

HEMI-SLANT SUBMERSIONS FROM ALMOST PRODUCT RIEMANNIAN MANIFOLDS

MEHMET AKIF AKYOL^{1*} AND YILMAZ GÜNDÜZALP²

ABSTRACT. In this paper, we introduce hemi-slant submersions from almost product Riemannian manifolds onto Riemannian manifolds. We give an example, investigate the geometry of foliations which are arisen from the definition of a Riemannian submersion. We also find necessary and sufficient conditions for a hemi-slant submersion to be totally geodesic.

1. INTRODUCTION

The theory of smooth maps between Riemannian manifolds has been widely studied in Riemannian geometry. Such maps are useful for comparing geometric structures between two manifolds. In this point of view, the study of Riemannian submersions between Riemannian manifolds was initiated by O'Neill [21] and Gray [12], see also [8] and [34]. Riemannian submersions have several applications in mathematical physics. Indeed, Riemannian submersions have their applications in the Yang-Mills theory ([6], [33]), Kaluza-Klein theory ([7], [16]), supergravity and superstring theories ([17], [20]), etc. Later such submersions were considered between manifolds with differentiable structures, see [8]. Furthermore, we have the following submersions: semi-Riemannian submersion and Lorentzian submersion [8], Riemannian submersion [12], almost Hermitian submersion [32], contact-complex submersion [19], quaternionic submersion [18], etc.

Recently, B. Şahin [26] introduced the notion of anti-invariant Riemannian submersions which are Riemannian submersions from almost Hermitian manifolds such that the vertical distributions are anti-invariant under the almost complex structure of the total manifold. Besides there are many other notions related with that of anti-invariant Riemannian submersions. see:([1], [2], [4], [5], [9], [10], [11], [13], [14], [15], [22], [23], [24], [25], [28], [29], [30]). In particular, B. Şahin [27] introduced the notion of semi-invariant Riemannian submersions and slant submersions when the base manifold is an almost Hermitian manifold. He showed that such submersions have rich geometric properties and they are useful for investigating the geometry of the total space. On the other hand, as a generalization of semi-invariant submersions and slant submersions, Taştan, Şahin and

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* Corresponding author.

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Yanan [31] introduced the notion of hemi-slant submersions. They showed that such submersions have rich geometric properties and they are useful for investigating the geometry of the total space. The present work, we define and study the notion of hemi-slant submersions from almost product Riemannian manifolds. The paper is organized as follows. In section 2, we recall some notions needed for this paper. In section 3, we define hemi-slant submersions from an almost product Riemannian manifold onto a Riemannian manifold. We also investigate the geometry of leaves of the distributions. Finally we give necessary and sufficient conditions for such submersions to be totally geodesic.

2. PRELIMINARIES

In this section, we define almost product Riemannian manifolds, recall the notion of Riemannian submersions between Riemannian manifolds and give a brief review of basic facts of Riemannian submersions.

Let M be a m -dimensional manifold with a tensor F of a type $(1,1)$ such that

$$F^2 = I, (F \neq I). \quad (2.1)$$

Then, we say that M is an almost product manifold with almost product structure F . We put

$$P = \frac{1}{2}(I + F), \quad Q = \frac{1}{2}(I - F).$$

Then we get

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad F = P - Q.$$

Thus P and Q define two complementary distributions P and Q . We easily see that the eigenvalues of F are $+1$ or -1 . If an almost product manifold M admits a Riemannian metric g such that

$$g(FX, FY) = g(X, Y) \quad (2.2)$$

for any vector fields X and Y on M , then M is called an almost product Riemannian manifold, denoted by (M, g, F) . Denote the Levi-Civita connection on M with respect to g by ∇^M . Then, M is called a locally product Riemannian manifold [34] if F is parallel with respect to ∇^M , i.e.,

$$\nabla_X^M F = 0, \quad X \in \Gamma(TM). \quad (2.3)$$

Let (M, g) and (N, g') be two Riemannian manifolds. A surjective C^∞ -map $\pi : M \rightarrow N$ is a C^∞ -submersion if it has maximal rank at any point of M . Putting $\mathcal{V}_x = \ker \pi_{*x}$, for any $x \in M$, we obtain an integrable distribution $\mathcal{V}$, which is called vertical distribution and corresponds to the foliation of M determined by the fibres of π . The complementary distribution $\mathcal{H}$ of $\mathcal{V}$, determined by the Riemannian metric g , is called horizontal distribution. A C^∞ -submersion $\pi : M \rightarrow N$ between two Riemannian manifolds (M, g) and (N, g') is called a Riemannian submersion if, at each point x of M , π_{*x} preserves the length of the horizontal vectors. A horizontal vector field X on M is said to be basic if X is π -related to a vector field X' on N . It is clear that every vector field X' on N

has a unique horizontal lift X to M and X is basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion $\pi : M \rightarrow N$ determines two $(1, 2)$ tensor fields T and A on M , by the formulas:

$$T(E, F) = T_E F = \mathcal{H} \nabla_{\mathcal{V}E}^M \mathcal{V}F + \mathcal{V} \nabla_{\mathcal{V}E}^M \mathcal{H}F \quad (2.4)$$

and

$$A(E, F) = A_E F = \mathcal{V} \nabla_{\mathcal{H}E}^M \mathcal{H}F + \mathcal{H} \nabla_{\mathcal{H}E}^M \mathcal{H}F \quad (2.5)$$

for any $E, F \in \Gamma(TM)$, where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections (see [8]). From (2.4) and (2.5), one can obtain

$$\nabla_V^M W = T_V W + \hat{\nabla}_V W; \quad (2.6)$$

$$\nabla_V^M X = T_V X + \mathcal{H}(\nabla_V^M X); \quad (2.7)$$

$$\nabla_X^M V = \mathcal{V}(\nabla_X^M V) + A_X V; \quad (2.8)$$

$$\nabla_X^M Y = A_X Y + \mathcal{H}(\nabla_X^M Y), \quad (2.9)$$

for any $X, Y \in \Gamma((\ker \pi_*)^\perp)$ and $V, W \in \Gamma(\ker \pi_*)$. Moreover, if X is basic then

$$\mathcal{H}(\nabla_V^M X) = \mathcal{V}(\nabla_X^M V) = A_X V. \quad (2.10)$$

We note that for $U, V \in \Gamma(\ker \pi_*)$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma((\ker \pi_*)^\perp)$, $A_X Y = \frac{1}{2} \mathcal{V}[X, Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X, Y \in \Gamma((\ker \pi_*)^\perp)$ and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U, V \in \Gamma(\ker \pi_*)$.

We now recall the following result which will be useful for later.

Lemma 2.1. (see [8],[21]). *If $\pi : M \rightarrow N$ is a Riemannian submersion and X, Y basic vector fields on M , π -related to X' and Y' on N , then we have the following properties*

- (1) $\mathcal{H}[X, Y]$ is a basic vector field and $\pi_* \mathcal{H}[X, Y] = [X', Y'] \circ \pi$;
- (2) $\mathcal{H}(\nabla_X^M Y)$ is a basic vector field π -related to $(\nabla_{X'}^N Y')$, where ∇^M and ∇^N are the Levi-Civita connection on M and N ;
- (3) $[E, U] \in \Gamma(\ker \pi_*)$, for any $U \in \Gamma(\ker \pi_*)$ and for any basic vector field E .

Let (M, g_M) and (N, g_N) be Riemannian manifolds and $\pi : M \rightarrow N$ is a smooth map. Then the second fundamental form of π is given by

$$(\nabla \pi_*)(X, Y) = \nabla_{\pi_* X} \pi_* Y - \pi_*(\nabla_X Y) \quad (2.11)$$

for $X, Y \in \Gamma(TM)$, where we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N . Recall that π is called a *totally geodesic* map if $(\nabla \pi_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ [3]. It is known that the second fundamental form is symmetric.

3. HEMI-SLANT SUBMERSIONS

In this section, we define hemi-slant submersions from an almost product Riemannian manifold onto a Riemannian manifold, investigate the integrability of distributions and obtain a necessary and sufficient condition for such submersions to be totally geodesic map.

Definition 3.1. Let (M, g_M, F) be an almost product Riemannian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $\pi : (M, g_M, F) \rightarrow (N, g_N)$ is called a hemi-slant submersion if the vertical distribution $\ker \pi_*$ of π admits two orthogonal complementary distributions D^θ and $D^\perp$ such that D^θ is slant and $D^\perp$ is anti-invariant, i.e, we have

$$\ker \pi_* = D^\theta \oplus D^\perp. \quad (3.1)$$

In this case, the angle θ is called the hemi-slant angle of the submersion. Suppose the dimension of distribution of $D^\perp$ (resp. D^θ) is m_1 (resp. m_2). Then we easily see the following particular cases.

- (a) If $m_2 = 0$, then M is an anti-invariant submersion [10].
- (b) If $m_1 = 0$ and $\theta = 0$, then M is an invariant submersion [11].
- (c) If $m_1 = 0$ and $\theta \neq 0, \frac{\pi}{2}$, then M is a proper slant submersion with slant angle θ [11].
- (d) If $\theta = \frac{\pi}{2}$, then M is an anti-invariant submersion.
- (e) If $m_1 \neq 0$ and $\theta = 0$, then M is an semi-invariant submersion.

We say that the hemi-slant submersion $\pi : (M, g_M, F) \rightarrow (N, g_N)$ is proper if $D^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$. As we have seen from above argument, anti-invariant submersions, semi-invariant submersions and slant submersions are all examples of hemi-slant submersions. Now, we present an example of proper hemi-slant submersions in locally product Riemannian manifolds and demonstrate that the method presented in this paper is effective. Note that given an Euclidean space $\mathbb{R}^8$ with coordinates $(x_1, \dots, x_8)$, we can canonically choose an almost product structure F on $\mathbb{R}^8$ as follows:

$$F(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = (-a_2, -a_1, a_4, a_3, -a_6, -a_5, a_8, a_7),$$

where $a_1, \dots, a_8 \in \mathbb{R}$.

Example 3.2. Let π be a submersion defined by

$$\pi : \begin{array}{ccc} \mathbb{R}^8 & \longrightarrow & \mathbb{R}^4 \\ (x_1, \dots, x_8) & \longmapsto & (x_1 \cos \alpha - x_4 \sin \alpha, x_2 \sin \beta - x_3 \cos \beta, \frac{x_5+x_7}{\sqrt{2}}, \frac{x_6+x_8}{\sqrt{2}}). \end{array}$$

Then it follows that

$$\begin{aligned} \ker \pi_* = \text{span}\{V_1 = \sin \alpha \partial x_1 + \cos \alpha \partial x_4, V_2 = \cos \beta \partial x_2 + \sin \beta \partial x_3, \\ V_3 = -\partial x_5 + \partial x_7, V_4 = -\partial x_6 + \partial x_8\} \end{aligned}$$

and

$$\begin{aligned} (\ker \pi_*)^\perp = \text{span}\{H_1 = \cos \alpha \partial x_1 - \sin \alpha \partial x_4, H_2 = \sin \beta \partial x_2 - \cos \beta \partial x_3, \\ H_3 = \frac{1}{\sqrt{2}}(\partial x_5 + \partial x_7), H_4 = \frac{1}{\sqrt{2}}(\partial x_6 + \partial x_8)\}. \end{aligned}$$

Thus it follows that $D^\theta = \text{span}\{V_1, V_2\}$ with the slant angle $\cos \theta = \sin(\beta - \alpha)$ and $D^\perp = \text{span}\{V_3, V_4\}$. Also by direct computations, we get

$$\begin{aligned} g(H_1, H_1) &= g'(\pi_* H_1, \pi_* H_1) \text{ and } g(H_2, H_2) = g'(\pi_* H_2, \pi_* H_2), \\ g(H_3, H_3) &= g'(\pi_* H_3, \pi_* H_3) \text{ and } g(H_4, H_4) = g'(\pi_* H_4, \pi_* H_4) \end{aligned}$$

which show that π is a Riemannian submersion. Thus π is a proper hemi-slant submersion.

Let π be a hemi-slant submersion from an almost product Riemannian manifold (M, g_M, F) onto a Riemannian manifold (N, g_N) . Then for $V \in \Gamma(\ker \pi_*)$, we put

$$V = \mathcal{P}V + \mathcal{Q}V, \quad (3.2)$$

where $\mathcal{P}V \in D^\theta$ and $\mathcal{Q}V \in D^\perp$ and write

$$FV = \phi V + \omega V, \quad (3.3)$$

where $\phi V \in \Gamma(\ker \pi_*)$ and $\omega V \in \Gamma((\ker \pi_*)^\perp)$. Also, for any $X \in \Gamma((\ker \pi_*)^\perp)$, we have

$$FX = \mathcal{B}X + \mathcal{C}X, \quad (3.4)$$

where $\mathcal{B}X \in \Gamma(\ker \pi_*)$ and $\mathcal{C}X \in \Gamma((\ker \pi_*)^\perp)$. We denote the complementary distribution to $\omega D^\theta \oplus FD^\perp$ in $(\ker \pi_*)^\perp$ by μ . It is invariant distribution of $(\ker \pi_*)^\perp$ with respect to F . Then, the horizontal distribution $(\ker \pi_*)^\perp$ is decomposed as

$$(\ker \pi_*)^\perp = \omega D^\theta \oplus FD^\perp \oplus \mu. \quad (3.5)$$

From (3.3), (3.4) and (3.5) we have

$$\phi D^\theta = D^\theta, \quad \phi D^\perp = \{0\}, \quad \mathcal{B}\omega D^\theta = D^\theta, \quad \mathcal{B}FD^\perp = D^\perp.$$

On the other hand, using (3.3), (3.4) and the fact that $F^2 = I$, we obtain

$$\phi^2 + \mathcal{B}\omega = I, \quad \mathcal{C}^2 + \omega\mathcal{B} = I, \quad \omega\phi + \mathcal{C}\omega = 0, \quad \mathcal{B}\mathcal{C} + \phi\mathcal{B} = 0.$$

Then by using (2.6), (2.7), (3.3) and (3.4) we get

$$(\nabla_U^M \phi)V = \mathcal{B}T_U V - T_U \omega V \quad (3.6)$$

$$(\nabla_U^M \omega)V = \mathcal{C}T_U V - T_U \phi V \quad (3.7)$$

for $U, V \in \Gamma(\ker \pi_*)$, where

$$(\nabla_U^M \phi)V = \hat{\nabla}_U \phi V - \phi \hat{\nabla}_U V$$

and

$$(\nabla_U^M \omega)V = A_{\omega V} U - \omega \hat{\nabla}_U V.$$

The proof of the following theorem is exactly the same as that one for hemi-slant submersions, see Theorem 3.4 of [22]. So, we omit it.

Theorem 3.3. *Let π be a Riemannian submersion from a locally product Riemannian manifold (M, g_M, F) onto a Riemannian manifold (N, g_N) . Then π is a hemi-slant submersion if and only if there exists a constant $[0, 1]$ and a distribution D on $\ker \pi_*$ such that*

- (a) $D = \{V \in \ker \pi_* \mid \phi^2 V = \lambda V\}$,
- (b) for any $W \in \ker \pi_*$ orthogonal to D , we have $\phi W = 0$.

Moreover, in this case $\lambda = \cos^2 \theta$, where θ is the slant angle of π .

Thus, from Theorem 3.3, for any $Z \in D^\theta$, we conclude that

$$\phi^2 Z = \cos^2 \theta Z. \quad (3.8)$$

On the other hand, for any $Z, W \in D^\theta$, using (2.1), (3.3) and (3.8), we get

$$g_M(\phi Z, \phi W) = \cos^2 \theta g_M(Z, W). \quad (3.9)$$

Also, using (2.1), (3.3) and (3.8), we find

$$g_M(\omega Z, \omega W) = \sin^2 \theta g_M(Z, W). \quad (3.10)$$

Next, we easily have the following lemma:

Lemma 3.4. *Let (M, g_M, F) be a locally product Riemannian manifold and (N, g_N) a Riemannian manifold. Let $\pi : (M, g_M, F) \rightarrow (N, g_N)$ be a hemi-slant submersion. Then we have*

(a)

$$\begin{aligned} A_X \mathcal{B}Y + \mathcal{H} \nabla_X^M \mathcal{C}Y &= \mathcal{C} \mathcal{H} \nabla_X^M Y + \omega A_X Y \\ \mathcal{V} \nabla_X^M \mathcal{B}Y + A_X \mathcal{C}Y &= \mathcal{B} \mathcal{H} \nabla_X^M Y + \phi A_X Y, \end{aligned}$$

(b)

$$\begin{aligned} T_U \phi V + A_{\omega V} U &= \mathcal{C} T_U V + \omega \hat{\nabla}_U V \\ \hat{\nabla}_U \phi V + T_U \omega V &= \mathcal{B} T_U V + \phi \hat{\nabla}_U V, \end{aligned}$$

(c)

$$\begin{aligned} A_X \phi U + \mathcal{H} \nabla_U^M \omega V &= \mathcal{C} A_X U + \omega \mathcal{V} \nabla_X^M U \\ \mathcal{V} \nabla_X^M \phi U + A_X \omega U &= \mathcal{B} A_X U + \phi \mathcal{V} \nabla_X^M U, \end{aligned}$$

for $X, Y \in \Gamma((\ker \pi_*)^\perp)$ and $U, V \in \Gamma(\ker \pi_*)$.

We now examine the integrability conditions for the anti-invariant distribution $D^\perp$ and the slant distribution D^θ .

Theorem 3.5. *Let π be a purely hemi-slant submersion from a locally product Riemannian manifold (M, g_M, F) onto a Riemannian manifold (N, g_N) . Then the anti-invariant distribution $D^\perp$ is integrable if and only if we have*

$$g_N((\nabla \pi_*)(U, FV) - (\nabla \pi_*)(V, FU), \mathcal{C}Z) = g_M(T_U FV - T_V FU, \mathcal{B}Z)$$

for $U, V \in \Gamma(D^\perp)$ and $Z \in \Gamma(D^\theta)$.

Proof. For $U, V \in \Gamma(D^\perp)$ and $Z \in \Gamma(D^\theta)$, using (2.1), (2.2) and (3.4), we get

$$\begin{aligned} g_M([U, V], Z) &= g_M(\nabla_U^M FV, \phi Z) + g_M(\nabla_U^M FV, \omega Z) - g_M(\nabla_V^M FU, \phi Z) \\ &\quad - g_M(\nabla_V^M FU, \omega Z). \end{aligned}$$

Since π is a hemi-slant submersion, from (2.7) and (2.11), we get

$$\begin{aligned} g_M([U, V], Z) &= g_M(T_U FV, \phi Z) + g_N(\pi_*(\nabla_U^M FV), \pi_*\omega Z) - g_M(T_V FU, \phi Z) \\ &\quad - g_N(\pi_*(\nabla_V^M FU), \omega Z) \\ &= g_M(T_U FV - T_V FU, \phi Z) + g_N((\nabla\pi_*)(V, FU) - (\nabla\pi_*)(U, FV), \omega Z) \end{aligned}$$

which proves assertion. $\square$

Theorem 3.6. *Let π be a purely hemi-slant submersion from a locally product Riemannian manifold (M, g_M, F) onto a Riemannian manifold (N, g_N) . Then the slant distribution D^θ is integrable if and only if we have*

$$g_M(T_W\omega\phi Z - T_Z\omega\phi W, U) = g_N((\nabla\pi_*)(Z, \omega W) - (\nabla\pi_*)(W, \omega Z), FU)$$

for $Z, W \in \Gamma(D^\theta)$ and $U \in \Gamma(D^\perp)$.

Proof. For $Z, W \in \Gamma(D^\theta)$ and $U \in \Gamma(D^\perp)$, using (2.1), (2.2) and (3.4), we get

$$\begin{aligned} g_M([Z, W], U) &= g_M(\nabla_Z^M \omega W, FU) + g_M(\nabla_Z^M F\phi W, U) - g_M(\nabla_W^M \omega Z, FZ) \\ &\quad - g_M(\nabla_W^M F\phi Z, U). \end{aligned}$$

If we take into account that π is a hemi-slant submersion, then from (2.7), (2.11) and (3.9), we get

$$\begin{aligned} g_M([Z, W], U) &= g_N(\pi_*(\nabla_Z^M \omega W), \pi_*FU) + \cos^2 \theta g_M(\nabla_Z^M W, U) + g_M(\nabla_Z^M \omega\phi W, U) \\ &\quad - g_N(\pi_*(\nabla_W^M \omega Z), \pi_*FU) - \cos^2 \theta g_M(\nabla_W^M Z, U) - g_M(\nabla_W^M \omega\phi Z, U) \\ &= g_N((\nabla\pi_*)(W, \omega Z) - (\nabla\pi_*)(Z, \omega W), FU) + \cos^2 \theta g_M([Z, W], U) \\ &\quad + g_M(T_Z\omega\phi W - T_W\omega\phi Z, U) \end{aligned}$$

or

$$\begin{aligned} \sin^2 \theta g_M([Z, W], U) &= g_N((\nabla\pi_*)(W, \omega Z) - (\nabla\pi_*)(Z, \omega W), FU) \\ &\quad + g_M(T_Z\omega\phi W - T_W\omega\phi Z, U) \end{aligned}$$

which proves assertion. $\square$

Now, we investigate the geometry of the leaves of the distributions $D^\perp$ and D^θ .

Theorem 3.7. *Let π be a purely hemi-slant submersion from a locally product Riemannian manifold (M, g_M, F) onto a Riemannian manifold (N, g_N) . Then $D^\perp$ defines a totally geodesic foliation on M if and only if*

$$g_N((\nabla\pi_*)(U, FV), \pi_*\omega Z) = g_M(T_U V, \omega\phi Z)$$

and

$$g_N((\nabla\pi_*)(U, FV), \pi_*\mathcal{C}X) = g_M(T_U FV, \mathcal{B}Z)$$

for $U, V \in \Gamma(D^\perp)$, $Z \in \Gamma(D^\theta)$ and $X \in \Gamma((\ker \pi_*)^\perp)$.

Proof. From the definition of a hemi-slant submersion, it follows that the anti-invariant distribution $D^\perp$ defines a totally geodesic foliation on M if and only

if $g_M(\nabla_U^M V, Z) = 0$ and $g_M(\nabla_U^M V, X) = 0$ for $U, V \in \Gamma(D^\perp)$, $Z \in \Gamma(D^\theta)$ and $X \in \Gamma((\ker \pi_*)^\perp)$, from (2.1) and (2.2), we get

$$g_M(\nabla_U^M V, Z) = g_M(\nabla_U^M V, F\phi Z) + g_M(\nabla_U^M FV, \omega Z).$$

Since π is a hemi-slant submersion, using (2.6), (2.11) and (3.9) we get

$$g_M(\nabla_U^M V, Z) = g_M(\nabla_U^M V, \cos^2 \theta Z) + g_M(\nabla_U^M V, \omega \phi Z) + g_N(\pi_*(\nabla_U^M FV), \pi_* \omega Z)$$

or

$$\sin^2 \theta g_M(\nabla_U^M V, Z) = g_M(T_U V, \omega \phi Z) - g_N((\nabla \pi_*)(U, FV), \pi_* \omega Z). \quad (3.11)$$

On the other hand, by using (3.4) we have

$$g_M(\nabla_U^M V, X) = g_M(\nabla_U^M FV, \mathcal{B}X) + g_M(\nabla_U^M FV, \mathcal{C}X).$$

If we take into account that π is a hemi-slant submersion, then by using (2.7) and (2.11) we get

$$g_M(\nabla_U^M V, X) = g_M(T_U FV, \mathcal{B}Z) - g_N((\nabla \pi_*)(U, FV), \pi_* \mathcal{C}X). \quad (3.12)$$

Thus proof follows from (3.11) and (3.12). $\square$

For the leaves of D^θ we have the following result.

Theorem 3.8. *Let π be a purely hemi-slant submersion from a locally product Riemannian manifold (M, g_M, F) onto a Riemannian manifold (N, g_N) . Then D^θ defines a totally geodesic foliation on M if and only if*

$$g_N((\nabla \pi_*)(Z, \omega W), \pi_* FV) = g_M(T_Z \omega \phi W, U)$$

and

$$g_N((\nabla \pi_*)(Z, \omega \phi W), \pi_* X) + g_N((\nabla \pi_*)(Z, \omega W), \pi_* \mathcal{C}X) = g_M(T_Z \mathcal{B}X, \omega W)$$

for $Z, W \in \Gamma(D^\theta)$, $U \in \Gamma(D^\perp)$ and $X \in \Gamma((\ker \pi_*)^\perp)$.

Proof. The slant distribution D^θ defines a totally geodesic foliation on M if and only if $g_M(\nabla_Z^M W, U) = 0$ and $g_M(\nabla_Z^M W, X) = 0$ for $Z, W \in \Gamma(D^\theta)$, $U \in \Gamma(D^\perp)$ and $X \in \Gamma((\ker \pi_*)^\perp)$, from (2.1) and (2.2), we get

$$g_M(\nabla_Z^M W, U) = g_M(\nabla_Z^M F\phi W, U) + g_M(\nabla_Z^M \omega W, FU).$$

Since π is a hemi-slant submersion, using (2.6), (2.11) and (3.9), we obtain

$$g_M(\nabla_Z^M W, U) = g_M(\nabla_Z^M \cos^2 \theta W, U) + g_M(\nabla_Z^M \omega \phi W, U) + g_N(\pi_*(\nabla_Z^M \omega W), \pi_* FU)$$

or

$$\sin^2 \theta g_M(\nabla_Z^M W, U) = g_M(T_Z \omega \phi W, U) - g_N((\nabla \pi_*)(Z, \omega W), \pi_* FU). \quad (3.13)$$

On the other hand, by using (3.3) and (3.4) we get

$$g_M(\nabla_Z^M W, X) = g_M(\nabla_Z^M \phi W, FX) + g_M(\nabla_Z^M \omega W, \mathcal{B}X) + g_M(\mathcal{H} \nabla_Z^M \omega W, \mathcal{C}X).$$

Using (2.6), (2.7), (3.3), (3.9) and if we take into account that π is a hemi-slant submersion, we obtain

$$\begin{aligned} g(\nabla_Z^M W, X) &= \cos^2 \theta g_M(\nabla_Z^M W, X) + g_N(\pi_*(\nabla_Z^M \omega \phi W), \pi_* X) \\ &\quad + g_M(T_Z \omega W, \mathcal{B}X) + g_N(\pi_*(\nabla_Z^M \omega W), \mathcal{C}X) \end{aligned}$$

or

$$\begin{aligned} \sin^2 \theta g(\nabla_Z^M W, X) &= g_M(T_Z \omega W, \mathcal{B}X) + g_N((\nabla \pi_*)(Z, \omega \phi W), \pi_* X) \\ &\quad + g_N((\nabla \pi_*)(Z, \omega W), \pi_* \mathcal{C}X). \end{aligned} \quad (3.14)$$

Thus proof follows from (3.13) and (3.14). $\square$

From Theorem 3.7 and Theorem 3.8, we have the following result.

Theorem 3.9. *Let $\pi : (M, g_M, F) \longrightarrow (N, g_N)$ be a purely hemi-slant submersion from a locally product Riemannian manifold (M, g_M, F) onto a Riemannian manifold (N, g_N) . Then the fibers of π are the locally product Riemannian manifold of leaves of $D^\perp$ and D^θ if and only if*

$$g_N((\nabla \pi_*)(U, FV), \pi_* \omega Z) = g_M(T_U V, \omega \phi Z)$$

and

$$g_N((\nabla \pi_*)(Z, \omega \phi W), \pi_* X) + g_N((\nabla \pi_*)(Z, \omega W), \pi_* \mathcal{C}X) = g_M(T_Z \mathcal{B}X, \omega W)$$

for any $U, V \in \Gamma(D^\perp)$, $Z, W \in \Gamma(D^\theta)$ and $X \in \Gamma((\ker \pi_*)^\perp)$.

For the geometry of leaves of the horizontal distribution $((\ker \pi_*)^\perp)$, we have the following theorem.

Theorem 3.10. *Let $\pi : (M, g_M, F) \longrightarrow (N, g_N)$ be a purely hemi-slant submersion from a locally product Riemannian manifold (M, g_M, F) onto a Riemannian manifold (N, g_N) . Then the distribution $(\ker \pi_*)^\perp$ defines a totally geodesic foliation on M if and only if*

$$A_{X_1} \mathcal{B}X_2 + \mathcal{H} \nabla_{X_1}^M \mathcal{C}X_2 \in \Gamma(FD^\perp \oplus \mu), \quad \mathcal{V} \nabla_{X_1}^M \mathcal{B}X_2 + A_{X_1} \mathcal{C}X_2 \in \Gamma(D^\perp)$$

for any $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$.

Proof. Since M is a locally product Riemannian manifold, from (2.1) and (2.2) we have $\nabla_{X_1}^M X_2 = F \nabla_{X_1}^M F X_2$ for $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$. Using (2.8), (2.9) and (3.4) we have

$$\begin{aligned} \nabla_{X_1}^M X_2 &= F(A_{X_1} \mathcal{B}X_2 + \mathcal{V} \nabla_{X_1}^M \mathcal{B}X_2) \\ &\quad + F(\mathcal{H} \nabla_{X_1}^M \mathcal{C}X_2 + A_{X_1} \mathcal{C}X_2). \end{aligned}$$

Then by using (3.3) and (3.4) we get

$$\begin{aligned} \nabla_{X_1}^M X_2 &= \mathcal{B}A_{X_1} \mathcal{B}X_2 + \mathcal{C}A_{X_1} \mathcal{B}X_2 + \phi \mathcal{V} \nabla_{X_1}^M \mathcal{B}X_2 \\ &\quad + \omega \mathcal{V} \nabla_{X_1}^M \mathcal{B}X_2 + \mathcal{B} \mathcal{H} \nabla_{X_1}^M \mathcal{C}X_2 + \mathcal{C} \mathcal{H} \nabla_{X_1}^M \mathcal{C}X_2 \\ &\quad + \phi A_{X_1} \mathcal{C}X_2 + \omega A_{X_1} \mathcal{C}X_2. \end{aligned}$$

Hence, we have $\nabla_{X_1}^M X_2 \in \Gamma((\ker \pi_*)^\perp)$ if and only if

$$\mathcal{B}(A_{X_1} \mathcal{B}X_2 + \mathcal{H} \nabla_{X_1}^M \mathcal{C}X_2) + \phi(\mathcal{V} \nabla_{X_1}^M \mathcal{B}X_2 + A_{X_1} \mathcal{C}X_2) = 0.$$

Thus $\nabla_{X_1}^M X_2 \in \Gamma((\ker \pi_*)^\perp)$ if and only if

$$\mathcal{B}(A_{X_1} \mathcal{B}X_2 + \mathcal{H} \nabla_{X_1}^M \mathcal{C}X_2) = 0 \text{ and } \phi(\mathcal{V} \nabla_{X_1}^M \mathcal{B}X_2 + A_{X_1} \mathcal{C}X_2) = 0,$$

which completes proof. $\square$

In the sequel we are going to investigate the geometry of leaves of the vertical distribution $\ker \pi_*$.

Theorem 3.11. *Let $\pi : (M, g_M, F) \longrightarrow (N, g_N)$ be a purely hemi-slant submersion from a locally product Riemannian manifold (M, g_M, F) onto a Riemannian manifold (N, g_N) . Then the distribution $(\ker \pi_*)$ defines a totally geodesic foliation on M if and only if*

$$T_{Z_1} \phi Z_2 + A_{\omega Z_2} Z_1 \in \Gamma(\ker \pi_*), \quad \hat{\nabla}_{Z_1} \phi Z_2 + T_{Z_1} \omega Z_2 \in \Gamma((\ker \pi_*)^\perp)$$

for any $Z_1, Z_2 \in \Gamma(\ker \pi_*)$.

Proof. For any $Z_1, Z_2 \in \Gamma(\ker \pi_*)$, using (2.2), (2.6), (2.7) and (3.3) we get

$$\begin{aligned} \nabla_{Z_1}^M Z_2 &= F \nabla_{Z_1}^M F Z_2 \\ &= F(\nabla_{Z_1}^M \phi Z_2 + \nabla_{Z_1}^M \omega Z_2) \\ &= F(T_{Z_1} \phi Z_2 + \hat{\nabla}_{Z_1} \phi Z_2 + A_{\omega Z_2} Z_1 + T_{Z_1} \omega Z_2) \\ &= \mathcal{B}T_{Z_1} \phi Z_2 + \mathcal{C}T_{Z_1} \phi Z_2 + \phi \hat{\nabla}_{Z_1} \phi Z_2 + \omega \hat{\nabla}_{Z_1} \phi Z_2 \\ &\quad + \mathcal{B}A_{\omega Z_2} Z_1 + \mathcal{C}A_{\omega Z_2} Z_1 + \phi T_{Z_1} \omega Z_2 + \omega T_{Z_1} \omega Z_2. \end{aligned}$$

From above equation, it follows that $(\ker \pi_*)$ defines a totally geodesic foliation if and only if

$$\mathcal{C}(T_{Z_1} \phi Z_2 + A_{\omega Z_2} Z_1) + \omega(\hat{\nabla}_{Z_1} \phi Z_2 + T_{Z_1} \omega Z_2) = 0.$$

Thus $\nabla_{Z_1}^M Z_2 \in \Gamma(\ker \pi_*)$ if and only if

$$\mathcal{C}(T_{Z_1} \phi Z_2 + A_{\omega Z_2} Z_1) = 0 \text{ and } \omega(\hat{\nabla}_{Z_1} \phi Z_2 + T_{Z_1} \omega Z_2) = 0,$$

which completes proof. $\square$

From Theorem 3.10 and Theorem 3.11, we have the following result.

Theorem 3.12. *Let $\pi : (M, g_M, F) \longrightarrow (N, g_N)$ be a purely hemi-slant submersion from a locally product Riemannian manifold (M, g_M, F) onto a Riemannian manifold (N, g_N) . Then the total space M is a locally product manifold of the leaves of $(\ker \pi_*)^\perp$ and $\ker \pi_*$, i.e., $M = M_{(\ker \pi_*)^\perp} \times M_{\ker \pi_*}$, if and only if*

$$A_{X_1} \mathcal{B}X_2 + \mathcal{H} \nabla_{X_1}^M \mathcal{C}X_2 \in \Gamma(FD^\perp \oplus \mu), \quad \mathcal{V} \nabla_{X_1}^M \mathcal{B}X_2 + A_{X_1} \mathcal{C}X_2 \in \Gamma(D^\perp)$$

and

$$T_{Z_1} \phi Z_2 + A_{\omega Z_2} Z_1 \in \Gamma(\ker \pi_*), \quad \hat{\nabla}_{Z_1} \phi Z_2 + T_{Z_1} \omega Z_2 \in \Gamma((\ker \pi_*)^\perp)$$

for any $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$ and $Z_1, Z_2 \in \Gamma(\ker \pi_*)$, where $M_{(\ker \pi_*)^\perp}$ and $M_{\ker \pi_*}$ are leaves of the distributions $(\ker \pi_*)^\perp$ and $\ker \pi_*$, respectively.

Now, we give necessary and sufficient conditions for a hemi-slant submersion to be totally geodesic. The Riemannian submersion map π is called totally geodesic map if the map π_* is parallel with respect to ∇ , i.e., $\nabla\pi_* = 0$. A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Theorem 3.13. *Let $\pi : (M, g_M, F) \longrightarrow (N, g_N)$ be a hemi-slant submersion from a locally product Riemannian manifold (M, g_M, F) onto a Riemannian manifold (N, g_N) . π is a totally geodesic map if and only if*

$$\begin{aligned}\omega T_{U_1}\omega V_1 + \mathcal{CH}\nabla_{U_1}^M\omega V_1 &= 0, \\ \mathcal{C}(T_U\phi W + A_{\omega W}V) + \omega(\hat{\nabla}_V\phi W + T_V\omega W) &= 0, \\ \mathcal{C}(T_V\mathcal{B}X + A_{\mathcal{C}X}V) + \omega(\hat{\nabla}_V\mathcal{B}X + T_V\mathcal{C}X) &= 0,\end{aligned}$$

for any $U_1, V_1 \in \Gamma(D_1)$, $W \in \Gamma(D_2)$, $U \in \Gamma(\ker\pi_*)$ and $X \in \Gamma((\ker\pi_*)^\perp)$.

Proof. For $X_1, X_2 \in \Gamma((\ker\pi_*)^\perp)$, since π is a Riemannian submersion, from (2.11) we obtain

$$(\nabla\pi_*)(X_1, X_2) = 0.$$

For $U_1, V_1 \in \Gamma(D^\perp)$, using (2.3) and (2.11) we have

$$(\nabla\pi_*)(U_1, V_1) = -\pi_*(F\nabla_{U_1}^M\omega V_1).$$

Then from (2.6) we arrive at

$$(\nabla\pi_*)(U_1, V_1) = -\pi_*(F(T_{U_1}\omega V_1 + \mathcal{H}\nabla_{U_1}^M\omega V_1)).$$

Using (3.3) and (3.4) in above equation we obtain

$$(\nabla\pi_*)(U_1, V_1) = -\pi_*(\phi T_{U_1}\omega V_1 + \omega T_{U_1}\omega V_1 + \mathcal{B}\mathcal{H}\nabla_{U_1}^M\omega V_1 + \mathcal{C}\mathcal{H}\nabla_{U_1}^M\omega V_1).$$

Since $\phi T_{U_1}\omega V_1 + \mathcal{B}\mathcal{H}\nabla_{U_1}^M\omega V_1 \in \Gamma(\ker\pi_*)$, we derive

$$(\nabla\pi_*)(U_1, V_1) = -\pi_*(\omega T_{U_1}\omega V_1 + \mathcal{C}\mathcal{H}\nabla_{U_1}^M\omega V_1).$$

Then, since π is a linear isomorphism between $(\ker\pi_*)^\perp$ and TM , $(\nabla\pi_*)(U_1, V_1) = 0$ if and only if

$$\omega T_{U_1}\omega V_1 + \mathcal{C}\mathcal{H}\nabla_{U_1}^M\omega V_1 = 0. \quad (3.15)$$

For $U \in \Gamma(\ker\pi_*)$ and $W \in \Gamma(D^\theta)$, using (2.3), (2.11) and (3.3), we have

$$\begin{aligned}(\nabla\pi_*)(U, W) &= \nabla_U^\pi\pi_*W - \pi_*(\nabla_U^M W) \\ &= -\pi_*(F\nabla_U^M FW) \\ &= -\pi_*(F\nabla_U^M(\phi W + \omega W)).\end{aligned}$$

Then from (2.7) we arrive at

$$(\nabla\pi_*)(U, W) = -\pi_*(F(T_U\phi W + \hat{\nabla}_V\phi W) + F(A_{\omega W}V + T_V\omega W)).$$

Using (3.3) and (3.4) in above equation we obtain

$$\begin{aligned}(\nabla\pi_*)(U, W) &= -\pi_*((\mathcal{B}T_U\phi W + \mathcal{C}T_U\phi W) + (\phi\hat{\nabla}_V\phi W + \omega\hat{\nabla}_V\phi W) \\ &\quad + (\mathcal{B}A_{\omega W}V + \mathcal{C}A_{\omega W}V) + (\phi T_V\omega W + \omega T_V\omega W)).\end{aligned}$$

Thus $(\nabla\pi_*)(V, W) = 0$ if and only if

$$\mathcal{C}(T_U\phi W + A_{\omega W}V) + \omega(\hat{\nabla}_V\phi W + T_V\omega W) = 0. \quad (3.16)$$

On the other hand, using (2.3), (2.6), (2.7) and (3.4) for any $V \in \Gamma(\ker\pi_*)$ and $X \in \Gamma((\ker\pi_*)^\perp)$, we get

$$\begin{aligned} (\nabla\pi_*)(V, X) &= \nabla_V^\pi\pi_*X - \pi_*(\nabla_V^M X) \\ &= -\pi_*(F\nabla_V^M F X) \\ &= -\pi_*(F\nabla_V^M (\mathcal{B}X + \mathcal{C}X)) \\ &= -\pi_*(\mathcal{B}T_V\mathcal{B}X + \mathcal{C}T_V\mathcal{B}X + \phi\hat{\nabla}_V\mathcal{B}X + \omega\hat{\nabla}_V\mathcal{B}X \\ &\quad + \mathcal{B}A_{\mathcal{C}X}V + \mathcal{C}A_{\mathcal{C}X}V + \phi T_V\mathcal{C}X + \omega T_V\mathcal{C}X). \end{aligned}$$

Thus $(\nabla\pi_*)(V, X) = 0$ if and only if

$$\mathcal{C}(T_V\mathcal{B}X + A_{\mathcal{C}X}V) + \omega(\hat{\nabla}_V\mathcal{B}X + T_V\mathcal{C}X) = 0. \quad (3.17)$$

The result then follows from (3.15), (3.16) and (3.17). $\square$

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¹ FACULTY OF SCIENCE AND ARTS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BİNGÖL, 12000, BİNGÖL, TURKEY

E-mail address: mehmetakifakyol@bingol.edu.tr

² FACULTY OF SCIENCE AND ARTS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DİCLE, 21280, DİYARBAKIR, TURKEY

E-mail address: y Gunduzalp@dicle.edu.tr