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MEASURE OF NONCOMPACTNESS AND SECOND ORDER EVOLUTION EQUATIONS

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ABSTRACT. In this article, we provide sufficient conditions for existence of mild solutions for second order semi-linear functional evolution equations in Banach spaces using the concept of a measure of noncompactness and Schauder's fixed point theorem. For illustration we give an example.

1. Introduction

This paper is concerned with the existence of mild solutions for the following second order evolution equation

$$y''(t) - A(t)y(t) = f(t, y(t)),\ a.e.\ t \in [0, T], \tag{1.1}$$

$$y(0) = y_0,\ y'(0) = y_1, \tag{1.2}$$

where $\{A(t)\}_{0 \le t < +\infty}$ is a family of linear closed operators from E into E that generate an evolution system of linear bounded operators $\{U(t, s)\}$ for $0 \le s \le t < +\infty$, $f : [0; T] \times E \to E$ be a Carathéodory function and $(E, |\cdot|)$ a real Banach space.

Evolution equations arise in many areas of applied mathematics [2, 14], this type of equations has received much attention in recent years [1]. There are many results concerning the second-order differential equations, see for exemple Fattorini [7], and Travis and Webb [13]. Among useful for the study of abstract second order equations is the existence of an evolution system $U(t, s)$ for the homogenous equation

$$y''(t) = A(t)y(t), \quad \text{for} \quad t \ge 0.$$

For this purpose there are many techniques to show the existence of $U(t, s)$ which has been developed by Kozak [10].

In this paper we use the technique of measures of noncompactness. It is well known that this method provides an excellent tool for obtaining existence of solutions of nonlinear differential equation. This technique works fruitfully for both integral and differential equations. More details are found in Akhmerov *et*

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al. [3], Alv́ares [4], Banaś and Goebel [5], Guo *et al.* [8], Kamenskii *et al.* [9], Olszowy [11], Olszowy and Węrychowicz [12], and the references therein.

The organization of this work is as follows. In Section 2, we recall some definitions and facts about evolution system. In Section 3, we give the existence of mild solutions of the problem (1.1)-(1.2). In Section 4 we present an example to illustrate our main result.

2. Preliminaries

Let E be a Banach space with the norm $|\cdot|$ and $C([0;T],E)$ be the Banach space of continuous functions y mapping $[0;T]$ into E with the usual supremum norm

$$\|y\| = \sup_{t\in[0;T]} |y(t)|.$$

Denote by $L^{\infty}([0;T],[0,\infty))$ the Banach space of essentially bounded measurable functions $\gamma : [0;T] \to [0,\infty)$ with the norm

$$\|\gamma\|_{\infty} = \inf\left\{c \geq 0 : |\gamma(t)| \leq c, \text{ a.e. } t \in [0;T]\right\},$$

and by B_R the closed ball centered 0 and radius R in $C([0;T],E)$.
$B(E)$ denotes the Banach space of linear bounded operators on E.

Definition 2.1. A map $f : [0,T] \times E \to E$is said to be Carathéodory if

(i) $t \to f(t,y)$ is measurable for all $y \in$E,
(ii) $y \to f(t,y)$ is continuous for almost each $t \in [0,T]$.

In what follows, let $\{A(t),\ t \geq 0\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ dense in E and independent of t.

In this work the existence of solution the problem (1.1)-(1.2) is related to the existence of an evolution operator $U(t,s)$ for the following homogeneous problem

$$y''(t) = A(t)y(t) \qquad t \in [0;T]. \tag{2.1}$$

This concept of evolution operator has been developed by Kozak [10].

Definition 2.2. A family U of bounded operators $U(t,s) : E \to E$, $(t,s) \in \Delta := \{(t,s) \in [0;T] \times [0;T] : s \leq t\}$, is called an evolution operator of the equation (2.1) if the following conditions hold:

(D_1): For any $x \in E$ the map $(t,s) \longmapsto U(t,s)x$ is continuously differentiable and

- **(a):** for any $t \in J$, $U(t,t) = 0$.
- **(b):** for all $(t,s) \in \Delta$ and for any $x \in E$, $\frac{\partial}{\partial t}U(t,s)x\Big|_{t=s} = x$ and $\frac{\partial}{\partial s}U(t,s)x\Big|_{t=s} = -x$.

(D_2): For all $(t,s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}U(t,s)x \in D(A(t))$, the map $(t,s) \longmapsto U(t,s)x$ is of class C^2 and

- **(a):** $\frac{\partial^2}{\partial t^2}U(t,s)x = A(t)U(t,s)x$.
- **(b):** $\frac{\partial^2}{\partial s^2}U(t,s)x = U(t,s)A(s)x$
- **(c):** $\frac{\partial^2}{\partial s \partial t}U(t,s)x\Big|_{t=s} = 0$.

(D_3): For all $(t,s)\in\Delta$, $\frac{\partial}{\partial s}U(t,s)x \in D(A(t))$, and there exist $\frac{\partial^3}{\partial t^2 \partial s}U(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}U(t,s)x$ such that

(a): $\frac{\partial^3}{\partial t^2 \partial s}U(t,s)x = A(t)\frac{\partial}{\partial s}(t)U(t,s)x$.
Moreover, the map $(t,s) \longmapsto A(t)\frac{\partial}{\partial s}(t)U(t,s)x$ is continuous.

(b): $\frac{\partial^3}{\partial s^2 \partial t}U(t,s)x = \frac{\partial}{\partial t}U(t,s)A(s)x$.

Throughout this paper, we will use the following definition of the concept of measure of noncompactness [5].

Definition 2.3. Let E be a Banach space and Ω_D the family of all nonempty and bounded subsets D of E. The measure of noncompactness is the map

$$\mu : \Omega_D \to [0;+\infty)$$

satisfying the following properties:

(i_1) $\mu(D) = 0$ if D is relatively compact,
(i_2) $\mu(\overline{D}) = \mu(D)$; $\overline{D}$ the closure of D,
(i_3) $\mu(C + D) \leq \mu(C) + \mu(D)$,
(i_4) $\mu(aD) = |a|\mu(D)$,
(i_5) $\mu(convD) = \mu(D)$; $convD$ the convex hull of D.

We will use a measure of noncompactness μ in the space $C([0;T], E)$ which was considered in [5]. Denote by $\omega^T(y,\varepsilon)$ the modulus of continuity of y on the interval $[0,T]$, i.e.

$$\omega^T(y,\varepsilon) = \sup\left\{|y(t) - y(s)| \,; t,s \in [0,T], |t-s| \leq \varepsilon\right\}.$$

Moreover, let us put

$$\omega^T(D,\varepsilon) = \sup\left\{\omega^T(y,\varepsilon); y \in D\right\}$$

$$\omega_0^T(D) = \lim_{\varepsilon\to 0}\sup \omega^T(D,\varepsilon).$$

Finally, we define

$$\mu^*(D) = \omega_0^T(D) + \sup_{t\in[0;T]} \mu(D(t)),$$

where μ is regular measure of noncompactness in E and

$$D(t) := \{y(t) : y \in D\}.$$

$\mu^*(D)$ is the measure of noncompactness in the space $C([0;T], E)$.

Lemma 2.4. [5] *If $\{D\}_{n=0}^{+\infty}$ is sequence of nonempty, bounded and closed subsets of E such that $D_{n+1} \subset D_n (n = 0,1,2,\ldots)$and if $\lim_{n\to\infty} \mu(D_n) = 0$, then the intersection*

$$D_\infty = \cap_{n=0}^{+\infty} D_n$$

is nonempty and compact.

Lemma 2.5. *(Cauchy formula). If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, then*

$$\int_a^t \int_a^{s_1} \int_a^{s_2} \cdots \int_a^{s_n} f(s_{n+1}) ds_{n+1} ds_n \cdots ds_1 = \frac{1}{n!} \int_a^t f(s)(t-s)^n ds$$

for each $t \geq a$.

Lemma 2.6. *([5]). If μ is a regular measure of noncompactness, then*

$$|\mu(D_1) - \mu(D_2)| \leq \mu(B(0,1)) d_H(D_1, D_2)$$

for any bounded subset $D_1, D_2 \in E$, where d_H is Hausdorff distance between D_1 and D_2.

Theorem 2.7 (Schauder fixed point theorem)**.** [6] *Let E be a Banach space, D compact convex subset of E and $N : D \to D$ continuous map. Then N has at least one fixed point in D.*

3. Main result

Definition 3.1. A function $y \in C([0;T], E)$ is called a mild solution to the problem (1.1)-(1.2) if y satisfies the integral equation

$$y(t) = -\frac{\partial}{\partial s} U(t,0) y_0 + U(t,0) y_1 + \int_0^t U(t,s) f(s, y(s)) ds. \tag{3.1}$$

To prove our results we introduce the following conditions:

(H_1)**:** There exists a constant $M \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq M.$$

(H_2)**:** There exists a constant $\tilde{M} \geq 0$ such that

$$\left\| \frac{\partial}{\partial s} U(t,s) \right\|_{B(E)} \leq \tilde{M}.$$

(H_3)**:** (A_1) There exist integrable function $p : [0;T] \to \mathbb{R}_+$ and a continuous nondecreasing function $\psi : [0, \infty) \to]0; \infty)$ such that:

$$|f(t,u)| \leq p(t) \psi(|u|) \text{ for a.e } t \in [0;T] \text{ and each } u \in E.$$

(A_2) There exists a measurable function $\gamma \in L^\infty([0;T], \mathbb{R}_+)$ such that for any nonempty set $D \subset E$ we have :

$$\mu(f([0;T] \times D)) \leq \gamma([0;T]) \mu(D).$$

(H_4)**:** There exists a constant $R > 0$ such that

$$\tilde{M}|y_0| + M|y_1| + M\psi(R) \|P\|_{L^1} \leq R.$$

Theorem 3.2. *If assumptions $(H_1) - (H_4)$ are satisfied, then the problem (1.1)-(1.2) admits at least one mild solution.*

Proof. Consider the operator $N : C([0;T], E) \to C([0;T], E)$ defined by

$$(Nu)(t) = -\frac{\partial}{\partial s}U(t,0)y_0 + U(t,0)y_1 + \int_0^t U(t,s)f(s,y(s))ds.$$

Let $s, t \in [0;T]$ with $t > s$. Then we have

$$\begin{aligned}
|(Ny)(t) - (Ny)(s)| &\leq \left|\frac{\partial}{\partial s}U(t,0)y_0 - \frac{\partial}{\partial s}U(s,0)y_0\right| + |U(t,0)y_0 - U(s,0)y_0| \\
&+ \int_0^s \|U(t,\tau) - U(s,\tau)\|_{B(E)} |f(\tau, y(\tau)| \, d\tau \\
&+ \int_s^t \|U(t,\tau)\|_{B(E)} |f(\tau, y(\tau)| \, d\tau \\
&\leq \tilde{\omega}^T(\frac{\partial}{\partial s}U(t,0)y_0, \varepsilon) + \omega^T(U(t,0)y_0, \varepsilon) \\
&+ A^T(U,\varepsilon) \int_0^s p(\tau)\psi(|y(\tau)|)d\tau \\
&+ M \sup\left\{\int_s^t p(\tau)\psi(|y(\tau)|)d\tau ; t \leq s \leq T, |t-s| \leq \varepsilon\right\},
\end{aligned}$$

where

$$\tilde{\omega}^T(\tfrac{\partial}{\partial s}U(t,0)y_0, \varepsilon) = \sup\left\{\left|\tfrac{\partial}{\partial s}U(t,0)y_0 - \tfrac{\partial}{\partial s}U(s,0)y_0\right| ; t \leq s \leq T, |t-s| \leq \varepsilon\right\},$$

$$\omega^T(U(t,0)y_0, \varepsilon) = \sup\left\{|U(t,0)y_0 - U(s,0)y_0| ; t \leq s \leq T, |t-s| \leq \varepsilon\right\},$$

$$A^T(U,\varepsilon) = \sup\left\{\|U(t,\tau) - U(s,\tau)\|_{B(E)} ; \tau \leq t \leq s \leq T, |t-s| \leq \varepsilon\right\}.$$

Putting

$$\begin{aligned}
\Omega(T,\varepsilon) &= \tilde{\omega}^T(\frac{\partial}{\partial s}U(t,0)y_0, \varepsilon) + \omega^T(U(t,0)y_0, \varepsilon) \\
&+ A^T(U,\varepsilon) \int_0^s p(\tau)\psi(|y(\tau)|)d\tau \\
&+ M \sup\left\{\int_s^t p(\tau)\psi(|y(\tau)|)d\tau ; t \leq s \leq T, |t-s| \leq \varepsilon\right\}.
\end{aligned}$$

We have

$$|(Ny)(t) - (Ny)(s)| \leq \Omega(T,\varepsilon)$$

i.e

$$\omega^T(Ny, \varepsilon) \leq \Omega(T,\varepsilon). \tag{3.2}$$

The assumptions $(H_1) - (H_4)$ yield that

$$\lim_{\varepsilon \to 0} \Omega(T,\varepsilon) = 0.$$

We define

$$D=\left\{y\in C([0;T],E):\|y\|\leq R \text{ and } \lim_{\varepsilon\to 0}\Omega(T,\varepsilon)=0\right\}.$$

The set D is nonempty convex and closed.

Now, we have

$$\begin{aligned}
|Ny(t)| &\leq \left\|\frac{\partial}{\partial s}U(t,0)\right\|_{B(E)}|y_0| \\
&+ \|U(t,s)\|_{B(E)}|y_1|+\|U(t,s)\|_{B(E)}\int_0^t p(s)\psi(|y(s)|)ds \\
&\leq \tilde{M}|y_0|+M|y_1|+M\int_0^t p(s)\psi(R)ds \\
&\leq \tilde{M}|y_0|+M|y_1|+M\psi(R)\|P\|_{L^1} \\
&\leq R. \qquad (3.3)
\end{aligned}$$

The conditions (3.2) and (3.3) ensure that the operator N transforms the set D into itself.

Step 1. N continuous.

Let $(y_n)_{n\in N}$ be a sequence in D such that $y_n\to y$ in D. We have

$$|Ny_n(t)-Ny(t)|\leq M\int_0^t|f(s,y_n(s))-f(s,y(s))|\,ds.$$

Since f is Caratheory we obtain by the Lebesgue dominated convergence theorem that

$$\lim_{t\to+\infty}\|N(y_n)-N(y)\|=0.$$

We deduce N is continuous.

Step 2. $D_\infty=\cap_{n=0}^{+\infty}D_n$ is compact.

In the sequel, we consider the sequence of sets $\{D_n\}_{n=0}^{+\infty}$ defined by induction as follows :

$$D_0=D,\ D_{n+1}=convN(D_n) \text{ for } n=0,1,2,\cdots \text{and } D_\infty=\cap_{n=0}^{+\infty}D_n$$

this sequence is nondecreasing, i.e. $D_n\subset D_{n+1}$ for each n.

Claim 1. $\lim_{n\to+\infty}\omega_0^T(D_n)=0$.

This is a consequence from the equicontinuity of the set D on compact intervals.

Claim 2. $\lim_{n\to+\infty}\mu^T(D_n)=0$.

Set

$$\alpha_n(t)=\mu(D_n(t)).$$

In view of Lemma 2.4 and Lemma 3.2 we have

$$|\alpha_n(t)-\alpha_n(s)|\leq\mu(B(0,1))\Omega(T,|t-s|)$$

which proves the continuity of $\alpha_n(t)$ on $[0, T]$.

Using the properties of μ, Lemma 2.6 and $(H_3)(A_2)$ we get

$$\begin{aligned}\alpha_{n+1}(t) &= \mu(CovN(D_n)) = \mu(N(D_n)) = \mu\left(\left\{\int_0^t U(t,s)f(s,y(s))ds : y \in D_n\right\}\right) \\ &\leq \|U(t,s)\|_{B(E)}\int_0^t \mu\left\{(f(s,y(s))) : y \in D_n\right\} ds \\ &\leq M\int_0^t \gamma(s)\alpha_n(s)ds.\end{aligned}$$

Using Lemma 2.5 we derive

$$\begin{aligned}\alpha_{n+1}(t) &\leq M^{n+1}\int_0^t \gamma(s_1)\int_0^{s_1}\gamma(s_2)\int_0^{s_2}\cdots\int_0^{s_n}\gamma(s_{n+1})\alpha_0(s_{n+1})ds_1ds_2\cdots ds_nds_{n+1} \\ &\leq M^{n+1}\tilde{\gamma}^{n+1}\int_0^t\int_0^{s_1}\int_0^{s_2}\cdots\int_0^{s_n}\alpha_0(s_{n+1})ds_{n+1}ds_n\cdots ds_2ds_1 \\ &\leq \frac{M^{n+1}\tilde{\gamma}^{n+1}(t)T^n}{n!}\int_0^t \alpha_0(s)ds,\end{aligned}$$

where $\tilde{\gamma}(t) = ess\sup\{\gamma(s) : s \leq t\}$.

Thus, we get

$$\lim_{n\to+\infty}\alpha_n(t) = 0.$$

Then

$$\lim_{n\to+\infty}\mu(D_n(t)) = 0.$$

Form Claim 1 and Claim 2 we conclude

$$\lim_{n\to+\infty}\omega_0^T(D_n) + \lim_{n\to+\infty}\sup_{t\in[0;T]}\mu(D_n(t)) = 0.$$

Lemma 2.4 implies that $D_\infty = \cap_{n=0}^{+\infty}D_n$ is nonempty, convex and compact. Thus by Schauder's fixed point theorem the operator $N : D_\infty \to D_\infty$ has at least one fixed point which is a mild solution of problem (1.1)-(1.2). □

4. An example

Consider the second order Cauchy problem

$$\begin{cases}\dfrac{\partial^2 z(t,\tau)}{\partial t^2} = \dfrac{\partial^2 z(t,\tau)}{\partial \tau^2} + a(t)\dfrac{\partial z(t,\tau)}{\partial t} \\ \qquad +e^{-t}\displaystyle\int_0^t \dfrac{z(t,\tau)}{1+z^2(t,\tau)}d\tau, & t\in[0,T],\ \tau\in[0,\pi], \\ z(t,0) = z(t,\pi) = 0 & t\in[0;T], \\ \dfrac{\partial z(0,\tau)}{\partial t} = \psi(\tau) & \tau\in[0,\pi]\end{cases} \tag{4.1}$$

where $a : [0;T] \to \mathbb{R}$ is a Hölder continuous function.

Let $E = L^2([0,\pi],\mathbb{C})$ the space of 2-integrable functions from $[0,\pi]$ into $\mathbb{C}$, and $H^2([0,\pi],\mathbb{C})$ denotes the Sobolev space of functions $x : [0,\pi] \to \mathbb{C}$ such that $x'' \in L^2([0,\pi],\mathbb{C})$. We consider the operator $A_1 y(\tau) = y''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R},\mathbb{C})$, infinitesimal generator of strongly continuous cosine function $C(t)$ on E. Moreover, we take $A_2(t)y(s) = a(t)y'(s)$ defined on $H^1([0,\pi],\mathbb{C})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$ which generates an evolution operator U defined by

$$U(t,s) = \sum_{n\in\mathbb{Z}} z_n(t,s)\langle x, w_n\rangle w_n,$$

where z_n is a solution to the following scalar initial value problem

$$\begin{cases} z''(t) = -n^2 z(t) + ina(t)z(t) \\ z(s) = 0, \qquad z'(s) = 1. \end{cases} \tag{4.2}$$

Set

$$y(t)(\tau) = z(t,\tau),\ t \in [0,T],\ \tau \in [0,\pi],$$

$$f(t,y)(\tau) = e^{-t}\int_0^t \frac{z(t,\tau)}{1+z^2(t,\tau)}d\tau,$$

and

$$\frac{d}{dt}y(0)(\tau) = \frac{\partial z(0,\tau)}{\partial t},\ \tau \in [0,\pi].$$

Consequently, (4.1) can be written in the abstract form (1.1)-(1.2) with $A(t)$ and f defined above. The existence of a mild solutions can be deduced from an application of Theorem 3.2.

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