

NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATION DRIVEN BY FRACTIONAL BROWNIAN MOTION AND POISSON POINT PROCESSES

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ABSTRACT. In this note we consider a class of neutral stochastic functional differential equations with finite delay driven simultaneously by a fractional Brownian motion and a Poisson point processes in a Hilbert space. We prove an existence and uniqueness result and we establish some conditions ensuring the exponential decay to zero in mean square for the mild solution by means of the Banach fixed point principle.

1. INTRODUCTION

The solution of Itô stochastic differential equations (SDEs) is a semimartingale and a Markov process. However, many real world phenomena are not always such processes since they have long-range after-effects. Following the work of Mandelbrot and Van Ness [15], there has been an increased interest in stochastic models involving fractional Brownian motion (fBm). Since a fBm $\{B^H(t) : t \in [0, T]\}$ is not a semimartingale if $H \neq \frac{1}{2}$ (see Biagini *al.* [1]), the classical Itô theory cannot be used to construct a stochastic calculus with respect to fBm. Since physical phenomena are often more naturally modeled by stochastic partial differential equations or stochastic integro-differential equations in which the randomness is described by a fBm, it is useful to study the existence, uniqueness of infinite dimensional equations with a fBm. Many research investigations of stochastic equations driven by a fBm in an infinite dimensional space have emerged recently (see [4, 3, 5, 6, 11, 12, 13]).

Moreover, Stochastic differential equations with Poisson jumps have become very popular in modeling the phenomena arising in the fields, such as economics, where the jump processes are widely used to describe the asset and commodity price dynamics (see [7]). Recently, there is observed an increasing interest in the study of stochastic differential equation with jumps. To be more precise, in [14], Luo and Taniguchi considered the existence and uniqueness to stochastic neutral delay evolution equations driven by Poisson jumps by the Banach fixed point theorem. Boufoussi and Hajji proved the existence and uniqueness for a class neutral

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functional stochastic differential equations driven both by the cylindrical Brownian motion and by the Poisson processes in a Hilbert space by using successive approximation [2]. In [8], Cui and Yan studied the existence and uniqueness of mild solutions to stochastic differential equations with infinite delay and Poisson jumps in the phase space $BC((-\infty, 0], H)$, and also Tam et al. established the existence and uniqueness of solution to neutral stochastic functional differential equations with Poisson jumps [18]. However, until now, there is no work on the existence and uniqueness of the solution to neutral stochastic functional differential equations driven simultaneously by a fractional Brownian motion and a Poisson processes. Therefore, motivated by the previous problems, our current consideration is on the study of existence, uniqueness and asymptotic behavior of mild solutions for a class of neutral functional stochastic differential equations with jumps described in the form:

$$\begin{cases} d[x(t) + g(t, x(t - r(t)))] = [Ax(t) + f(t, x(t - \rho(t)))]dt + \sigma(t)dB^H(t) \\ \quad + \int_{\mathcal{U}} h(t, x(t - \theta(t)), y)\tilde{N}(dt, dy), \quad 0 \leq t \leq T, \\ x(t) = \varphi(t), \quad -\tau \leq t \leq 0. \end{cases} \quad (1.1)$$

where $\varphi \in \mathcal{D} := D([-\tau, 0], X)$ the space of càdlàg functions from $[-\tau, 0]$ into X equipped with the supremum norm $|\varphi|_{\mathcal{D}} = \sup_{s \in [-\tau, 0]} \|\varphi(s)\|_X$ and A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t \geq 0}$, in a Hilbert space X , B^H is a fractional Brownian motion on a real and separable Hilbert space Y , $r, \rho, \theta : [0, +\infty) \rightarrow [0, \tau]$ ($\tau > 0$) are continuous and $f, g : [0, +\infty) \times X \rightarrow X$, $\sigma : [0, +\infty) \rightarrow \mathcal{L}_2^0(Y, X)$, $h : [0, +\infty) \times X \times \mathcal{U} \rightarrow X$ are appropriate functions. Here $\mathcal{L}_2^0(Y, X)$ denotes the space of all Q -Hilbert-Schmidt operators from Y into X (see section 2 below).

On the other hand, to the best of our knowledge, there is no paper which investigates the study of neutral stochastic functional differential equations with delays driven both by fractional Brownian motion and by Poisson point processes. Thus, we will make the first attempt to study such problem in this paper.

Our results are inspired by the one in [5] where the existence and uniqueness of mild solutions to model (1.1) with $h = 0$ is studied, as well as some results on the asymptotic behavior.

The rest of this paper is organized as follows, In Section 2 we introduce some notations, concepts, and basic results about fractional Brownian motion, Poisson point processes, Wiener integral over Hilbert spaces and we recall some preliminary results about analytic semi-groups and fractional power associated to its generator. In section 3 by the Banach fixed point theorem we consider a sufficient condition for the existence, uniqueness and exponential decay to zero in mean square for mild solutions of equation (1.1).

2. PRELIMINARIES

In this section, we collect some notions, conceptions and lemmas on Wiener integrals with respect to an infinite dimensional fractional Brownian and we recall some basic results about analytical semi-groups and fractional powers of their infinitesimal generators, which will be used throughout the whole of this paper. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space satisfying the usual condition, which means that the filtration is right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets.

Let $(\mathcal{U}, \mathcal{E}, \nu(du))$ be a σ -finite measurable space. Given a stationary Poisson point process $(p_t)_{t > 0}$, which is defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathcal{U}$ and with characteristic measure ν (see [10]). We will denote by $N(t, du)$ be the counting measure of p_t such that $\hat{N}(t, A) := \mathbb{E}(N(t, A)) = t\nu(A)$ for $A \in \mathcal{E}$. Define $\tilde{N}(t, du) := N(t, du) - t\nu(du)$, the Poisson martingale measure generated by p_t .

Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{\beta^H(t), t \in [0, T]\}$ the one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. This means by definition that β^H is a centered Gaussian process with covariance function:

$$R_H(s, t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Moreover β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s) d\beta(s), \quad (2.1)$$

where $\beta = \{\beta(t) : t \in [0, T]\}$ is a Wiener process, and $K_H(t; s)$ is the kernel given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(\cdot, \cdot)$ denotes the Beta function. We put $K_H(t, s) = 0$ if $t \leq s$.

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. In fact $\mathcal{H}$ is the closure of set of indicator functions $\{1_{[0;t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0;t]}, 1_{[0;s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

The mapping $1_{[0;t]} \rightarrow \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ by the previous isometry.

We recall that for $\psi, \varphi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H - 1) \int_0^T \int_0^T \psi(s) \varphi(t) |t - s|^{2H-2} ds dt.$$

Let us consider the operator K_H^* from $\mathcal{H}$ to $L^2([0, T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r, s) dr.$$

We refer to [16] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0, T])$. Moreover for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^T (K_H^* \varphi)(t) d\beta(t).$$

It follows from [16] that the elements of $\mathcal{H}$ may be not functions but distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|_{|\mathcal{H}|}^2 := \alpha_H \int_0^T \int_0^T |\psi(s)| |\psi(t)| |s - t|^{2H-2} ds dt < \infty,$$

where $\alpha_H = H(2H - 1)$. The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and we have the following inclusions (see [16])

Lemma 2.1.

$$\mathbb{L}^2([0, T]) \subseteq \mathbb{L}^{1/H}([0, T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H}$$

and for any $\varphi \in \mathbb{L}^2([0, T])$, we have

$$\|\varphi\|_{|\mathcal{H}|}^2 \leq 2HT^{2H-1} \int_0^T |\varphi(s)|^2 ds.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y, X)$ be the space of bounded linear operator from Y to X . For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathcal{L}(Y, X)$. Let $Q \in \mathcal{L}(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\text{tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty$ where $\lambda_n \geq 0$ ($n = 1, 2, \dots$) are non-negative real numbers and $\{e_n\}$ ($n = 1, 2, \dots$) is a complete orthonormal basis in Y . Let $B^H = (B^H(t))$, independent of the Poisson point process, be Y -valued fBm on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance Q as

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where β_n^H are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle B^H(t), x \rangle \langle B^H(s), y \rangle = R(s, t) \langle Q(x), y \rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0, T].$$

In order to define Wiener integrals with respect to the Q -fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y, X)$ of all Q -Hilbert-Schmidt operators $\psi : Y \rightarrow X$. We recall that $\psi \in \mathcal{L}(Y, X)$ is called a Q -Hilbert-Schmidt operator, if

$$\|\psi\|_{\mathcal{L}_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$ is a separable Hilbert space.

Now, let $\phi(s); s \in [0, T]$ be a function with values in $\mathcal{L}_2^0(Y, X)$, The Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} (K_H^*(\phi e_n)(s) d\beta_n(s), \quad (2.2)$$

where β_n is the standard Brownian motion used to present β_n^H as in (2.1).

Now, we end this subsection by stating the following result which is fundamental to prove our result. It can be proved by similar arguments as those used to prove Lemma 2 in [6].

Lemma 2.2. *If $\psi : [0, T] \rightarrow \mathcal{L}_2^0(Y, X)$ satisfies $\int_0^T \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds < \infty$ then the above sum in (2.2) is well defined as a X -valued random variable and we have*

$$\mathbb{E} \left\| \int_0^t \psi(s) dB^H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds.$$

Now we turn to state some notations and basic facts about the theory of semigroups and fractional power operators.

Let $A : D(A) \rightarrow X$ be the infinitesimal generator of an analytic semigroup, $(S(t))_{t \geq 0}$, of bounded linear operators on X . For the theory of strongly continuous semigroup, we refer to [17] and [9]. We will point out here some notations and properties that will be used in this work. It is well known that there exist $M \geq 1$ and $\lambda \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\lambda t}$ for every $t \geq 0$.

If $(S(t))_{t \geq 0}$ is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A , then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X , and the expression

$$\|h\|_\alpha = \|(-A)^\alpha h\|,$$

defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|\cdot\|_\alpha$, then the following properties are well known (cf. [17], p. 74).

Lemma 2.3. *Suppose that the preceding conditions are satisfied.*

- (1) *Let $0 < \alpha \leq 1$. Then X_α is a Banach space.*
- (2) *If $0 < \beta \leq \alpha$ then the injection $X_\alpha \hookrightarrow X_\beta$ is continuous.*
- (3) *For every $0 < \alpha \leq 1$ there exists $M_\alpha > 0$ such that*

$$\|(-A)^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\lambda t}, \quad t > 0, \quad \lambda > 0.$$

3. MAIN RESULTS

In this section, we consider existence, uniqueness and exponential stability of mild solution to Equation (1.1). Our main method is the Banach fixed point principle. First we define the space Π of the càdlàg processes $x(t)$ as follows:

Definition 3.1. Let the space Π denote the set of all càdlàg processes $x(t)$ such that $x(t) = \varphi(t)$ $t \in [-\tau, 0]$ and there exist some constants $M^* = M^*(\varphi, a) > 0$ and $a > 0$

$$\mathbb{E} \|x(t)\|^2 \leq M^* e^{-at}, \quad \forall t \in [0, T].$$

Definition 3.2. $\|\cdot\|_\Pi$ denotes the norm in Π which is defined by

$$\|x\|_\Pi := \sup_{t \geq 0} \mathbb{E} \|x(t)\|_X^2 \quad \text{for } x \in \Pi.$$

Remark 3.3. It is routine to check that Π is a Banach space endowed with the norm $\|\cdot\|_\Pi$.

In order to obtain our main result, we assume that the following conditions hold.

- (H.1) A is the infinitesimal generator of an analytic semigroup, $(S(t))_{t \geq 0}$, of bounded linear operators on X . Further, to avoid unnecessary notations, we suppose that $0 \in \rho(A)$, and that, see Lemma 2.3,

$$\|S(t)\| \leq M e^{-\lambda t} \quad \text{and} \quad \|(-A)^{1-\beta} S(t)\| \leq \frac{M_{1-\beta}}{t^{1-\beta}}$$

for some constants $M, \lambda, M_{1-\beta}$ and every $t \in [0, T]$.

- (H.2) There exist positive constant $K_1 > 0$ such that, for all $t \in [0, T]$ and $x, y \in X$

$$\|f(t, x) - f(t, y)\|^2 \leq K_1 \|x - y\|^2.$$

- (H.3) There exist constants $0 < \beta < 1, K_2 > 0$ such that the function g is X_β -valued and satisfies for all $t \in [0, T]$ and $x, y \in X$

$$\|(-A)^\beta g(t, x) - (-A)^\beta g(t, y)\|^2 \leq K_2 \|x - y\|^2.$$

- (H.4) The function $(-A)^\beta g$ is continuous in the quadratic mean sense:

For all $x \in D([0, T], \mathbb{L}^2(\Omega, X))$, $\lim_{t \rightarrow s} \mathbb{E} \|(-A)^\beta g(t, x(t)) - (-A)^\beta g(s, x(s))\|^2 = 0$.

- (H.5) There exists some $\gamma > 0$ such that the function $\sigma : [0, +\infty) \rightarrow \mathcal{L}_2^0(Y, X)$ satisfies

$$\int_0^T e^{2\gamma s} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds < \infty, \quad \forall T > 0.$$

- (H.6) There exist positive constant $K_3 > 0$ such that, for all $t \in [0, T]$ and $x, y \in X$

$$\int_{\mathcal{U}} \|h(t, x, z) - h(t, y, z)\|_X^2 \nu(dz) \leq K_3 \|x - y\|_X^2.$$

We further assume that $g(t, 0) = f(t, 0) = h(t, 0, z) = 0$ for all $t \geq 0$ and $z \in \mathcal{U}$. Moreover, we suppose that $\mathbb{E} |\varphi|_{\mathcal{D}}^2 < \infty$.

Similar to the deterministic situation we give the following definition of mild solutions for equation (1.1).

Definition 3.4. A X -valued process $\{x(t), t \in [-\tau, T]\}$, is called a mild solution of equation (1.1) if

- i) $x(\cdot)$ has càdlàg path, and $\int_0^T \|x(t)\|^2 dt < \infty$ almost surely;
- ii) $x(t) = \varphi(t), -\tau \leq t \leq 0$.

iii) For arbitrary $t \in [0, T]$, we have

$$\left\{ \begin{array}{l} x(t) = S(t)(\varphi(0) + g(0, \varphi(-r(0)))) - g(t, x(t-r(t))) \\ \quad - \int_0^t AS(t-s)g(s, x(s-r(s)))ds + \int_0^t S(t-s)f(s, x(s-\rho(s)))ds \\ \quad + \int_0^t S(t-s)\sigma(s)dB^H(s) \\ \quad + \int_0^t \int_{\mathcal{U}} S(t-s)h(s, x(s-\theta(s)), y)\tilde{N}(ds, dy) \quad \mathbb{P} - a.s \end{array} \right.$$

We can now state the main result of this paper.

Theorem 3.5. *Suppose that $(\mathcal{H}.1) - (\mathcal{H}.6)$ hold and that*

$$4(K_2\|(-A)^{-\beta}\|^2 + K_2M_{1-\beta}^2\lambda^{-2\beta}\Gamma(\beta)^2 + K_1M^2\lambda^{-2} + M^2K_3(2\lambda)^{-1}) < 1,$$

where $\Gamma(\cdot)$ is the Gamma function, $M_{1-\alpha}$ is the corresponding constant in Lemma 2.3.

If the initial value $\varphi(t)$ satisfies

$$\mathbb{E}\|\varphi(t)\|^2 \leq M_0\mathbb{E}|\varphi|_{\mathcal{D}}^2e^{-at}, \quad t \in [-\tau, 0],$$

for some $M_0 > 0$, $a > 0$; then, for all $T > 0$, the equation (1.1) has a unique mild solution on $[-\tau, T]$ and is exponential decay to zero in mean square, i.e., there exists a pair of positive constants $a > 0$ and $M^* = M^*(\varphi, a)$ such that

$$\mathbb{E}\|x(t)\|^2 \leq M^*e^{-at}, \quad \forall t \geq 0.$$

Proof. Define the mapping Ψ on Π as follows:

$$\Psi(x)(t) := \varphi(t), \quad t \in [-\tau, 0],$$

and for $t \in [0, T]$

$$\begin{aligned} \Psi(x)(t) &= S(t)(\varphi(0) + g(0, \varphi(-r(0)))) - g(t, x(t-r(t))) - \int_0^t AS(t-s)g(s, x(s-r(s)))ds \\ &\quad + \int_0^t S(t-s)f(s, x(s-\rho(s)))ds + \int_0^t S(t-s)\sigma(s)dB^H(s) \\ &\quad + \int_0^t \int_{\mathcal{U}} S(t-s)h(s, x(s-\theta(s)), y)\tilde{N}(ds, dy). \end{aligned}$$

Then it is clear that to prove the existence of mild solutions to equation (1.1) is equivalent to find a fixed point for the operator Ψ .

We will show by using Banach fixed point theorem that Ψ has a unique fixed point. First we show that $\Psi(\Pi) \subset \Pi$.

Let $x(t) \in \Pi$, then we have

$$\begin{aligned} \mathbb{E}\|\Psi(x)(t)\|^2 &\leq 6\mathbb{E}\|S(t)(\varphi(0) + g(0, \varphi(-r(0))))\|^2 \\ &\quad + 6\mathbb{E}\|g(t, x(t-r(t)))\|^2 + 6\mathbb{E}\left\|\int_0^t AS(t-s)g(s, x(s-r(s)))ds\right\|^2 \end{aligned}$$

$$\begin{aligned}
& + 6\mathbb{E}\left\|\int_0^t S(t-s)f(s-\rho(s))ds\right\|^2 + 6\mathbb{E}\left\|\int_0^t S(t-s)\sigma(s)dB^H(s)\right\|^2 \\
& + 6\mathbb{E}\left\|\int_0^t \int_{\mathcal{U}} S(t-s)h(s, x(s-\theta(s)), y)\tilde{N}(ds, dy)\right\|^2 \\
& := 6(I_1 + I_2 + I_3 + I_4 + I_5 + I_6).
\end{aligned} \tag{3.1}$$

Now, let us estimate the terms on the right of the inequality (3.1).

Let $M^* = M^*(\varphi, a) > 0$ and $a > 0$ such that

$$\mathbb{E}\|x(t)\|^2 \leq M^*e^{-at}, \quad \forall t \in [0, T].$$

Without loss of generality we may assume that $0 < a < \lambda$. Then, by assumption (H.1) we have

$$I_1 \leq M^2\mathbb{E}\|\varphi(0) + g(0, \varphi(-r(0)))\|^2 e^{-\lambda t} \leq C_1 e^{-\lambda t}, \tag{3.2}$$

where $C_1 = M^2\mathbb{E}\|\varphi(0) + g(0, \varphi(-r(0)))\|^2 < +\infty$.

By using assumption (H.3) and the fact that the operator $(-A)^{-\beta}$ is bounded, we obtain that

$$\begin{aligned}
I_2 & \leq \|(-A)^{-\beta}\|^2 \mathbb{E}\|(-A)^{\beta}g(t, x(t-r(t))) - (-A)^{\beta}g(t, 0)\|^2 \\
& \leq K_2\|(-A)^{-\beta}\|^2 \mathbb{E}\|x(t-r(t))\|^2 \\
& \leq K_2\|(-A)^{-\beta}\|^2 (M^*e^{-(t-r(t))} + \mathbb{E}\|\varphi(t-r(t))\|^2) \\
& \leq K_2\|(-A)^{-\beta}\|^2 (M^* + M_0\mathbb{E}|\varphi|_{\mathcal{D}}^2)e^{-(t-r(t))} \\
& \leq K_2\|(-A)^{-\beta}\|^2 (M^* + M_0\mathbb{E}|\varphi|_{\mathcal{D}}^2)e^{-at}e^{a\tau} \\
& \leq C_2 e^{-at},
\end{aligned} \tag{3.3}$$

where $C_2 = K_2\|(-A)^{-\beta}\|^2 (M^* + M_0\mathbb{E}|\varphi|_{\mathcal{D}}^2)e^{a\tau} < +\infty$.

To estimate I_3 , we use the trivial identity

$$c^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-ct} dt, \quad \forall c > 0. \tag{3.4}$$

Using Hölder's inequality, Lemma 2.3 together with assumption (H.3) and the identity (3.4), we get

$$\begin{aligned}
I_3 & \leq \mathbb{E}\left\|\int_0^t AS(t-s)g(s, x(s-r(s)))ds\right\|^2 \\
& \leq \int_0^t \|(-A)^{1-\beta}S(t-s)\|ds \int_0^t \|(-A)^{1-\beta}S(t-s)\| \mathbb{E}\|(-A)^{\beta}g(s, x(s-r(s)))\|^2 ds
\end{aligned}$$

$$\begin{aligned}
&\leq M_{1-\beta}^2 K_2 \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} \mathbb{E} \|x(s-r(s))\|^2 ds \\
&\leq M_{1-\beta}^2 K_2 \lambda^{-\beta} \Gamma(\beta) \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} (M^* + M_0 \mathbb{E} |\varphi|_{\mathcal{D}}^2) e^{-as} e^{a\tau} ds \\
&\leq M_{1-\beta}^2 K_2 \lambda^{-\beta} \Gamma(\beta) (M^* + M_0 \mathbb{E} |\varphi|_{\mathcal{D}}^2) e^{-at} e^{a\tau} \int_0^t (t-s)^{\beta-1} e^{(a-\lambda)(t-s)} ds \\
&\leq M_{1-\beta}^2 K_2 \lambda^{-\beta} \Gamma^2(\beta) (\lambda-a)^{-1} (M^* + M_0 \mathbb{E} |\varphi|_{\mathcal{D}}^2) e^{a\tau} e^{-at} \\
&\leq C_3 e^{-at},
\end{aligned} \tag{3.5}$$

where $C_3 = M_{1-\beta}^2 K_2 \lambda^{-\beta} \Gamma^2(\beta) (\lambda-a)^{-1} (M^* + M_0 \mathbb{E} |\varphi|_{\mathcal{D}}^2) e^{a\tau} < +\infty$.

Similar computations can be used to estimate the term I_4 .

$$\begin{aligned}
I_4 &\leq \mathbb{E} \left\| \int_0^t S(t-s) f(s, x(s-\rho(s))) ds \right\|^2 \\
&\leq M^2 K_1 \int_0^t e^{-\lambda(t-s)} ds \int_0^t e^{-\lambda(t-s)} \mathbb{E} \|x(s-\rho(s))\|^2 ds \\
&\leq M^2 K_1 \lambda^{-1} \int_0^t e^{-\lambda(t-s)} (M^* + M_0 \mathbb{E} |\varphi|_{\mathcal{D}}^2) e^{-as} e^{a\tau} ds \\
&\leq M^2 K_1 \lambda^{-1} (M^* + M_0 \mathbb{E} |\varphi|_{\mathcal{D}}^2) e^{-at} e^{a\tau} \int_0^t e^{(a-\lambda)(t-s)} ds \\
&\leq M^2 K_1 \lambda^{-1} (\lambda-a)^{-1} (M^* + M_0 \mathbb{E} |\varphi|_{\mathcal{D}}^2) e^{-at} e^{a\tau} \\
&\leq C_4 e^{-at}.
\end{aligned} \tag{3.6}$$

By using Lemma 2.2, we get that

$$\begin{aligned}
I_5 &\leq \mathbb{E} \left\| \int_0^t S(t-s) \sigma(s) dB^H(s) \right\|^2 \\
&\leq 2M^2 H t^{2H-1} \int_0^t e^{-2\lambda(t-s)} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds,
\end{aligned} \tag{3.7}$$

If $\gamma < \lambda$, then the following estimate holds

$$\begin{aligned}
I_5 &\leq 2M^2 H t^{2H-1} \int_0^t e^{-2\lambda(t-s)} e^{-2\gamma(t-s)} e^{2\gamma(t-s)} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds \\
&\leq 2M^2 H t^{2H-1} e^{-2\gamma t} \int_0^t e^{-2(\lambda-\gamma)(t-s)} e^{2\gamma s} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds
\end{aligned}$$

$$\begin{aligned}
&\leq 2M^2 HT^{2H-1} e^{-2\gamma t} \int_0^T e^{-2(\lambda-\gamma)(t-s)} e^{2\gamma s} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds \\
&\leq 2M^2 HT^{2H-1} e^{-2\gamma t} \int_0^T e^{2\gamma s} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds.
\end{aligned} \tag{3.8}$$

If $\gamma > \lambda$, then the following estimate holds

$$I_5 \leq 2M^2 HT^{2H-1} e^{-2\lambda t} \int_0^T e^{2\gamma s} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds. \tag{3.9}$$

In virtue of (3.7), (3.8) and (3.9) we obtain

$$I_5 \leq C_5 e^{-\min(\lambda, \gamma)t} \tag{3.10}$$

where $C_5 = 2M^2 HT^{2H-1} \int_0^T e^{2\gamma s} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds < +\infty$.

On the other hand, by assumptions (H.1) and (H.6), we get

$$\begin{aligned}
I_6 &\leq \mathbb{E} \left\| \int_0^t \int_{\mathcal{U}} S(t-s) h(s, x(s-\theta(s)), y) \tilde{N}(ds, dy) \right\|^2 \\
&\leq M^2 \mathbb{E} \int_0^t e^{-2\lambda(t-s)} \int_{\mathcal{U}} \|h(s, x(s-\theta(s)), y)\|^2 \nu(dy) ds \\
&\leq M^2 K_3 \int_0^t e^{-2\lambda(t-s)} \mathbb{E} \|x(s-\theta(s))\|^2 ds \\
&\leq M^2 K_3 \int_0^t e^{-2\lambda(t-s)} (M^* + M_0 \mathbb{E} |\varphi|_{\mathcal{D}}^2) e^{-as} e^{a\tau} ds \\
&\leq M^2 K_3 (M^* + M_0 \mathbb{E} |\varphi|_{\mathcal{D}}^2) e^{-at} e^{a\tau} \int_0^t e^{(-2\lambda+a)(t-s)} ds \\
&\leq M^2 K_3 (M^* + M_0 \mathbb{E} |\varphi|_{\mathcal{D}}^2) e^{a\tau} (2\lambda - a)^{-1} e^{-at} \\
&\leq C_6 e^{-at},
\end{aligned} \tag{3.11}$$

where $C_6 = M^2 K_3 (M^* + M_0 \mathbb{E} |\varphi|_{\mathcal{D}}^2) e^{a\tau} (2\lambda - a)^{-1} < +\infty$.

Inequalities (3.2), (3.3), (3.5), (3.6), (3.10) and (3.11) together imply that

$$\mathbb{E} \|\Psi(x)(t)\|^2 \leq \overline{M} e^{-\bar{a}t}, \quad t \geq 0.$$

for some $\overline{M} > 0$ and $\bar{a} > 0$.

Next we show that $\Psi(x)(t)$ is càdlàg process on Π . Let $0 < t < T$ and $h > 0$ be sufficiently small. Then for any fixed $x(t) \in \Pi$, we have

$$\mathbb{E} \|\Psi(x)(t+h) - \Psi(x)(t)\|^2$$

$$\begin{aligned}
&\leq 6\mathbb{E}\|(S(t+h) - S(t))(\varphi(0) + g(0, \varphi(-r(0))))\|^2 \\
&+ 6\mathbb{E}\|g(t+h, x(t+h-r(t+h))) - g(t, x(t-r(t)))\|^2 \\
&+ 6\mathbb{E}\left\|\int_0^{t+h} AS(t+h-s)g(s, x(s-r(s)))ds - \int_0^t AS(t-s)g(s, x(s-r(s)))ds\right\|^2 \\
&+ 6\mathbb{E}\left\|\int_0^{t+h} S(t+h-s)f(s-\rho(s))ds - \int_0^t S(t-s)f(s-\rho(s))ds\right\|^2 \\
&+ 6\mathbb{E}\left\|\int_0^{t+h} S(t+h-s)\sigma(s)dB^H(s) - \int_0^t S(t-s)\sigma(s)dB^H(s)\right\|^2 \\
&+ 6\mathbb{E}\left\|\int_0^{t+h} \int_{\mathcal{U}} S(t+h-s)h(s, x(s-\theta(s)), y)\tilde{N}(ds, dy) \right. \\
&\quad \left. - \int_0^t \int_{\mathcal{U}} S(t-s)h(s, x(s-\theta(s)), y)\tilde{N}(ds, dy)\right\|^2 \\
&= 6 \sum_{1 \leq i \leq 6} I_i(h).
\end{aligned}$$

For $i = 1, 2, \dots, 5$, the terms $I_i(h)$ can be treated in the same way as in the proof of Theorem 5 in [5].

For the term $I_6(h)$, we have by assumption $(\mathcal{H}.1)$

$$\begin{aligned}
I_6(h) &\leq 2\mathbb{E}\left\|\int_0^t \int_{\mathcal{U}} (S(t+h-s) - S(t-s))h(s, x(s-\theta(s)), y)\tilde{N}(ds, dy)\right\|^2 \\
&+ 2\mathbb{E}\left\|\int_t^{t+h} \int_{\mathcal{U}} S(t+h-s)h(s, x(s-\theta(s)), y)\tilde{N}(ds, dy)\right\|^2 \\
&\leq 2M^2\|S(h) - I\|\mathbb{E}\int_0^t \int_{\mathcal{U}} e^{-2\lambda(t-s)}\|h(s, x(s-\theta(s)), y)\|^2\nu(dy)ds \\
&+ 2M^2\mathbb{E}\int_t^{t+h} \int_{\mathcal{U}} e^{-2\lambda(t+h-s)}\|h(s, x(s-\theta(s)), y)\|^2\nu(dy)ds \quad (3.12)
\end{aligned}$$

By assumption $(\mathcal{H}.6)$, we have

$$\begin{aligned}
\mathbb{E}\int_0^t \int_{\mathcal{U}} e^{-2\lambda(t-s)}\|h(s, x(s-\theta(s)), y)\|^2\nu(dy)ds &\leq K_3 \int_0^t e^{-2\lambda(t-s)}\mathbb{E}\|x(s-\theta(s))\|^2 ds \\
&\leq K_3 \int_0^t e^{-2\lambda(t-s)}(M^* + M_0\mathbb{E}|\varphi|_{\mathcal{D}}^2)e^{-as}e^{a\tau} ds \\
&\leq K_3(M^* + M_0\mathbb{E}|\varphi|_{\mathcal{D}}^2)e^{a\tau}(2\lambda - a)^{-1}e^{-at}. \quad (3.13)
\end{aligned}$$

Inequality (3.13) imply that there exist a constant $B > 0$ such that

$$\mathbb{E} \int_0^t \int_{\mathcal{U}} e^{-2\lambda(t-s)} \|h(s, x(s - \theta(s)), y)\|^2 \nu(dy) ds \leq B. \quad (3.14)$$

Using the strong continuity of $S(t)$ together with inequalities (3.12) and (3.14) we obtain that $I_6(h) \rightarrow 0$ as $h \rightarrow 0$.

The above arguments show that $\Psi(x)(t)$ is càdlàg process. Then, we conclude that $\Psi(\Pi) \subset \Pi$.

Now, we are going to show that $\Psi : \Pi \rightarrow \Pi$ is a contraction mapping. For this end, fix $x, y \in \Pi$, we have

$$\begin{aligned} \mathbb{E} \|\Psi(x)(t) - \Psi(y)(t)\|^2 &\leq 4\mathbb{E} \|g(t, x(t - r(t))) - g(t, y(t - r(t)))\|^2 \\ &\quad + 4\mathbb{E} \left\| \int_0^t AS(t-s)(g(s, x(s - r(s))) - g(s, y(s - r(s)))) ds \right\|^2 \\ &\quad + 4\mathbb{E} \left\| \int_0^t S(t-s)(f(s, x(s - \rho(s))) - f(s, y(s - \rho(s)))) ds \right\|^2 \\ &\quad + 4\mathbb{E} \left\| \int_0^t S(t-s) \left(\int_{\mathcal{U}} h(s, x(s - \theta(s)), z) - h(s, y(s - \theta(s)), z) \tilde{N}(ds, dz) \right) \right\|^2 \\ &:= 4(J_1 + J_2 + J_3 + J_4). \end{aligned} \quad (3.15)$$

We estimate the various terms of the right hand of (3.15) separately.

For the first term, we have

$$\begin{aligned} J_1 &\leq \mathbb{E} \|g(t, x(t - r(t))) - g(t, y(t - r(t)))\|^2 \\ &\leq K_2 \|(-A)^{-\beta}\|^2 \mathbb{E} \|x(s - r(s)) - y(s - r(s))\|^2 \\ &\leq K_2 \|(-A)^{-\beta}\|^2 \sup_{s \geq 0} \mathbb{E} \|x(s) - y(s)\|^2. \end{aligned} \quad (3.16)$$

For the second term, combining Lemma 2.3 and Hölder's inequality, we get

$$\begin{aligned} J_2 &\leq \mathbb{E} \left\| \int_0^t AS(t-s)(g(s, x(s - r(s))) - g(s, y(s - r(s)))) ds \right\|^2 \\ &\leq K_2 M_{1-\beta}^2 \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} \mathbb{E} \|x(s - r(s)) - y(s - r(s))\|^2 ds \\ &\leq K_2 M_{1-\beta}^2 \lambda^{-\beta} \Gamma(\beta) \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \left(\sup_{s \geq 0} \mathbb{E} \|x(s) - y(s)\|^2 \right) \end{aligned}$$

$$\leq K_2 M_{1-\beta}^2 \lambda^{-2\beta} \Gamma(\beta)^2 \sup_{s \geq 0} \mathbb{E} \|x(s) - y(s)\|^2. \quad (3.17)$$

For the third term, by assumption $(\mathcal{H}.2)$, we get that

$$\begin{aligned} J_3 &\leq \mathbb{E} \left\| \int_0^t S(t-s) (f(s, x(s-\rho(s))) - f(s, y(s-\rho(s)))) ds \right\|^2 \\ &\leq K_1 M^2 \int_0^t e^{-\lambda(t-s)} ds \int_0^t e^{-\lambda(t-s)} \mathbb{E} \|x(s-\rho(s)) - y(s-\rho(s))\|^2 ds \\ &\leq K_1 M^2 \lambda^{-2} \sup_{s \geq 0} \mathbb{E} \|x(s) - y(s)\|^2. \end{aligned} \quad (3.18)$$

For the last term, by using assumption $(\mathcal{H}.6)$, we get

$$\begin{aligned} J_4 &\leq \mathbb{E} \int_0^t S(t-s) \left(\int_{\mathcal{U}} h(s, x(s-\theta(s)), z) - h(s, y(s-\theta(s)), z) \tilde{N}(ds, dz) \right)^2 \\ &\leq M^2 \mathbb{E} \int_0^t e^{-2\lambda(t-s)} \left(\int_{\mathcal{U}} \|h(s, x(s-\theta(s)), z) - h(s, y(s-\theta(s)), z)\|^2 \nu(dz) ds \right) \\ &\leq M^2 K_3 (2\lambda)^{-1} \sup_{s \geq 0} \mathbb{E} \|x(s) - y(s)\|^2. \end{aligned} \quad (3.19)$$

Thus, inequality (3.16), (3.17), (3.18) and (3.19) together imply

$$\begin{aligned} \sup_{t \in [0, T]} \mathbb{E} \|\Psi(x)(t) - \Psi(y)(t)\|^2 &\leq 4 \left(K_2 \|(-A)^{-\beta}\|^2 + K_2 M_{1-\beta}^2 \lambda^{-2\beta} \Gamma(\beta)^2 + K_1 M^2 \lambda^{-2} \right. \\ &\quad \left. + M^2 K_3 (2\lambda)^{-1} \right) \left(\sup_{t \geq 0} \mathbb{E} \|x(t) - y(t)\|^2 \right). \end{aligned}$$

Therefore by the condition of the theorem it follows that Ψ is a contractive mapping. Thus by the Banach fixed point theorem Ψ has the fixed point $x(t) \in \Pi$, which is a unique mild solution to (1.1) satisfying $x(s) = \varphi(s)$ on $[-\tau, 0]$.

By the definition of the space Π this solution is exponentially stable in mean square. This completes the proof. $\square$

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