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GENERALIZATION OF TITCHMARSH'S THEOREM FOR THE GENERALIZED FOURIER TRANSFORM IN THE SPACE $L^2_{\alpha,\beta}(\mathbb{R})$

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Abstract. In this paper, we prove the generalization of Titchmarsh's theorem for the generalized Fourier transform for functions satisfying the $(\psi, 2, k)$-Cherednik-Opdam Lipschitz condition in the space $L^2_{\alpha,\beta}(\mathbb{R})$.

1. Introduction and Preliminaries

In [3], E. C. Titchmarsh characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

Theorem 1.1. [3] *Let $\delta \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents*
(i) $\quad \|f(t+h) - f(t)\| = O(h^\delta), \quad as \quad h \to 0,$

(ii) $$\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\delta}) \quad as \quad r \to \infty,$$
where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we prove the generalization of Theorem 1.1 for the generalized Fourier transform for functions satisfying the $(\psi, 2, k)$-Cherednik-Opdam Lipschitz condition in the space $L^2_{\alpha,\beta}(\mathbb{R})$. For this purpose, we use the generalized translation operator.

In this section, we develop some results from harmonic analysis related to the differential-difference operator $T^{(\alpha,\beta)}$. Further details can be found in [1] and [2]. In the following we fix parameters α, β subject to the constraints $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha > \frac{-1}{2}$.
Let $\rho = \alpha + \beta + 1$ and $\lambda \in \mathbb{C}$. The Opdam hypergeometric functions $G^{(\alpha,\beta)}_\lambda$ on $\mathbb{R}$ are eigenfunctions $T^{(\alpha,\beta)} G^{(\alpha,\beta)}_\lambda(x) = i\lambda G^{(\alpha,\beta)}_\lambda(x)$ of the differential-difference

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operator

$$T^{(\alpha,\beta)}f(x) = f'(x) + [(2\alpha+1)\coth x + (2\beta+1)\tanh x]\frac{f(x)-f(-x)}{2} - \rho f(-x),$$

that are normalized such that $G_\lambda^{(\alpha,\beta)}(0) = 1$. In the notation of Cherednik one would write $T^{(\alpha,\beta)}$ as

$$T(k_1+k_2)f(x) = f'(x) + \left\{\frac{2k_1}{1+e^{-2x}} + \frac{4k_2}{1-e^{-4x}}\right\}(f(x)-f(-x)) - (k_1+2k_2)f(x),$$

with $\alpha = k_1 + k_2 - \frac{1}{2}$ and $\beta = k_2 - \frac{1}{2}$. Here k_1 is the multiplicity of a simply positive root and k_2 the (possibly vanishing) multiplicity of a multiple of this root. By [1] or [2], the eigenfunction $G_\lambda^{(\alpha,\beta)}$ is given by

$$G_\lambda^{(\alpha,\beta)}(x) = \varphi_\lambda^{\alpha,\beta}(x) - \frac{1}{\rho - i\lambda}\frac{\partial}{\partial x}\varphi_\lambda^{\alpha,\beta}(x) = \varphi_\lambda^{\alpha,\beta}(x) + \frac{\rho}{4(\alpha+1)}\sinh(2x)\varphi_\lambda^{\alpha+1,\beta+1}(x),$$

where $\varphi_\lambda^{\alpha,\beta}(x) ={}_2 F_1(\frac{\rho+i\lambda}{2}; \frac{\rho-i\lambda}{2}; \alpha+1; -\sinh^2 x)$ is the classical Jacobi function.

Lemma 1.2. [4] *The following inequalities are valid for Jacobi functions* $\varphi_\lambda^{\alpha,\beta}(x)$
(i) $|\varphi_\lambda^{\alpha,\beta}(x)| \leq 1$.
(ii) $1 - \varphi_\lambda^{\alpha,\beta}(x) \leq x^2(\lambda^2+\rho^2)$.
(iii) there is a constant $c > 0$ *such that*

$$1 - \varphi_\lambda^{\alpha,\beta}(x) \geq c,$$

for $\lambda x \geq 1$.

Denote $L^2_{\alpha,\beta}(\mathbb{R})$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{2,\alpha,\beta} = \left(\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x)dx\right)^{1/2} < +\infty,$$

where

$$A_{\alpha,\beta}(x) = (\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

The generalized Fourier transform of $f \in C_c(\mathbb{R})$ is defined by

$$\mathcal{H}f(\lambda) = \int_{\mathbb{R}} f(x)G_\lambda^{(\alpha,\beta)}(-x)A_{\alpha,\beta}(x)dx \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

The inverse transform is given as

$$\mathcal{H}^{-1}g(x) = \int_{\mathbb{R}} g(\lambda)G_\lambda^{(\alpha,\beta)}(x)\left(1 - \frac{\rho}{i\lambda}\right)\frac{d\lambda}{8\pi|c_{\alpha,\beta}(\lambda)|^2},$$

here

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\lambda))}.$$

The corresponding Plancherel formula was established in [1], to the effect that

$$\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x)dx = \int_0^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda),$$

where $\check{f}(x) := f(-x)$ and $d\sigma$ is the measure given by

$$d\sigma(\lambda) = \frac{d\lambda}{16\pi|c_{\alpha,\beta}(\lambda)|^2}.$$

According to [2] there exists a family of signed measures $\mu_{x,y}^{(\alpha,\beta)}$ such that the product formula

$$G_\lambda^{(\alpha,\beta)}(x)G_\lambda^{(\alpha,\beta)}(y) = \int_{\mathbb{R}} G_\lambda^{(\alpha,\beta)}(z)d\mu_{x,y}^{(\alpha,\beta)}(z),$$

holds for all $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, where

$$d\mu_{x,y}^{(\alpha,\beta)}(z) = \begin{cases} \mathcal{K}_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz & \text{if } xy \neq 0, \\ \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0, \end{cases}$$

and

$$\begin{aligned} \mathcal{K}_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}|\sinh x.\sinh y.\sinh z|^{-2\alpha}\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1} \\ &\times \ [1-\sigma_{x,y,z}^\chi + \sigma_{x,z,y}^\chi + \sigma_{z,y,x}^\chi + \frac{\rho}{\beta+\frac{1}{2}}\coth x.\coth y.\coth z(\sin\chi)^2] \times (\sin\chi)^{2\beta}d\chi \end{aligned}$$

if $x, y, z \in \mathbb{R}\backslash\{0\}$ satisfy the triangular inequality $||x|-y|| < |z| < |x|+|y|$, and $\mathcal{K}_{\alpha,\beta}(x,y,z) = 0$ otherwise. Here

$$\forall x, y, z \in \mathbb{R}, \chi \in [0,1], \sigma_{x,y,z}^\chi = \begin{cases} \frac{\cosh x+\cosh y-\cosh z\cos\chi}{\sinh x\sinh y} & \text{if } xy \neq 0, \\ \\ 0 & \text{if } xy = 0, \end{cases}$$

and $g(x,y,z,\chi) = 1-\cosh^2 x - \cosh^2 y.\cosh^2 z + 2\cosh x.\cosh y.\cosh z.\cos\chi$.

Lemma 1.3. [2] *For all $x, y \in \mathbb{R}$, we have*
(i) $\mathcal{K}_{\alpha,\beta}(x,y,z) = \mathcal{K}_{\alpha,\beta}(y,x,z)$.
(ii) $\mathcal{K}_{\alpha,\beta}(x,y,z) = \mathcal{K}_{\alpha,\beta}(-x,z,y)$.
(iii) $\mathcal{K}_{\alpha,\beta}(x,y,z) = \mathcal{K}_{\alpha,\beta}(-z,y,-x)$.

The product formula is used to obtain explicit estimates for the generalized translation operators

$$\tau_x^{(\alpha,\beta)}f(y) = \int_{\mathbb{R}} f(z)d\mu_{x,y}^{(\alpha,\beta)}(z).$$

It is known from [2] that

$$\mathcal{H}\tau_x^{(\alpha,\beta)}f(\lambda) = G_\lambda^{(\alpha,\beta)}(x)\mathcal{H}f(\lambda), \tag{1.1}$$

$$\mathcal{H}(T^{(\alpha,\beta)}f)(\lambda) = i\lambda\mathcal{H}f(\lambda), \tag{1.2}$$

for $f \in C_c(\mathbb{R})$.
For $L^2_{\alpha,\beta}(\mathbb{R})$, we define the finite differences of first and higher order as follows:

$$\begin{aligned} \Delta_h^1 f &= \Delta_h f = (\tau_h^{(\alpha,\beta)} + \tau_{-h}^{(\alpha,\beta)} - 2I)f, \\ \Delta_h^k f &= \Delta_h(\Delta_h^{k-1}f) = (\tau_h^{(\alpha,\beta)} + \tau_{-h}^{(\alpha,\beta)} - 2I)^k f, \quad k = 2, 3, ..., \end{aligned}$$

where I is the unit operator in the space $L^2_{\alpha,\beta}(\mathbb{R})$.
We denote by $W^{2,k}_{\alpha,\beta}$, $k \in \mathbb{N}^*$, the Sobolev space constructed by the operator $T^{(\alpha,\beta)}$, i.e.,

$$W^{2,k}_{\alpha,\beta} = \{f \in L^2_{\alpha,\beta}(\mathbb{R}); (T^{(\alpha,\beta)})^j f \in L^2_{\alpha,\beta}(\mathbb{R}), j = 0,1,2,...,k\},$$

where $(T^{(\alpha,\beta)})^0 f = f$, $(T^{(\alpha,\beta)})^1 f = T^{(\alpha,\beta)} f$, $(T^{(\alpha,\beta)})^m f = T^{(\alpha,\beta)}((T^{(\alpha,\beta)})^{m-1} f)$, $m = 2,3,...$

2. Main results

In this section we give the main result of this paper. We need first to define $(\psi,2,k)$-Cherednik-Opdam Lipschitz class.

Definition 2.1. A function $W^{2,k}_{\alpha,\beta}$, $k \in \mathbb{N}^*$, is said to be in the $(\psi,2,k)$-Cherednik-Opdam Lipschitz class, denoted by $Lip(\psi,2,k)$, if

$$\|\Delta^k_h (T^{(\alpha,\beta)})^m f\|_{2,\alpha,\beta} = O(\psi(h)), \quad \text{as} \quad h \to 0,$$

where $m = 0,1,...,k$ and ψ is a continuous increasing function on $[0,\infty)$ satisfying $\psi(0) = 0$, $\psi(ts) = \psi(t)\psi(s)$ for all $t,s \in [0,\infty)$ and this function verify

$$\int_0^{1/h} s\psi(s^{-2})ds = O(h^{-2}\psi(h^2)), \quad h \to 0.$$

Lemma 2.2. *If $f \in C_c(\mathbb{R})$, then*

$$\mathcal{H}\check{\tau}^{(\alpha,\beta)}_x f(\lambda) = G^{(\alpha,\beta)}_\lambda(-x)\mathcal{H}\check{f}(\lambda). \tag{2.1}$$

Proof. For $f \in C_c(\mathbb{R})$, we have

$$\begin{aligned}
\mathcal{H}\check{\tau}^{(\alpha,\beta)}_x f(\lambda) &= \int_{\mathbb{R}} \tau^{(\alpha,\beta)}_x f(-y) G^{(\alpha,\beta)}_\lambda(-y) A_{\alpha,\beta}(y)dy \\
&= \int_{\mathbb{R}} \tau^{(\alpha,\beta)}_x f(y) G^{(\alpha,\beta)}_\lambda(y) A_{\alpha,\beta}(y)dy \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(z)\mathcal{K}_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz\right] G^{(\alpha,\beta)}_\lambda(y) A_{\alpha,\beta}(y)dy \\
&= \int_{\mathbb{R}} f(z) \left[\int_{\mathbb{R}} G^{(\alpha,\beta)}_\lambda(y)\mathcal{K}_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(y)dy\right] A_{\alpha,\beta}(z)dz.
\end{aligned}$$

Since $\mathcal{K}_{\alpha,\beta}(x,y,z) = \mathcal{K}_{\alpha,\beta}(-x,z,y)$, it follows from the product formula that

$$\begin{aligned}
\mathcal{H}\check{\tau}^{(\alpha,\beta)}_x f(\lambda) &= G^{(\alpha,\beta)}_\lambda(-x)\int_{\mathbb{R}} f(z) G^{(\alpha,\beta)}_\lambda(z) A_{\alpha,\beta}(z)dz \\
&= G^{(\alpha,\beta)}_\lambda(-x)\int_{\mathbb{R}} f(-z) G^{(\alpha,\beta)}_\lambda(-z) A_{\alpha,\beta}(z)dz \\
&= G^{(\alpha,\beta)}_\lambda(-x)\mathcal{H}\check{f}(\lambda).
\end{aligned}$$

$\square$

Lemma 2.3. *For $W^{2,k}_{\alpha,\beta}$, $k \in \mathbb{N}^*$, then*

$$\|\Delta_h^k (T^{(\alpha,\beta)})^m f\|^2_{2,\alpha,\beta} = 2^{2k} \int_0^{+\infty} \lambda^{2m} |\varphi_\lambda^{\alpha,\beta}(h) - 1|^{2k} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2 \right) d\sigma(\lambda),$$

where $m = 0, 1, ..., k$.

Proof. From formulas (1.1) and (2.1), we have

$$\mathcal{H}(\Delta_h^k f)(\lambda) = (G_\lambda^{(\alpha,\beta)}(h) + G_\lambda^{(\alpha,\beta)}(-h) - 2)^k \mathcal{H}(f)(\lambda),$$

and

$$\mathcal{H}(\check{\Delta}_h^k f)(\lambda) = (G_\lambda^{(\alpha,\beta)}(-h) + G_\lambda^{(\alpha,\beta)}(h) - 2)^k \mathcal{H}(\check{f})(\lambda).$$

Since

$$G_\lambda^{(\alpha,\beta)}(h) = \varphi_\lambda^{\alpha,\beta}(h) + \frac{\rho}{4(\alpha+1)} \sinh(2h) \varphi_\lambda^{\alpha+1,\beta+1}(h),$$

and $\varphi_\lambda^{\alpha,\beta}$ is even, then

$$\mathcal{H}(\Delta_h^k f)(\lambda) = 2^k (\varphi_\lambda^{\alpha,\beta}(h) - 1)^k \mathcal{H}(f)(\lambda)$$

and

$$\mathcal{H}(\check{\Delta}_h^k f)(\lambda) = 2^k (\varphi_\lambda^{\alpha,\beta}(h) - 1)^k \mathcal{H}(\check{f})(\lambda).$$

Furthermore, we obtain by the formula (1.2)

$$\mathcal{H}((T^{(\alpha,\beta)})^m f)(\lambda) = (i\lambda)^m \mathcal{H}f(\lambda).$$

Now by Plancherel Theorem, we have the result. □

Theorem 2.4. *Let $W^{2,k}_{\alpha,\beta}$, $k \in \mathbb{N}^*$. Then the following are equivalents*

(a) $f \in Lip(\psi, 2, k)$,

(b) $\displaystyle\int_r^{+\infty} \lambda^{2m} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2 \right) d\sigma(\lambda) = O(\psi(r^{-2})), \quad as \quad r \to \infty,$

where $m = 0, 1, ..., k$.

Proof. $(a) \Rightarrow (b)$ Let $f \in Lip(\psi, 2, k)$. Then we have

$$\|\Delta_h^k (T^{(\alpha,\beta)})^m f\|_{2,\alpha,\beta} = O(\psi(h)) \quad \text{as} \quad h \to 0.$$

From Lemma 2.3, we have

$$\|\Delta_h^k (T^{(\alpha,\beta)})^m f\|^2_{2,\alpha,\beta} = 2^{2k} \int_0^{+\infty} \lambda^{2m} |\varphi_\lambda^{\alpha,\beta}(h) - 1|^{2k} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2 \right) d\sigma(\lambda).$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h}]$, then $\lambda h \geq 1$ and (iii) of Lemma 1.2 implies that

$$1 \leq \frac{1}{c^{2k}} |1 - \varphi_\lambda^{\alpha,\beta}(h)|^{2k}.$$

Then

$$\begin{aligned}
\int_{\frac{1}{h}}^{\frac{2}{h}} \lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda) &\le \frac{1}{c^{2k}}\int_{\frac{1}{h}}^{\frac{2}{h}} \lambda^{2m}|1-\varphi_\lambda^{\alpha,\beta}(h)|^{2k}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda)\\
&\le \frac{1}{c^{2k}}\int_0^{+\infty} \lambda^{2m}|1-\varphi_\lambda^{\alpha,\beta}(h)|^{2k}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda)\\
&\le \frac{1}{2^{2k}c^{2k}}\|\Delta_h^k(T^{(\alpha,\beta)})^m f\|_{2,\alpha,\beta}^2\\
&= O(\psi(h^2)).
\end{aligned}$$

We obtain

$$\int_r^{2r} \lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda) \le C\psi(r^{-2}), \quad r\to\infty,$$

where C is a positive constant. Now,

$$\begin{aligned}
\int_r^{+\infty} \lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda) &= \sum_{i=0}^{\infty}\int_{2^i r}^{2^{i+1}r} \lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda)\\
&\le C\psi(r^{-2})\sum_{i=0}^{\infty}(\psi(2^{-2}))^i\\
&\le CC_\psi\psi(r^{-2}),
\end{aligned}$$

where $C_\psi=(1-\psi(2^{-2}))^{-1}$ since $\psi(2^{-2})<1$.
Consequently

$$\int_r^{+\infty} \lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda) = O(\psi(r^{-2})), \quad as \quad r\to\infty.$$

$(b)\Rightarrow(a)$. Suppose now that

$$\int_r^{+\infty} \lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda) = O(\psi(r^{-2})), \quad as \quad r\to\infty,$$

and write

$$\|\Delta_h^k(T^{(\alpha,\beta)})^m f\|_{2,\alpha,\beta}^2 = 2^{2k}(I_1+I_2),$$

where

$$I_1=\int_0^{\frac{1}{h}} \lambda^{2m}|1-\varphi_\lambda^{\alpha,\beta}(h)|^{2k}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda,$$

and

$$I_2=\int_{\frac{1}{h}}^{+\infty} \lambda^{2m}|1-\varphi_\lambda^{\alpha,\beta}(h)|^{2k}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda.$$

Firstly, we use the formula $|\varphi_\lambda^{\alpha,\beta}(h)|\le 1$ and

$$I_2\le 2^{2k}\int_{\frac{1}{h}}^{+\infty} \lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda) = O(\psi(h^2)), \quad as \quad h\to 0.$$

To estimate I_1, we use the inequalities (i) and (ii) of Lemma 1.2

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{h}} \lambda^{2m}|1-\varphi_\lambda^{\alpha,\beta}(h)|^{2k-1}|1-\varphi_\lambda^{\alpha,\beta}(h)|\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda \\ &\leq 2^{2k-1}\int_0^{\frac{1}{h}} \lambda^{2m}|1-\varphi_\lambda^{\alpha,\beta}(h)|\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda \\ &\leq 2h^2\int_0^{\frac{1}{h}}(\lambda^2+\rho^2)\lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda = I_3+I_4, \end{aligned}$$

where

$$I_3 = 2h^2\rho^2\int_0^{\frac{1}{h}}\lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda,$$

and

$$I_4 = 2h^2\int_0^{\frac{1}{h}}\lambda^2\lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda.$$

Note that

$$\begin{aligned} I_3 &\leq 2h^2\rho^2\int_0^{+\infty}\lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda \\ &= 2h^2\rho^2\|(T^{(\alpha,\beta)})^m f\|_{2,\alpha,\beta}^2 = O(\psi(h^2)), \quad as \quad h\to 0. \end{aligned}$$

For a while, we put

$$\phi(s) = \int_s^{+\infty}\lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda).$$

Using integration by parts, we find that

$$\begin{aligned} h^2\int_0^{1/h}\lambda^2\lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda &= h^2\int_0^{1/h}-s^2\phi'(s)ds \\ &= h^2\left(-\frac{1}{h^2}\phi(\frac{1}{h})+2\int_0^{1/h}s\phi(s)ds\right) \\ &= -\phi(\frac{1}{h})+2h^2\int_0^{1/h}s\phi(s)ds. \end{aligned}$$

Since $\phi(s)=O(\psi(s^{-2}))$, we have $s\phi(s)=O(s\psi(s^{-2}))$ and

$$\int_0^{1/h}s\phi(s)ds = O\left(\int_0^{1/h}s\psi(s^{-2})ds\right) = O(h^{-2}\psi(h^2)).$$

Then

$$h^2\int_0^{1/h}\lambda^2\lambda^{2m}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda \leq 2C_1h^2h^{-2}\psi(h^2),$$

where C_1 is a positive constant.
Finally

$$I_4 = O(\psi(h^2)),$$

which completes the proof of the theorem. $\square$

Corollary 2.5. *Let $f \in W^{2,k}_{\alpha,\beta}$ such that $f \in Lip(\psi, 2, k)$. Then*

$$\int_r^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) = O(r^{-2m}\psi(r^{-2})), \quad as \quad r \to \infty,$$

where $m = 0, 1, ..., k$.

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