Gulf Journal of Mathematics
Vol 4, Issue 3 (2016) 42-48

# A NOTE ON THE IDEAL CONVERGENCE OF INFINITE PRODUCT

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ABSTRACT. In this paper, we give the relationship between $\mathcal{I}$-convergent infinite product and $\mathcal{I}$-convergent infinite series. We also give some results related to these concepts.

## 1. Introduction and notation

The concept of statistical convergence which is a generalization of convergence is defined by Fast [2] and by Steinhaus [1] in 1951. It has been studied so far and is still being studied. Especially, Salat [3], Friday [4] and many authors have contributed significantly to development of statistical convergence. The statistical convergence was studied in metric space by Kostryko et al [26] and it was investigated in topological space by Di Maio and Kocinac [27]. Caserta et al. [29] gave statistical version of several classical kinds of convergence of sequences of functions between metric spaces and different function spaces.

Kostyrko et al [9] introduced ideal convergence which is a generalization of statistical convergence. Then, they showed that whether $\mathcal{I}$-convergence satisfied general axiom of convergence or not and fundemantal arithmetical properties of $\mathcal{I}$-convergence. $\mathcal{I}$-boundedness of a sequence was defined and it was stated in [11] that $\mathcal{I}$-convergence implies $\mathcal{I}$-boundedness . Salat et al.[12] gave the definition of $\mathcal{I}$-monotonic sequences and proved that an $\mathcal{I}$-monotonic sequence is $\mathcal{I}$-convergent if and only if it is $\mathcal{I}$-bounded. Lahiri et al. [28] extended the idea of $\mathcal{I}$-convergence of sequences to a topological space.

For more papers about ideal convergence of sequences, one can see [13]-[16].

Using the means of statistical convergence of sequence of partial sums, the notion of statistical convergent series was introduced in [17] and some properties of statistical convergent series were studied by Tripathy [18].

*Date*: Received: Dec 23, 2015; Accepted: Apr 4, 2016.
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2010 *Mathematics Subject Classification.* 40A35, 40A20, 40A05.
*Key words and phrases.* Infinite product; Infinite series; $\mathcal{I}$-convergence; absolutely $\mathcal{I}$-convergence.

The purpose of this paper is to satisfy the relationship of ideal convergence between infinite product and infinite series. We also introduce the idea of infinite product which is absolutely $\mathcal{I}$-convergent. In section 2, we give some basic facts and preliminaries. The main results are given in section 3.

In this section, we give some basic definitions and notations.

**Definition 1.1.** [20] Let $X \neq \emptyset$. A class $\mathcal{I} \subseteq 2^X$ of subsets of $X$ is said to be an ideal in $X$ provided that the following conditions hold:

(1) $\emptyset \in \mathcal{I}$,
(2) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$,
(3) $A \in \mathcal{I}, B \subset A$ imply $B \in \mathcal{I}$.

An ideal is called proper if $X \notin \mathcal{I}$.

**Definition 1.2.** [11] A proper ideal $\mathcal{I}$ in $X$ is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

**Definition 1.3.** [22] Let $X \neq \emptyset$. A non-empty class $\mathcal{F} \subseteq 2^X$ is said to be a filter on $X$ provided that the following conditions hold:

(1) $\emptyset \notin \mathcal{F}$,
(2) $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$,
(3) $A \in \mathcal{F}, A \subset B$ imply $B \in \mathcal{F}$.

**Proposition 1.4.** [11] *Let $\mathcal{I}$ be a proper ideal in $X \neq \emptyset$. Then a class of sets*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : \exists A \in \mathcal{I}; \text{such that } M = X \backslash A\}$$

*is a filter in $X$ (the filter associated with the ideal $\mathcal{I}$).*

**Definition 1.5.** [9] Let $\mathcal{I}$ be a proper ideal in $\mathbb{N}$. A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to $\xi \in \mathbb{R}$ if for every $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - \xi| \geq \varepsilon\}$ belongs to $\mathcal{I}$. Then we denote $\mathcal{I} - \lim x_n = \xi$.

**Definition 1.6.** [11] A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-bounded if there is an $M > 0$ such that $\{n \in \mathbb{N} : |x_n| > M\} \in \mathcal{I}$.

**Definition 1.7.** [12] A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-monotonic increasing ( $\mathcal{I}$-monotonic decreasing ) if there is a set $\{k_1, k_2, ... : \ k_1 < k_2 < ...\} \in \mathcal{F}(\mathcal{I})$ such that $x_{k_i} \leq x_{k_{i+1}}$ $(x_{k_i} \geq x_{k_{i+1}})$ for every $i \in \mathbb{N}$.

**Proposition 1.8.** [12] *A real sequence $x = \{x_n\}_{n \in \mathbb{N}}$ is $\mathcal{I}$-monotonic increasing ( $\mathcal{I}$-monotonic decreasing ) if and only if $x$ can be written in the form*

$$x = y + z,$$

*where $y = \{y_n\}$ is nondecreasing (nonincreasing) real sequence and $z = \{z_n\}$ is a sequence such that $\{n \in \mathbb{N} : z_n \neq 0\} \in \mathcal{I}$.*

**Corollary 1.9.** [12] *An $\mathcal{I}$-monotonic sequence is $\mathcal{I}$-convergent if and only if it is $\mathcal{I}$-bounded.*

**Definition 1.10.** The infinite product $\prod_{n=0}^{\infty} u_n$ is called convergent if from a certain stage on, say for all $n > k$, all factors $u_n \neq 0$ and the partial products beginning after the $k^{th}$ factor

$$p_n = u_{k+1}.u_{k+2}...u_n \qquad (n > k)$$

tend to a limit $p$ different from zero.

For more definitions and theorems related to infinite products, one can see [23] and [24].

We use that $1 + x \leq e^x$ for all $x$.

## 2. Main results

Throughout this section, we assume that $\mathcal{I}$ is an admissible ideal.

In [25], the $\mathcal{I}$-convergence of a series $\sum_{n=1}^{\infty} a_n$ is defined as $\mathcal{I}$-convergence of the sequence of its partial sums.

**Definition 2.1.** An infinite product $\prod_{n=1}^{\infty} u_n$ is said to be $\mathcal{I}$-convergent if the sequence of its partial products is $\mathcal{I}$-convergent to an number different from zero.

**Theorem 2.2.** *An infinite product* $\prod\limits_{n=1}^{\infty}(1 + a_n)$ *with positive terms* $a_n$ *is* $\mathcal{I}$-*convergent if and only if the infinite series* $\sum\limits_{n=1}^{\infty} a_n$ *is* $\mathcal{I}$-*convergent.*

*Proof.* Let $\prod\limits_{n=1}^{\infty}(1 + a_n)$ be $\mathcal{I}$-convergent. Hence the sequence of partial products $\{P_n\} = \{(1 + a_1)(1 + a_2)...(1 + a_n)\}$ is $\mathcal{I}$-convergent. Therefore $\{P_n\}$ is $\mathcal{I}$-bounded, i.e. there exists $M > 0$ such that

$$A = \{n : |P_n| > M\} \in \mathcal{I}.$$

Our aim is to show that the sequence of partial sums $\{S_n\} = \{\sum\limits_{k=1}^{n} a_k\}$ is $\mathcal{I}$-convergent. It can be easily seen that $\{S_n\}$ is $\mathcal{I}$-monotone from Proposition 1.8.

Suppose that $n \notin A$. Then $|P_n| = |(1 + a_1)(1 + a_2)...(1 + a_n)| \leq M$. Since $|S_n|$=$|a_1 + a_2 + ... + a_n| \leq |(1 + a_1)(1 + a_2)...(1 + a_n)| \leq M$, we have $|S_n| \leq M$ which means $n \notin B = \{n : |S_n| > M\}$. Accordingly, $B \subseteq A$ and so $B \in \mathcal{I}$. Thus $\{S_n\}$ is $\mathcal{I}$-bounded. Consequently, $\{S_n\}$ is $\mathcal{I}$-convergent, that is $\sum\limits_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent.

Conversely, let $\sum\limits_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent. Thus $\{S_n\}$ is $\mathcal{I}$-bounded; there exists $M > 0$ such that

$$B = \{n : |S_n| > M\} \in \mathcal{I}.$$

Since the inequality $1 + a_i < e^{a_i}$ holds for every $a_i > 0$, we obtain

$$(1 + a_1)(1 + a_2)...(1 + a_n) < e^{a_1+a_2+...+a_n}.$$

Let $n \notin B$. From the last inequality, it has seen that $n \notin A = \{n : |P_n| > M\}$. In that case $\{P_n\}$ is $\mathcal{I}$-bounded. Since $\{P_n\}$ is also $\mathcal{I}$-monotone increasing, $\{P_n\}$ is $\mathcal{I}$-convergent. Hence $\prod\limits_{n=1}^{\infty}(1+a_n)$ is $\mathcal{I}$-convergent. $\square$

**Example 2.3.** $\prod\limits_{n=1}^{\infty}\left(1+\frac{1}{n^3}\right)$ infinite product is $\mathcal{I}$-convergent, since $\sum\limits_{n=1}^{\infty}\frac{1}{n^3}$ infinite series is $\mathcal{I}$-convergent.

**Theorem 2.4.** *An infinite product* $\prod\limits_{n=1}^{\infty}(1-a_n)$ *with* $0 \leq a_n < 1$ *is $\mathcal{I}$-convergent if and only if the series* $\sum\limits_{n=1}^{\infty} a_n$ *is $\mathcal{I}$-convergent.*

*Proof.* Let $\prod\limits_{n=1}^{\infty}(1-a_n)$ be $\mathcal{I}$-convergent. Then the sequence of partial products $\{T_n\} = \{(1-a_1)(1-a_2)\dots(1-a_n)\}$ is $\mathcal{I}$-convergent. Since $\{T_n\}$ is also $\mathcal{I}$-monotone decreasing, it is $\mathcal{I}$-bounded. Thus $\{\frac{1}{T_n}\}$ is $\mathcal{I}$-bounded. Then there exists $M > 0$ such that

$$C = \left\{n : \left|\frac{1}{(1-a_1)(1-a_2)\dots(1-a_n)}\right| > M\right\} \in \mathcal{I}.$$

From the inequality $1 + a_n \leq \frac{1}{1-a_n}$, we have the following inequality

$$(1+a_1)(1+a_2)\dots(1+a_n) \leq \frac{1}{(1-a_1)(1-a_2)\dots(1-a_n)} \tag{2.1}$$

for $0 \leq a_n < 1$.

Assume that $n \notin C$ and so $\left|\frac{1}{(1-a_1)(1-a_2)\dots(1-a_n)}\right| \leq M$. By using the inequality (2.1), we obtain

$$n \notin A = \{n : |(1+a_1)(1+a_2)\dots(1+a_n)| > M\}.$$

Consequently, $\{P_n\}$ is $\mathcal{I}$-convergent. From Theorem 2.2, $\sum\limits_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent.

Conversely, let $\sum\limits_{n=1}^{\infty} a_n$ be $\mathcal{I}$-convergent. By Theorem 2.2, $\prod\limits_{n=1}^{\infty}(1+a_n)$ is $\mathcal{I}$-convergent. Hence $\{\frac{1}{P_n}\}$ is $\mathcal{I}$-bounded, i.e. there exists $M > 0$ such that

$$D = \left\{n : \left|\frac{1}{P_n}\right| > M\right\} \in \mathcal{I}.$$

Let $n \notin D$. Then we have $n \notin \{n : |T_n| > M\}$ by using (2.1). Hence $\{T_n\}$ is $\mathcal{I}$-bounded. This implies that $\{T_n\}$ is $\mathcal{I}$-convergent since it is also $\mathcal{I}$-monotone decreasing. $\square$

**Example 2.5.** $\prod\limits_{n=2}^{\infty}\left(1-\frac{2}{n(n+1)}\right)$ infinite product is $\mathcal{I}$-convergent. Because, $\sum\limits_{n=2}^{\infty}\frac{2}{n(n+1)}$ is $\mathcal{I}$-convergent. Also, we have

$$P_k = \prod_{n=2}^{k} \left(1 - \frac{2}{n\,(n+1)}\right) = \prod_{n=2}^{k} \left(\frac{(n-1)\,(n+2)}{n\,(n+1)}\right) = \frac{1}{3} \cdot \frac{k+2}{k}.$$

So, $\prod\limits_{n=2}^{\infty} \left(1 - \frac{2}{n(n+1)}\right)$ is $\mathcal{I}$-convergent to $\frac{1}{3}$.

**Definition 2.6.** If the infinite product $\prod\limits_{n=1}^{\infty} (1 + |a_n|)$ is $\mathcal{I}$-convergent, then the product $\prod\limits_{n=1}^{\infty} (1 + a_n)$ is said to be absolutely $\mathcal{I}$-convergent.

**Theorem 2.7.** *If an infinite product* $\prod\limits_{n=1}^{\infty} (1 + a_n)$ *is absolutely* $I$*-convergent, then* $\prod\limits_{n=1}^{\infty} (1 + a_n)$ *is* $I$*-convergent.*

*Proof.* Let $\prod\limits_{n=1}^{\infty} (1 + |a_n|)$ be $\mathcal{I}$-convergent. Then given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $p > q > N$

$$A = \{p : |(1 + |a_{q+1}|)\,(1 + |a_{q+2}|) \ldots (1 + |a_p|) - 1| \geq \varepsilon\} \in \mathcal{I}.$$

Suppose that $p \notin A$. Since the inequality

$$|(1 + a_{q+1})\,(1 + a_{q+2}) \ldots (1 + a_p) - 1| \leq |(1 + |a_{q+1}|)\,(1 + |a_{q+2}|) \ldots (1 + |a_p|) - 1|$$

holds, we obtain $p \notin B = \{p : |(1 + a_{q+1})\,(1 + a_{q+2}) \ldots (1 + a_p) - 1| \geq \varepsilon\}$. Hence it is $\mathcal{I}$-convergent. □

**Theorem 2.8.** *An infinite product* $\prod\limits_{n=1}^{\infty} (1 + a_n)$ *is absolutely* $\mathcal{I}$*-convergent if and only if* $\sum\limits_{n=1}^{\infty} a_n$ *is absolutely* $\mathcal{I}$*-convergent.*

*Proof.* If $\prod\limits_{n=1}^{\infty} (1 + |a_n|)$ is $I$-convergent, then the sequence of partial product $\{Q_n\} = \{(1 + |a_1|)\,(1 + |a_2|) \ldots (1 + |a_n|)\}$ is $\mathcal{I}$-bounded. Therefore, there exist $M > 0$ such that $A = \{n : |Q_n| > M\} \in \mathcal{I}$. Let $S_n = \sum\limits_{k=1}^{n} |a_k|$ and $n \notin A$. It seems to be $|Q_n| = |(1 + |a_1|)(1 + |a_2|)\ldots(1 + |a_n|)| \leq M$. Hence we have

$$||a_1| + |a_2| + \ldots + |a_n|| \leq |(1 + |a_1|)\,(1 + |a_2|) \ldots (1 + |a_n|)| \leq M. \tag{2.2}$$

It is clear that $n \notin B = \{n : ||a_1| + |a_2| + \ldots + |a_n|| > M\}$ and so $\{S_n\}$ is $\mathcal{I}$-bounded. $\{S_n\}$ is $\mathcal{I}$-monotone increasing. Consequently; $\sum\limits_{n=1}^{\infty} |a_n|$ is $\mathcal{I}$-convergent. If $\sum\limits_{n=1}^{\infty} |a_n|$ is $\mathcal{I}$-convergent, then $\{S_n\} = \{\sum\limits_{k=1}^{n} |a_k|\}$ is $\mathcal{I}$-convergent. Since we have $1 + |a_1| < e^{a_1}$, we can obtain

$$(1 + |a_1|)\,(1 + |a_2|) \ldots (1 + |a_n|) < e^{|a_1| + |a_2| + \ldots + |a_n|}.$$

There exist $M > 0$ such that,

$$B = \{n : ||a_1| + |a_2| + ... + |a_n|| > M\} \in I,$$

since $\{S_n\}$ is $I$-bounded. Suppose that $n \notin B$. Then, there exist such that $K > 0$ and hence $|P_n| < K$ is $\mathcal{I}$-bounded. Since $\{P_n\}$ is $\mathcal{I}$-monotone increasing, we obtain that $\{P_n\}$ is $\mathcal{I}$-convergent, easily. □

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