p}_i \mid i \in \{1,2\}\}$ is the set of all maximal N-primes of the zero ideal in T. As $\frac{\mathfrak{m}}{Vm}$ is not a B-prime of the zero ideal in R, we get that $\mathfrak{p}_i$ is not a B-prime of the zero ideal in T for each $i \in \{1,2\}$. Hence, we obtain from Proposition 3.11 that $G(T)$ is connected and $diam(G(T)) = 2$. It is clear that $nil(T) = nil(R) \times nil(R)$. As $nil(R) \notin \mathbb{A}(R)$, we obtain that $nil(T) \notin \mathbb{A}(T)$. Hence, we obtain from Lemma 3.3 that $r(G(T)) = 2$.
(2) As R is not reduced, it follows that $S = R \times \mathbb{Z}_4$ is not reduced. Observe that $Z(S) = (Z(R) \times \mathbb{Z}_4) \cup (R \times Z(\mathbb{Z}_4)) = (\frac{\mathfrak{m}}{Vm} \times \mathbb{Z}_4) \cup (R \times 2\mathbb{Z}_4)$. Let $\mathfrak{p}_1 = \frac{\mathfrak{m}}{Vm} \times \mathbb{Z}_4$ and let $\mathfrak{p}_2 = R \times 2\mathbb{Z}_4$. It is clear that $Max(S) = \{\mathfrak{p}_i \mid i \in \{1,2\}\}$. From $Z(S) = \bigcup_{i=1}^{2} \mathfrak{p}_i$, it follows that $\{\mathfrak{p}_i \mid i \in \{1,2\}\}$ is the set of all maximal N-primes of the zero ideal in S. As $\frac{\mathfrak{m}}{Vm}$ is not a B-prime of the zero ideal in R, we get that $\mathfrak{p}_1$ is not a B-prime of the zero ideal in S. Now, we obtain from Proposition 3.11 that $G(S)$ is connected and $diam(G(S)) = 2$. It is clear that $\mathfrak{p}_2 = ((0+Vm, 0) :_S (0+Vm, 2))$ is a B-prime of the zero ideal in S. Observe that $nil(S) = nil(R) \times nil(\mathbb{Z}_4) = \frac{\mathfrak{m}}{Vm} \times 2\mathbb{Z}_4$. Let $I \in \mathbb{A}(S)^*$. We assert that there exists a non-zero $x \in nil(S)$ such that $Ix = (0+Vm, 0)$. Observe that if $I \subseteq \mathfrak{p}_2$, then with $x = (0+Vm, 2)$, we get that x is a non-zero element of $nil(S)$ and $Ix = (0+Vm, 0)$. Suppose that $I \not\subseteq \mathfrak{p}_2$. As $I \subseteq Z(S) = \bigcup_{i=1}^{2} \mathfrak{p}_i$, we obtain that $I \subseteq \mathfrak{p}_1 = \frac{\mathfrak{m}}{Vm} \times \mathbb{Z}_4$. From $I \subseteq \mathfrak{p}_1$ but $I \not\subseteq \mathfrak{p}_2$, it follows that $I = I_1 \times \mathbb{Z}_4$, where I_1 is an ideal of R with $I_1 \subset \frac{\mathfrak{m}}{Vm}$. Suppose that $I_1 = (0+Vm)$. Let $m' \in \mathfrak{m}$ be such that $m' \notin Vm$. Then $(m'+Vm, 0)$ is a non-zero element of $nil(S)$ and $I(m'+Vm, 0) = (0+Vm, 0)$. Suppose that $I_1 \neq (0+Vm)$. Since $I \in \mathbb{A}(S)^*$, there exists a non-zero $(r, n) \in R \times \mathbb{Z}_4$ such that $I(r, n) = (0+Vm, 0)$. This implies that $I_1 r = (0+Vm)$ and $n = 0$. As $I_1 \neq (0+Vm)$, it follows that $r \notin U(R)$ and so, $r \in \frac{\mathfrak{m}}{Vm}$. Hence, $(r, 0)$ is a non-zero element of $nil(S)$ and $I(r, 0) = (0+Vm, 0)$. This shows that for any $I \in \mathbb{A}(S)^*$, there exists a non-zero $x \in nil(S)$ such that $Ix = (0+Vm, 0)$. Therefore, we obtain from Corollary 3.5 that $r(G(S)) = 2$.
(3) Since $\mathbb{Z}_4$ is not reduced, it follows that $T_1 = R \times \mathbb{Z}_4$ is not reduced. It is clear that $Z(T_1) = (Z(R) \times \mathbb{Z}_4) \cup (R \times Z(\mathbb{Z}_4))$. In the notation of Example 2.14, $Z(R) = \mathfrak{m}$, the unique maximal ideal of R and observe that $Z(\mathbb{Z}_4) = 2\mathbb{Z}_4$. Therefore, $Z(T_1) = (\mathfrak{m} \times \mathbb{Z}_4) \cup (R \times 2\mathbb{Z}_4)$. Let $\mathfrak{P}_1 = \mathfrak{m} \times \mathbb{Z}_4$ and let $\mathfrak{P}_2 = R \times 2\mathbb{Z}_4$. It is clear that $Max(T_1) = \{\mathfrak{P}_i \mid i \in \{1,2\}\}$. From $Z(T_1) = \bigcup_{i=1}^{2} \mathfrak{P}_i$, it follows that $\{\mathfrak{P}_i \mid i \in \{1,2\}\}$ is the set of all maximal N-primes of the zero ideal in T_1. In the notation of Example 2.14, the zero ideal of R equals $(0+I)$. We claim that $\mathfrak{m}$ is not a B-prime of zero ideal in R. For if $\mathfrak{m}$ is a B-prime of zero ideal in R,

then there exists $r \in \mathfrak{m}$ such that $r \neq 0 + I$ and $\mathfrak{m} = ((0 + I) :_R r)$. This implies that $r = 0 + I$, since R is reduced and $r^2 = 0 + I$. This is a contradiction and so, $\mathfrak{m}$ is not a B-prime of $(0 + I)$ in R and so, $\mathfrak{P}_1 = \mathfrak{m} \times \mathbb{Z}_4$ is not a B-prime of the zero ideal in T_1. It is clear that $\mathfrak{P}_2 = ((0 + I, 0) :_{T_1} (0 + I, 2))$ is a B-prime of $(0 + I, 0)$ in T_1. We know from Proposition 3.11 that $diam(G(T_1)) = 2$. Let $i \in \mathbb{N}$. In the notation of Example 2.14, $\mathfrak{p}_i \in Min(R) \cap \mathbb{A}(R)^*$. Hence, from the proof of (3) $\Rightarrow$ (1) of Corollary 2.10 that $e(\mathfrak{p}_i) = 1$ in $G(R)$. It is clear that $\mathfrak{p}_i \times \mathbb{Z}_4 \in \mathbb{A}(T_1)^*$. We verify that $e(\mathfrak{p}_i \times \mathbb{Z}_4) = 1$ in $G(T_1)$. Let $A = I_1 \times I_2 \in \mathbb{A}(T_1)^*$ be such that $A \neq \mathfrak{p}_i \times \mathbb{Z}_4$. If $(\mathfrak{p}_i \times \mathbb{Z}_4)A \neq (0 + I, 0)$, then $\mathfrak{p}_i \times \mathbb{Z}_4$ and A are adjacent in $G(T_1)$. Suppose that $(\mathfrak{p}_i \times \mathbb{Z}_4)(A = I_1 \times I_2) = (0+I, 0)$. Then $I_2 = (0)$ and $\mathfrak{p}_i I_1 = (0+I)$. It is clear that $I_1 \neq (0+I)$ and since R is reduced, we obtain that $I_1 \neq \mathfrak{p}_i$. From $e(\mathfrak{p}_i) = 1$ in $G(R)$, it follows that $\mathfrak{p}_i + I_1 \notin \mathbb{A}(R)$ and so, $(\mathfrak{p}_i \times \mathbb{Z}_4) + A \notin \mathbb{A}(T_1)$. Therefore, $\mathfrak{p}_i \times \mathbb{Z}_4$ and A are adjacent in $G(T_1)$ for any $A \in \mathbb{A}(T_1)^*$ with $A \neq \mathfrak{p}_i \times \mathbb{Z}_4$. This shows that $e(\mathfrak{p}_i \times \mathbb{Z}_4) = 1$ in $G(T_1)$ and so, $r(G(T_1)) = 1$. $\square$

Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the statement of Proposition 3.9. Assume that $\mathfrak{p}_i$ is a B-prime of (0) in R for each $i \in \{1, 2\}$. Let $u, v \in R$ be such that $\mathfrak{p}_1 = ((0) :_R u)$ and $\mathfrak{p}_2 = ((0) :_R v)$. It follows from [11, Lemma 3.6] that $uv = 0$. Hence, $u \in \mathfrak{p}_2$ and $v \in \mathfrak{p}_1$. Since R is not reduced, we obtain that $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$. Let $x \in \bigcap_{i=1}^{2} \mathfrak{p}_i$ with $x \neq 0$. Observe that $(u+v)x = 0$. From $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$, it follows that either $u+v \in \mathfrak{p}_1$ or $u+v \in \mathfrak{p}_2$. If $u+v \in \mathfrak{p}_1$, then $u \in \mathfrak{p}_1$ and so, $u \in \bigcap_{i=1}^{2} \mathfrak{p}_i$. If $u+v \in \mathfrak{p}_2$, then $v \in \mathfrak{p}_2$ and so, $v \in \bigcap_{i=1}^{2} \mathfrak{p}_i$. With the assumption that both u and v belong to $\bigcap_{i=1}^{2} \mathfrak{p}_i$, we prove in Proposition 3.13 that $diam(G(R)) = 3$ and $r(G(R)) = 2$.

Proposition 3.13. *Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the statement of Proposition 3.9. Suppose that there exist $u, v \in \bigcap_{i=1}^{2} \mathfrak{p}_i$ such that $\mathfrak{p}_1 = ((0) :_R u)$ and $\mathfrak{p}_2 = ((0) :_R v)$. Then $diam(G(R)) = 3$ and $r(G(R)) = 2$.*

Proof. We know from Proposition 3.9 that $G(R)$ is connected and $diam(G(R)) \leq 3$. It is already noted in the paragraph which appears just preceding the statement of this proposition that $uv = 0$. From $u \in \bigcap_{i=1}^{2} \mathfrak{p}_i$, it follows that $(Ru + Rv)u = (0)$. Now, $Ru, Rv \in \mathbb{A}(R)^*$ are such that $Ru \neq Rv$, $(Ru)(Rv) = (0)$, and $Ru + Rv \in \mathbb{A}(R)$. Hence, Ru and Rv are not adjacent in $G(R)$. We claim that there exists no path of length two between Ru and Rv in $G(R)$. Suppose that there exists $I \in \mathbb{A}(R)^*$ such that $Ru - I - Rv$ is a path of length two between Ru and Rv in $G(R)$. As $I \subseteq Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$, it follows that either $I \subseteq \mathfrak{p}_1$ or $I \subseteq \mathfrak{p}_2$. If $I \subseteq \mathfrak{p}_1$, then $(Ru)I = (0)$ and $(Ru + I)u = (0)$.

This implies that Ru and I are not adjacent in $G(R)$. This is a contradiction. If $I \subseteq \mathfrak{p}_2$, then $I(Rv) = (0)$ and $(I + Rv)v = (0)$. This implies that I and Rv are not adjacent in $G(R)$. This is again a contradiction. Therefore, there exists no path of length two between Ru and Rv in $G(R)$. Hence, $d(Ru, Rv) \geq 3$ in $G(R)$. This implies that $diam(G(R)) \geq 3$ and so, $diam(G(R)) = 3$.

We next verify that $r(G(R)) = 2$. We know from Lemma 3.10 that $r(G(R)) \leq 2$. Let $I \in \mathbb{A}(R)^*$. We assert that $e(I) \geq 2$ in $G(R)$. Suppose that $e(I) = 1$ in $G(R)$. Then I is adjacent to any $J \in \mathbb{A}(R)^* \backslash \{I\}$ in $G(R)$. This implies (as is remarked in the paragraph which appears just preceding the statement of Proposition 2.8) that $diam(G(R)) \leq 2$ and this is in contradiction to the fact that $diam(G(R)) = 3$. Therefore, $e(I) \geq 2$ in $G(R)$. Hence, $r(G(R)) \geq 2$ and so, $r(G(R)) = 2$. $\square$

Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the statement of Proposition 3.9. Assume that $\mathfrak{p}_i$ is a B-prime of (0) in R for each $i \in \{1, 2\}$. Suppose that $\mathfrak{p}_1 = ((0) :_R u)$ with $u \in \bigcap_{i=1}^{2} \mathfrak{p}_i$ and $\mathfrak{p}_2 I \neq (0)$ for any non-zero ideal I of R with $I \subseteq \bigcap_{i=1}^{2} \mathfrak{p}_i$. We prove in Proposition 3.14 that $diam(G(R)) = 2$ and $r(G(R)) = 1$.

Proposition 3.14. *Let $R, \mathfrak{p}_1, \mathfrak{p}_2$ be as in the statement of Proposition 3.9. Suppose that $\mathfrak{p}_i$ is a B-prime of (0) in R for each $i \in \{1, 2\}$. Let $u \in \bigcap_{i=1}^{2} \mathfrak{p}_i$ be such that $\mathfrak{p}_1 = ((0) :_R u)$ and suppose that $\mathfrak{p}_2 I \neq (0)$ for any non-zero ideal I of R with $I \subseteq \bigcap_{i=1}^{2} \mathfrak{p}_i$. Then $diam(G(R)) = 2$ and $r(G(R)) = 1$.*

Proof. We know from Proposition 3.9 that $G(R)$ is connected and $2 \leq diam(G(R)) \leq 3$. As the maximal N-prime $\mathfrak{p}_2$ of (0) in R is a B-prime of (0) in R, we get that $\mathfrak{p}_2 \in \mathbb{A}(R)$. Let I be a non-zero ideal of R such that $I \subseteq nil(R)$. Then $I \subseteq \bigcap_{i=1}^{2} \mathfrak{p}_i$. By hypothesis, we get that $\mathfrak{p}_2 I \neq (0)$. It now follows from Lemma 3.6 that $diam(G(R)) = 2$ and $r(G(R)) = 1$. $\square$

We provide Example 3.15 to illustrate Propositions 3.13 and 3.14.

Example 3.15. (1) Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then R is not reduced, R has exactly two maximal N-primes of $(0) \times (0)$, $diam(G(R)) = 3$, and $r(G(R)) = 2$.
(2) Let $T = \mathbb{Z}_4 \times F$, where F is a field. Then T is not reduced, T has exactly two maximal N-primes of $(0) \times (0)$, $diam(G(T)) = 2$, and $r(G(T)) = 1$.

Proof. (1) As $(2, 0)^2 = (0, 0)$ but $(2, 0) \neq (0, 0)$, it follows that R is not reduced. It is clear that $Max(R) = \{\mathfrak{m}_1 = 2\mathbb{Z}_4 \times \mathbb{Z}_4, \mathfrak{m}_2 = \mathbb{Z}_4 \times 2\mathbb{Z}_4\}$ and that $Z(R) = \bigcup_{i=1}^{2} \mathfrak{m}_i$. Therefore, $\{\mathfrak{m}_i \mid i \in \{1, 2\}\}$ is the set of all maximal N-primes of $(0) \times (0)$ in R. From $\mathfrak{m}_1 = ((0) \times (0) :_R (2, 0))$ and $\mathfrak{m}_2 = ((0) \times (0) :_R (0, 2))$, we get that $\mathfrak{m}_i$ is a B-prime of $(0) \times (0)$ in R for each $i \in \{1, 2\}$. As $(2, 0), (0, 2) \in \bigcap_{i=1}^{2} \mathfrak{m}_i$, we obtain from Proposition 3.13 that $diam(G(R)) = 3$ and $r(G(R)) = 2$.

(2) It is clear that $(2,0) \neq (0,0)$ but $(2,0)^2 = (0,0)$. Therefore, T is not reduced. Now, $Max(T) = \{\mathfrak{m}_1 = 2\mathbb{Z}_4 \times F, \mathfrak{m}_2 = \mathbb{Z}_4 \times (0)\}$. From $Z(T) = \bigcup_{i=1}^{2} \mathfrak{m}_i$, we obtain that $\{\mathfrak{m}_i \mid i \in \{1,2\}\}$ is the set of all maximal N-primes of $(0) \times (0)$ in T. As $\mathfrak{m}_1 = ((0) \times (0) :_T (2,0))$ and $\mathfrak{m}_2 = ((0) \times (0) :_T (0,1))$, it follows that $\mathfrak{m}_i$ is a B-prime of $(0) \times (0)$ in T for each $i \in \{1,2\}$. Observe that $(2,0) \in \bigcap_{i=1}^{2} \mathfrak{m}_i = 2\mathbb{Z}_4 \times (0)$. It is clear that $\bigcap_{i=1}^{2} \mathfrak{m}_i$ is the only non-zero ideal of T contained in $\bigcap_{i=1}^{2} \mathfrak{m}_i$. As $\mathfrak{m}_2(\bigcap_{i=1}^{2} \mathfrak{m}_i) = 2\mathbb{Z}_4 \times (0) \neq (0) \times (0)$, we obtain from Proposition 3.14 that $diam(G(R)) = 2$ and $r(G(R)) = 1$. $\square$

Let R be a non-reduced ring. If R admits more than two maximal N-primes of (0), then we prove in Proposition 3.16 that $G(R)$ is connected and $diam(G(R)) = 2$.

Proposition 3.16. *Let R be a ring which is not reduced. If R has more than two maximal N-primes of (0), then $G(R)$ is connected and $diam(G(R)) = 2$.*

Proof. We know from [26, Proposition 5.1(i)] that $(\mathbb{AG}(R))^c$ is connected and $diam((\mathbb{AG}(R))^c) = 2$. Since $(\mathbb{AG}(R))^c$ is a spanning subgraph of $G(R)$, we get that $G(R)$ is connected and $diam(G(R)) \leq 2$. As R is not reduced, we know from Lemma 3.1 that there exist $I, J \in \mathbb{A}(R)^*$ such that $d(I,J) \geq 2$ in $G(R)$. Therefore, $diam(G(R)) \geq 2$ and so, $diam(G(R)) = 2$. $\square$

Example 3.17. (1) Let $R_1 = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. Then R_1 has exactly three maximal N-primes of its zero ideal and $G(R_1)$ is connected with $diam(G(R_1)) = r(G(R_1)) = 2$.
(2) Let R be as in Example 3.8(2). Let $R_2 = R \times R \times R$. Then R has exactly three maximal N-primes of its zero ideal and $G(R_2)$ is connected with $diam(G(R_2)) = r(G(R_2)) = 2$.
(3) Let $R_3 = \mathbb{Z}_4 \times \mathbb{Z}_4 \times F$, where F is a field. Then R_3 has exactly three maximal N-primes of its zero ideal and $G(R_3)$ is connected with $diam(G(R_3)) = 2$ and $r(G(R_3)) = 1$.

Proof. As $\mathbb{Z}_4$ is not reduced, it follows that $R_1 = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ is not reduced. It is clear that $Max(R_1) = \{\mathfrak{m}_1 = 2\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4, \mathfrak{m}_2 = \mathbb{Z}_4 \times 2\mathbb{Z}_4 \times \mathbb{Z}_4, \mathfrak{m}_3 = \mathbb{Z}_4 \times \mathbb{Z}_4 \times 2\mathbb{Z}_4\}$ and that $Z(R_1) = \bigcup_{i=1}^{3} \mathfrak{m}_i$. Hence, $\{\mathfrak{m}_i \mid i \in \{1,2,3\}\}$ is the set of all maximal N-primes of $(0) \times (0) \times (0)$ in R_1. This shows that R_1 has exactly three maximal N-primes of $(0) \times (0) \times (0)$. It now follows from Proposition 3.16 that $G(R_1)$ is connected and $diam(G(R_1)) = 2$. Let $A \in \mathbb{A}(R_1)^*$. Then $A = A_1 \times A_2 \times A_3$, where A_i is an ideal of $\mathbb{Z}_4$ for each $i \in \{1,2,3\}$ and there exists at least one $i \in \{1,2,3\}$ such that $A_i \subseteq 2\mathbb{Z}_4$. If $A_1 \subseteq 2\mathbb{Z}_4$, then $A(2,0,0) = (0,0,0)$; if $A_2 \subseteq 2\mathbb{Z}_4$, then $A(0,2,0) = (0,0,0)$; and if $A_3 \subseteq 2\mathbb{Z}_4$, then $A(0,0,2) = (0,0,0)$. Thus given $A \in \mathbb{A}(R_1)^*$, there exists a non-zero $x \in 2\mathbb{Z}_4 \times 2\mathbb{Z}_4 \times 2\mathbb{Z}_4 = nil(R_1)$

such that $Ax = (0,0,0)$. Hence, we obtain from Lemma 3.4 that $e(A) \geq 2$ in $G(R_1)$ for each $A \in \mathbb{A}(R_1)^*$. Therefore, $r(G(R_1)) \geq 2$ and so, $r(G(R_1)) = 2$.
(2) In the notation of Example 3.8(2), R is not reduced, $nil(R) = \frac{\mathfrak{m}}{Vm}$ is the unique maximal ideal of R and is the unique maximal N-prime of the zero ideal in R, and $nil(R) \notin \mathbb{A}(R)$. Since R is not reduced, it follows that $R_2 = R \times R \times R$ is not reduced. It is clear that $Max(R_2) = \{\mathfrak{m}_1 = \frac{\mathfrak{m}}{Vm} \times R \times R, \mathfrak{m}_2 = R \times \frac{\mathfrak{m}}{Vm} \times R, \mathfrak{m}_3 = R \times R \times \frac{\mathfrak{m}}{Vm}\}$ and that $Z(R_2) = \bigcup_{i=1}^{3} \mathfrak{m}_i$. Hence, $\{\mathfrak{m}_i \mid i \in \{1,2,3\}\}$ is the set of all maximal N-primes of the zero ideal in R_2. Thus R_2 has exactly three maximal N-primes of its zero ideal. From $nil(R) \notin \mathbb{A}(R)$, it follows that $nil(R_2) = nil(R) \times nil(R) \times nil(R) \notin \mathbb{A}(R_2)$. Therefore, we obtain from Lemma 3.3 that $G(R_2)$ is connected and $diam(G(R_2)) = r(G(R_2)) = 2$.
(3) Since $\mathbb{Z}_4$ is not reduced, it follows that $R_3 = \mathbb{Z}_4 \times \mathbb{Z}_4 \times F$ is not reduced. It is clear that $Max(R_3) = \{\mathfrak{m}_1 = 2\mathbb{Z}_4 \times \mathbb{Z}_4 \times F, \mathfrak{m}_2 = \mathbb{Z}_4 \times 2\mathbb{Z}_4 \times F, \mathfrak{m}_3 = \mathbb{Z}_4 \times \mathbb{Z}_4 \times (0)\}$ and that $Z(R_3) = \bigcup_{i=1}^{3} \mathfrak{m}_i$. Hence, $\{\mathfrak{m}_i \mid i \in \{1,2,3\}\}$ is the set of all maximal N-primes of $(0) \times (0) \times (0)$ in R_3. Thus R_3 has exactly three maximal N-primes of its zero ideal. Now, $\mathfrak{m}_3 = Z_4 \times \mathbb{Z}_4 \times (0) \in \mathbb{A}(R_3)^*$. As $nil(R_3) = 2\mathbb{Z}_4 \times 2\mathbb{Z}_4 \times (0)$ and $\mathfrak{m}_3 I \neq (0) \times (0) \times (0)$ for any non-zero ideal I of R_3 with $I \subseteq nil(R_3)$, it follows from Lemma 3.6 that $G(R_3)$ is connected with $diam(G(R_3)) = 2$ and $r(G(R_3)) = 1$. $\square$

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