Gulf Journal of Mathematics ISSN: 2309-4966
Vol 13, Issue 2 (2022) 67-77 https://doi.org/10.56947/gjom.v13i2.723

COEFFICIENT BOUNDS FOR REGULAR AND BI-UNIVALENT FUNCTIONS LINKED WITH GEGENBAUER POLYNOMIALS

SONDEKOLA RUDRA SWAMY[1*] AND WAGGAS GALIB ATSHAN[2]

Abstract. Making use of Gegenbauer polynomials, we initiate and explore two sets of normalized regular and bi-univalent (or bi-Schlicht) functions in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ linked with Gegenbauer polynomials. We investigate certain coefficients bounds and the Fekete-Szegö functional for functions in these families. We also present few interesting observations and provide relevant connections of the results investigated.

1. Introduction and preliminaries

Let the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ be symbolized by $\mathfrak{D}$, $\mathbb{C}$ being the collection of all complex numbers. Let $\mathbb{N} =: \{1, 2, 3, ...\}$ and $\mathbb{R}$ be the set of real numbers. The set of normalized regular functions in $\mathfrak{D}$ that have the power series of the form

$$g(z) = z + d_2z^2 + d_3z^3 + ... = z + \sum_{j=2}^{\infty} d_j z^j, \tag{1.1}$$

be indicated by $\mathcal{A}$ and the set of all functions of $\mathcal{A}$ that are univalent (or schlicht) in $\mathfrak{D}$ is symbolized by $\mathcal{S}$. As per the Koebe theorem (see[9]) any function $g \in \mathcal{S}$ has an inverse function given by

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + ..., \tag{1.2}$$

such that $z = g^{-1}(g(z))$, $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)\, and\, r_0(g) \geq 1/4$, $z, \omega \in \mathfrak{D}$. A function g of $\mathcal{A}$ is said to be bi-univalent (or bi-schlicht) in $\mathfrak{D}$ if g and its inverse g^{-1} are both univalent (or schlicht) in $\mathfrak{D}$. The set of bi-univalent functions having the form (1.1) is indicated by Σ. Lewin [19] investigated the family Σ and proved that $|d_2| < 1.51$ for elements of Σ . Brannan and Clunie [6] claimed that $|d_2| < \sqrt{2}$ for $g \in \Sigma$. Later, Tan [35] found the initial coefficient bounds of bi-univalent functions. Brannan and Taha [7] proposed certain well-known subsets of Σ, which are analogues to starlike and convex functions. The momentum on investigation of the family Σ was gained in recent years, which is due to the paper of Srivastava et al.[28] and that has led to a large number of papers in recent times. Some

Date: Received: Oct 26, 2022; Accepted: Apr 11, 2022.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 30C45, 33C45; Secondary 11B39.
Key words and phrases. Fekete-Szegö functional, Regular function, Bi-univalent function, Gegenbauer polynomials.

interesting results concerning initial bounds for certain special sets of Σ have been examined in [13], [25],[26] and [27].

Recently, Kiepiela et al. [17] examined the Gegenbauer polynomials (or ultraspherical polynomials) $C_j^{\alpha}(x)$. They are orthagonal polynomials on [-1,1] that can be defined by the recurrence relation

$$C_j^{\alpha}(x)=\frac{2x(j+\alpha-1)C_{j-1}^{\alpha}(x)-(j+2\alpha-2)C_{j-2}^{\alpha}(x))}{j}, C_0^{\alpha}(x)=1\,,\, C_1^{\alpha}(x)=2\alpha x \tag{1.3}$$

where $j\in\mathbb{N}\backslash\{1\}$. It is easy to see from (1.3) that $C_2^{\alpha}(x)=2\alpha(1+\alpha)x^2-\alpha$. For $\alpha\in\mathbb{R}\backslash\{0\}$, a generating function of the sequence $C_j^{\alpha}(x)$, $j\in\mathbb{N}$, is defined by (see [2]):

$$\mathcal{H}_{\alpha}(x,z):=\sum_{j=0}^{\infty}C_j^{\alpha}(x)z^j=\frac{1}{(1-2xz+z^2)^{\alpha}}, \tag{1.4}$$

where $z\in\mathfrak{D}$ and $x\in$ [-1,1].

Two particular cases of $C_j^{\alpha}(x)$ are $i)\,C_j^{1}(x)$, the second kind Chebyshev polynomials and $ii)\,C_j^{\frac{1}{2}}(x)$, the Legendre polynomials (See [3]). Gegenbauer polynomials, Fibonacci polynomials, Pell-Lucas polynomials, Chebyshev polynomials, Horadam polynomials, Fermat-Lucas polynomials and generalizations of them have potential applications in branches such as architecture, physics, combinatorics, number theory, statistics and engineering. These polynomials have been studied in [11],[12], [14], [15] and [37].

The recent research trends are the outcomes of the study of functions $\in\Sigma$ linked with any of the above mentioned polynomials, can be seen in [4], [20], [24],[29], [30], [33] and [32]. Generally interest was shown to estimate the initial Taylor-Maclaurin coefficients and the celebrated inequality of Fekete-Szegö for the special subfamilies of Σ. The popular Fekete-Szegö problem for certian subfamilies of Σ associated with Gegenbauer polynomials have been investigated in the works of [2], [3], [34] and [36]. However, there is little work on bi-univalent functions linked with Gegenbauer polynomials. To initiate and explore the study on bi-univalent functions linked with Gegenbauer polynomials, we present two special families of Σ associated with Gegenbauer polynomials $C_j^{\alpha}(x)$ as in (1.3). Throughout this paper, an inverse function $g^{-1}(\omega)=f(\omega)$ is as in (1.2) and te generating function $\mathcal{H}_{\alpha}(x,z)$ is as in (1.4).

For regular functions g and f in $\mathfrak{D}$, g is said to subordinate to f, if there is a Schwarz function ψ in $\mathfrak{D}$, such that $\psi(0)=0$, $|\psi(z)|<1$ and $g(z)=f(\psi(z))$, $z\in\mathfrak{D}$. This subordination is indicated as $g\prec f$. Specifically, when $f\in\mathcal{S}$ in $\mathfrak{D}$, then $g(z)\prec f(z)$ *is equivalent to* $g(0)=f(0)$ and $g(\mathfrak{D})\subset f(\mathfrak{D})$.

Definition 1.1. A function g in Σ having the power series (1.1) is said to be in the family $S\mathfrak{G}_{\Sigma}^{\alpha}(x,\gamma,\mu)$, $0\leq\gamma\leq1$, $\mu\geq\gamma, 1/2<x\leq1$ and α a nonzero real constant, if

$$\frac{zg'(z)+\mu z^2g''(z)}{\gamma zg'(z)+(1-\gamma)g(z)}\prec\mathcal{H}_{\alpha}(x,z),\; z\in\mathfrak{D}$$

and

$$\frac{\omega f'(\omega)+\mu\omega^2 f''(\omega)}{\gamma\omega f'(\omega)+(1-\gamma)f(\omega)} \prec \mathcal{H}_\alpha(x,\omega),\ \omega\ \in \mathfrak{D}.$$

Definition 1.2. A function g in Σ having the power series (1.1) is said to be in the family $R\mathfrak{B}^\alpha_\Sigma(x,\gamma,\mu)$, $0\le\gamma\le 1$, $\mu\ge\gamma, 1/2<x\le 1$ and α a nonzero real constant, if

$$\frac{zg'(z)+\mu z^2 g''(z)}{\gamma z g'(z)+(1-\gamma)z} \prec \mathcal{H}_\alpha(x,z),\ z\in\mathfrak{D}$$

and

$$\frac{\omega f'(\omega)+\mu\omega^2 f''(\omega)}{\gamma\omega f'(\omega)+(1-\gamma)\omega} \prec \mathcal{H}_\alpha(x,\omega),\ \omega\ \in \mathfrak{D}.$$

The above defined families $S\mathfrak{G}^\alpha_\Sigma(\gamma,\mu,x)$ and $R\mathfrak{B}^\alpha_\Sigma(x,\gamma,\mu)$ are of special interest for they contain many well-known as well as new subfamilies of $\sum$ for particular values of γ and μ, as illustrated below:

1. $SL^\alpha_\Sigma(x,\mu)\equiv S\mathfrak{G}^\alpha_\Sigma(x,0,\mu)$, $\mu\ge 0$, is the class of functions $g\in\Sigma$ satisfying

$$\left(\frac{zg'(z)}{g(z)}\right)\left(1+\mu\left(\frac{zg''(z)}{g'(z)}\right)\right) \prec \mathcal{H}_\alpha(x,z),\ z\ \in\mathfrak{D}$$

and

$$\left(\frac{\omega f'(\omega)}{f(\omega)}\right)\left(1+\mu\left(\frac{\omega f''(\omega)}{f'(\omega)}\right)\right) \prec \mathcal{H}_\alpha(x,\omega),\ \omega\in\mathfrak{D}.$$

2. $SM^\alpha_\Sigma(x,\gamma)\equiv S\mathfrak{G}^\alpha_\Sigma(x,\gamma,1)$, is the class of functions $g\in\Sigma$ satisfying

$$\frac{zg'(z)+z^2 g''(z)}{\gamma z g'(z)+(1-\gamma)g(z)} \prec \mathcal{H}_\alpha(x,z),\ z\in\mathfrak{D}$$

and

$$\frac{\omega f'(\omega)+\omega^2 f''(\omega)}{\gamma\omega f'(\omega)+(1-\gamma)f(\omega)} \prec \mathcal{H}_\alpha(x,\omega),\ \omega\ \in \mathfrak{D}.$$

3. $RP^\alpha_\Sigma(x,\mu)\equiv R\mathfrak{B}^\alpha_\Sigma(x,0,\mu)$, $\mu\ge 0$, is the set of functions $g\in\Sigma$ satisfying

$$g'(z)+\mu z g''(z)\prec\mathcal{H}_\alpha(x,z),\ z\in\mathfrak{D}\quad\text{and}\quad f'(\omega)+\mu\omega f''(\omega)\prec\mathcal{H}_\alpha(x,\omega),\ \omega\in\mathfrak{D},$$

4. $RQ^\alpha_\Sigma(x,\gamma)\equiv R\mathfrak{B}^\alpha_\Sigma(x,\gamma,1)$, is the class of functions $g\in\Sigma$ satisfying

$$\frac{zg'(z)+z^2 g''(z)}{\gamma z g'(z)+(1-\gamma)z} \prec \mathcal{H}_\alpha(x,z),\ z\in\mathfrak{D}$$

and

$$\frac{\omega f'(\omega)+\omega^2 f''(\omega)}{\gamma\omega f'(\omega)+(1-\gamma)\omega} \prec \mathcal{H}_\alpha(x,\omega),\ \omega\ \in \mathfrak{D}.$$

5. $R\mathfrak{N}^\alpha_\Sigma(x,\mu)\equiv S\mathfrak{G}^\alpha_\Sigma(x,1,\mu)$, $\mu\ge 1$ (or $R\mathfrak{N}^\alpha_\Sigma(x,\mu)\equiv R\mathfrak{B}^\alpha_\Sigma(x,1,\mu)$, $\mu\ge 1$), is the class of functions $g\in\Sigma$ satisfying

$$1+\mu\left(\frac{zg''(z)}{g'(z)}\right)\prec\mathcal{H}_\alpha(x,z),\ z\in\mathfrak{D}\quad\text{and}\quad 1+\mu\left(\frac{\omega f''(\omega)}{f'(\omega)}\right)\prec\mathcal{H}_\alpha(x,\omega),\ \omega\ \in\mathfrak{D},$$

6. $RP^\alpha_\Sigma(x,0)\equiv R^\alpha_\Sigma(x)$ and $SL^\alpha_\Sigma(x,0)\equiv S^\alpha_\Sigma(x)$ were examined by Amourah et al. [3]. $R\mathfrak{N}^\alpha_\Sigma(x,1)\equiv K^\alpha_\Sigma(x)$ was investigated by Amourah et al. [2].

In Section 2, we derive the estimates for $|d_2|$, $|d_3|$ and the inequality of Fekete-Szegö [10] and [5] for functions of the form (1.1) $\in S\mathfrak{S}_{\Sigma}^{\alpha}(x,\gamma,\mu)$. In Section 3, we derive the estimates for $|d_2|$, $|d_3|$ and the inequality of Fekete- Szegö for functions of the form (1.1) $\in R\mathfrak{B}_{\Sigma}^{\alpha}(x,\gamma,\mu)$. We also present interesting consequences and provide relevant connections of our results.

2. The function family $S\mathfrak{S}_{\Sigma}^{\alpha}(x,\gamma,\mu)$

In the following theorem, we find the initial coefficients bounds and the inequality of Fekete-Szegö for functions in $S\mathfrak{S}_{\Sigma}^{\alpha}(x,\gamma,\mu)$.

Theorem 2.1. *Let $0\le\gamma\le 1$, $\mu\ge\gamma, 1/2<x\le 1$ and α a nonzero real constant. If a function g of the form (1.1) $\in S\mathfrak{S}_{\Sigma}^{\alpha}(x,\gamma,\mu)$, then*

$$|d_2|\le\frac{2|\alpha|x\sqrt{2x}}{\sqrt{|(1-\gamma+2\mu)^2(1-2x^2)+2((\gamma-1)^2-4\mu(\mu-1))\alpha x^2|}},\tag{2.1}$$

$$|d_3|\le\frac{4\alpha^2x^2}{(1-\gamma+2\mu)^2}+\frac{|\alpha|x}{1-\gamma+3\mu}\tag{2.2}$$

and for $\delta\in\mathbb{R}$

$$|d_3-\delta d_2^2|\le\begin{cases}\frac{|\alpha|x}{1-\gamma+3\mu} & ;\ |1-\delta|\le\mathfrak{J}\\ \frac{8\alpha^2x^3\,|1-\delta|}{|(1-\gamma+2\mu)^2(1-2x^2)+2((\gamma-1)^2-4\mu(\mu-1))\alpha x^2|} & ;\ |1-\delta|\ge\mathfrak{J},\end{cases}\tag{2.3}$$

where

$$\mathfrak{J}=\left|\frac{(1-\gamma+2\mu)^2(1-2x^2)+2((\gamma-1)^2-4\mu(\mu-1))\alpha x^2}{8(1-\gamma+3\mu)\alpha x^2}\right|.\tag{2.4}$$

Proof. Let $g\in S\mathfrak{S}_{\Sigma}^{\alpha}(x,\gamma,\mu)$. Then, for two regular functions $\mathfrak{M}$, $\mathfrak{N}$ with $\mathfrak{M}(0)=0, |\mathfrak{M}(z)|<1, \mathfrak{N}(0)=0$ and $|\mathfrak{N}(\omega)|<1$, $z,\omega\in\mathfrak{D}$ and on account of Definition 1.1, we can write

$$\frac{zg'(z)+\mu z^2g''(z)}{\gamma zg'(z)+(1-\gamma)g(z)}=\mathcal{H}_\alpha(x,\mathfrak{M}(z))$$

and

$$\frac{\omega f'(\omega)+\mu\omega^2f''(\omega)}{\gamma\omega f'(\omega)+(1-\gamma)f(\omega)}=\mathcal{H}_\alpha(x,\mathfrak{N}(\omega)).$$

Equivalently, we have

$$\frac{zg'(z)+\mu z^2g''(z)}{\gamma zg'(z)+(1-\gamma)g(z)}=1+C_1^\alpha(x)+C_2^\alpha(x)\mathfrak{m}(z)+C_3^\alpha(x)(\mathfrak{m}(z))^2+...\tag{2.5}$$

and

$$\frac{\omega f'(\omega)+\mu\omega^2f''(\omega)}{\gamma\omega f'(\omega)+(1-\gamma)f(\omega)}=1+C_1^\alpha(x)+C_2^\alpha(x)\mathfrak{n}(\omega)+C_3^\alpha(x)(\mathfrak{n}(\omega))^2+....\tag{2.6}$$

From (2.5) and (2.6), in view of (1.3), we obtain

$$\frac{zg'(z)+\mu z^2g''(z)}{\gamma zg'(z)+(1-\gamma)g(z)}=1+C_1^\alpha(x)\mathfrak{m}_1z+[C_1^\alpha(x)\mathfrak{m}_2+C_2^\alpha(x)\mathfrak{m}_1^2]z^2+...\tag{2.7}$$

and

$$\frac{\omega f'(\omega) + \mu\omega^2 f''(\omega)}{\gamma\omega f'(\omega) + (1-\gamma)f(\omega)} = 1 + C_1^\alpha(x)\mathfrak{n}_1\omega + [C_1^\alpha(x)\mathfrak{n}_2 + C_1^\alpha(x)\mathfrak{n}_1^2]\omega^2 + \dots. \tag{2.8}$$

It is known that if $|\mathfrak{M}(z)| = |\mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + ...| < 1, z \in \mathfrak{D}$ and $|\mathfrak{N}(\omega)| = |\mathfrak{n}_1\omega + \mathfrak{n}_2\omega^2 + \mathfrak{n}_3\omega^3 + ...| < 1, \omega \in \mathfrak{D}$, then

$$|\mathfrak{m}_i| \leq 1 \text{ and } |\mathfrak{n}_i| \leq 1\, (i \in \mathbb{N}). \tag{2.9}$$

We get the following by equating the corresponding coefficients in (2.7) and (2.8):

$$(1-\gamma+2\mu)d_2 = C_1^\alpha(x)\mathfrak{m}_1, \tag{2.10}$$

$$2(1-\gamma+3\mu)d_3 - (1+\gamma)(1-\gamma+2\mu)\, d_2^2 = C_1^\alpha(x)\mathfrak{m}_2 + C_2^\alpha(x)\mathfrak{m}_1^2, \tag{2.11}$$

$$-\,(1-\gamma+2\mu)\, d_2 = C_1^\alpha(x)\mathfrak{n}_1 \tag{2.12}$$

and

$$2(1-\gamma+3\mu)(2d_2^2 - d_3) - (1+\gamma)(1-\gamma+2\mu)\, d_2^2 = C_1^\alpha(x)\mathfrak{n}_2 + C_2^\alpha(x)\mathfrak{n}_1^2. \tag{2.13}$$

It follows from (2.10) and (2.12) that

$$\mathfrak{m}_1 = -\mathfrak{n}_1 \tag{2.14}$$

and

$$2(1-\gamma=2\mu)^2 d_2^2 = (\mathfrak{m}_1^2 + \mathfrak{n}_1^2)(C_1^\alpha(x))^2. \tag{2.15}$$

If we add (2.11) and (2.13), then we obtain

$$2((1-\gamma)(1-\gamma+2\mu)+2\mu)d_2^2 = C_1^\alpha(x)(\mathfrak{m}_2+\mathfrak{n}_2) + C_2^\alpha(x)(\mathfrak{m}_1^2+\mathfrak{n}_1^2). \tag{2.16}$$

Substituting the value of $\mathfrak{m}_1^2 + \mathfrak{n}_1^2$ from (2.15) in (2.16), we get

$$d_2^2 = \frac{(C_1^\alpha(x))^3(\mathfrak{m}_2+\mathfrak{n}_2)}{2\left[((1-\gamma)(1-\gamma+2\mu)+2\mu)(C_1^\alpha(x))^2 - (1-\gamma+2\mu)^2 C_2^\alpha(x)\right]}, \tag{2.17}$$

which gets the desired result (2.1) on using (2.9).

After subtracting (2.13) from (2.11) and then using (2.14), we obtain

$$d_3 = d_2^2 + \frac{C_1^\alpha(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{4(1-\gamma+3\mu)}. \tag{2.18}$$

Then in view of (2.15), equation (2.18) becomes

$$d_3 = \frac{(C_1^\alpha(x))^2(\mathfrak{m}_1^2+\mathfrak{n}_1^2)}{2(1-\gamma+2\mu)^2} + \frac{C_1^\alpha(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{4(1-\gamma+3\mu)},$$

which yields (2.2) on applying (2.9).

From (2.17) and (2.18), for $\delta \in \mathbb{R}$, we get

$$|d_3 - \delta d_2^2| =$$

$$|C_1^\alpha(x)|\left|\left(T(\delta,x) + \frac{1}{4(1-\gamma+3\mu)}\right)\mathfrak{m}_2 + \left(T(\delta,x) - \frac{1}{4(1-\gamma+3\mu)}\right)\mathfrak{n}_2\right|,$$

where

$$T(\delta,x) = \frac{(1-\delta)\,(C_1^\alpha(x))^2}{2\left[((1-\gamma)(1-\gamma+2\mu)+2\mu)(C_1^\alpha(x))^2 - (1-\gamma+2\mu)^2 C_2^\alpha(x)\right]}.$$

In view of (1.3), we obtain

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|C_1^\alpha(x)|}{2(1-\gamma+3\mu)} & ;\ 0 \leq |T(\delta,x)| \leq \frac{1}{4(1-\gamma+3\mu)} \\ 2|C_1^\alpha(x)||T(\delta,x)| & ;\ |T(\delta,x)| \geq \frac{1}{4(1-\gamma+3\mu)}, \end{cases}$$

which enable us to conclude (2.3) with $\mathfrak{J}$ as in (2.4). This completes the proof of Theorem 2.1. □

Setting $\gamma = 0$ in the above theorem, we have

Corollary 2.2. *Let $\mu \geq 0, 1/2 < x \leq 1$ and α a nonzero real constant. If a function g of the form (1.1) $\in SL_\Sigma^\alpha(x,\mu)$, then*

$$|d_2| \leq \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|(2\mu+1)^2(1-2x^2) - 2(4\mu(\mu-1)-1)\alpha x^2|}},$$

$$|d_3| \leq \frac{4\alpha^2x^2}{(2\mu+1)^2} + \frac{|\alpha|x}{3\mu+1}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|\alpha|x}{3\mu+1} & ; |1-\delta| \leq \mathfrak{J}_1 \\ \frac{8\alpha^2x^3\,|1-\delta|}{|(2\mu+1)^2(1-2x^2)-2(4\mu(\mu-1)-1)\alpha x^2|} & ; |1-\delta| \geq \mathfrak{J}_1. \end{cases}$$

where

$$\mathfrak{J}_1 = \leq \left| \frac{(2\mu+1)^2(1-2x^2) - 2(4\mu(\mu-1)-1)\alpha x^2}{8(2\mu+1)\alpha x^2} \right|.$$

Remark 2.3. Corollary 2.2 coincide with Corollary 8 of [3], when $\mu = 0$.

Taking $\mu = 1$ in Theorem 2.1, we obtain

Corollary 2.4. *Let $0 \leq \gamma \leq 1$, $1/2 < x \leq 1$ and α a nonzero real constant. If a function g of the form (1.1) $\in SM_\Sigma^\alpha(x,\gamma)$, then*

$$|d_2| \leq \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|(3-\gamma)^2(1-2x^2) + 2(\gamma-1)^2\alpha x^2|}},$$

$$|d_3| \leq \frac{4\alpha^2x^2}{(3-\gamma)^2} + \frac{|\alpha|x}{4-\gamma}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|\alpha|x}{4-\gamma} & ;\ |1-\delta| \leq \left|\frac{(3-\gamma)^2(1-2x^2)+2(\gamma-1)^2\alpha x^2}{8(4-\gamma)\alpha x^2}\right| \\ \frac{8\alpha^2x^3\,|1-\delta|}{|(3-\gamma)^2(1-2x^2)+2(\gamma-1)^2\alpha x^2|} & ;\ |1-\delta| \geq \left|\frac{(3-\gamma)^2(1-2x^2)+2(\gamma-1)^2\alpha x^2}{8(4-\gamma)\alpha x^2}\right|. \end{cases}$$

Remark 2.5. Corollary 2.4 state the correct form for $|d_2|$ in Theorem 3.1 [2], when $\gamma = 1$.

3. THE FUNCTION FAMILY $R\mathfrak{B}_{\Sigma}^{\alpha}(x,\gamma,\mu)$

In the next theorem, we find the first two Taylor-Maclaurin coefficients and the inequality of Fekete-Szegö for functions in $R\mathfrak{B}_{\Sigma}^{\alpha}(x,\gamma,\mu)$.

Theorem 3.1. *Let $0 \leq \gamma \leq 1$, $\mu \geq \gamma, 1/2 < x \leq 1$ and α a nonzero real constant. If a function g of the form (1.1) $\in R\mathfrak{B}_{\Sigma}^{\alpha}(x,\ \gamma,\ \mu)$, then*

$$|d_2| \leq \frac{|\alpha|x\sqrt{2x}}{\sqrt{|(1-\gamma+\mu)^2(1-2x^2)+(2\gamma^2-3\gamma+1+2\mu-2\mu^2)\alpha x^2|}}, \tag{3.1}$$

$$|d_3| \leq \frac{(\alpha x)^2}{(1-\gamma+\mu)^2}+\frac{2|\alpha|x}{3(1-\gamma+2\mu)} \tag{3.2}$$

and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \leq \begin{cases} \frac{2|\alpha|x}{3(1-\gamma+2\mu)} & ;\ |1-\delta| \leq\ \Omega \\ \frac{|1-\delta|2\alpha^2x^3}{|(1-\gamma+\mu)^2(1-2x^2)+(2\gamma^2-3\gamma+1+2\mu-2\mu^2)\alpha x^2|} & ;\ |1-\delta| \geq \Omega, \end{cases} \tag{3.3}$$

where

$$\Omega = \left|\frac{(1-\gamma+\mu)^2(1-2x^2)+(\gamma^2-3\gamma+1+2\mu-2\mu^2)\alpha x^2}{3(1-\gamma+2\mu)\alpha x^2}\right|.$$

Proof. Let $g \in R\mathfrak{B}_{\Sigma}^{\alpha}(x,\gamma,\ \mu)$. Then, for some regular functions $\mathfrak{M}$ and $\mathfrak{N}$ such that $\mathfrak{M}(0)=0, |\mathfrak{M}(z)| = |\mathfrak{m}_1 z+\mathfrak{m}_2 z^2+\mathfrak{m}_3 z^3+...| < 1, \mathfrak{N}(0)=0$ and $|\mathfrak{N}(\omega)| = |\mathfrak{n}_1\omega+\mathfrak{n}_2\omega^2+\mathfrak{n}_3\omega^3+...| < 1,\ z,\omega \in \mathfrak{D}$ and on account of Definition 1.2, we can write

$$\frac{zg'(z)+\mu z^2 g''(z)}{\gamma zg'(z)+(1-\gamma)z} = \mathcal{H}_\alpha(x,\mathfrak{M}(z)),\ z \in \mathfrak{D}$$

and

$$\frac{\omega f'(\omega)+\mu\omega^2 f''(\omega)}{\gamma\omega f'(\omega)+(1-\gamma)\omega} = \mathcal{H}_\alpha(x,\mathfrak{N}(\omega)),\ \omega\ \in \mathfrak{D}.$$

Following the procedure similar to the proof of Theorem **??**, one gets

$$2(1-\gamma+\mu)d_2 = C_1^\alpha(x)\mathfrak{m}_1 \tag{3.4}$$

$$3(1-\gamma+2\mu)d_3-4(1-\gamma+\mu)\gamma d_2^2 = C_1^\alpha(x)\mathfrak{m}_2+C_2^\alpha(x)\mathfrak{m}_1^2 \tag{3.5}$$

$$-(1-\gamma+\mu)d_2 = H_2(x)\mathfrak{n}_1 \tag{3.6}$$

$$3(1-\gamma+2\mu)(2d_2^2-d_3)-4(1-\gamma+\mu)\gamma d_2^2 = C_1^\alpha(x)\mathfrak{n}_2+C_2^\alpha(x)\mathfrak{n}_1^2. \tag{3.7}$$

The results (3.1)-(3.3) now follow from (3.4)-(3.7) by adopting the procedure as in Theorem 3.1. □

By specializing $\gamma = 0$ in the above theorem, we get

Corollary 3.2. *Let $\mu \geq 0, 1/2 < x \leq 1$ and α a nonzero real constant. If a function g of the form (1.1) $\in RP_{\Sigma}^{\alpha}(x,\mu)$, then*

$$|d_2| \leq \frac{|\alpha|x\sqrt{2x}}{\sqrt{|(1+\mu)^2(1-2x^2)+(1+2\mu-2\mu^2)\alpha x^2|}},$$

$$|d_3| \leq \frac{\alpha^2x^2}{(1+\mu)^2}+\frac{2|\alpha|x}{3(1+2\mu)}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{2|\alpha|x}{3(1+2\mu)} & ; |1-\delta| \leq \Omega_1 \\ \frac{|1-\delta| 2\alpha^2 x^3}{|(1+\mu)^2(1-2x^2)+(1+2\mu-2\mu^2)\alpha x^2|} & ; |1-\delta| \geq \Omega_1. \end{cases}$$

where

$$\Omega_1 = \left| \frac{(1+\mu)^2(1-2x^2)+(1+2\mu-2\mu^2)\alpha x^2}{3(1+2\mu)\alpha x^2} \right|$$

Remark 3.3. Corollary 3.2 coincide with Corollary 9 of [3] when $\mu = 0$.

Corollary 3.4. *If a function g of the form (1.1) $\in RQ_{\Sigma}^{\alpha}(x, \gamma)$, then*

$$|d_2| \leq \frac{|\alpha| x\sqrt{2x}}{\sqrt{|(2-\gamma)^2(1-2x^2)+(2\gamma^2-3\gamma+1)\alpha x^2|}},$$

$$|d_3| \leq \frac{(\alpha x)^2}{(2-\gamma)^2} + \frac{2|\alpha|x}{3(3-\gamma)}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{2|\alpha|x}{3(3-\gamma)} & ; |1-\delta| \leq \Omega_2 \\ \frac{|1-\delta| 2\alpha^2 x^3}{|(2-\gamma)^2(1-2x^2)+(2\gamma^2-3\gamma+1)\alpha x^2|} & ; |1-\delta| \geq \Omega_2, \end{cases}$$

where

$$\Omega_2 = \left| \frac{(2-\gamma)^2(1-2x^2)+(\gamma^2-3\gamma+1)\alpha x^2}{3(3-\gamma)\alpha x^2} \right|.$$

Allowing $\gamma = 1$ in Theorem 3.1, we obtain

Corollary 3.5. *Let $\mu \geq 1, 1/2 < x \leq 1$ and α a nonzero real constant. If a function g of the form (1.1) $\in R\mathfrak{N}_{\Sigma}^{\alpha}(x, \mu)$, then*

$$|d_2| \leq \frac{|\alpha| x\sqrt{2x}}{\sqrt{|\mu^2(1-2x^2)+2\mu(1-\mu)\alpha x^2|}},$$

$$|d_3| \leq \frac{\alpha^2 x^2}{\mu^2} + \frac{|\alpha|x}{3\mu}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|\alpha|x}{3\mu} & ; |1-\delta| \leq \left|\frac{\mu(1-2x^2)+2(1-2\mu)\alpha x^2}{6\alpha x^2}\right| \\ \frac{|1-\delta| 2\alpha^2 x^3}{|\mu^2(1-2x^2)+2\mu(1-2\mu)\alpha x^2|} & ; |1-\delta| \geq \left|\frac{\mu(1-2x^2)+2(1-2\mu)\alpha x^2}{6\alpha x^2}\right|. \end{cases}$$

Remark 3.6. Corollary 3.5 corrects the statement of $|d_2|$ in the Theorem 3.1 of [2] when $\mu = 1$.

4. Conclusion

Two special families of regular and bi-univalent (or bi-schlicht) functions linked with Gegenbauer polynomials are initiated and explored. Bounds of the first two coefficients $|d_2|$, $|d_3|$ and the celebrated Fekete- Szegö functional have been fixed for each of the two families. Through corollaries of our main results, we have highlighted many interesting new consequences. The special families examined in this research paper linked with Gegenbauer polynomials could inspire further research related to other aspects such as families using q-derivative operator [8],[18] and [31], q-integral operator [16], meromorphic bi-univalent function families associated with Al-Oboudi differential operator [22] and families using integro-differential operators [21]. Hankel determinant[23] for the defined special families could also be examined.

Acknowledgement. The authors would like to thank the editor and referees for their helpful suggestions which essentially improved the presentation of this paper

References

1. Ş. Altınkaya and S. Yalçın, On the Chebyshev polynomial coefficient problem of some subclasses of bi-univalent functions, Gulf Journal of Mathematics, 5(3) (2017), 34-40.
2. A. Amourah, A. Alamoush and M. Al-Kaseasbeh, Gegenbauer polynomials and biunivalent functions, Palestine J. Math., 10(2)(2021),625-632.
3. A. Amourah, B. A. Frasin and T. Abdeljawad, Fekete-Szegö inequality for analytic and biunivalent functions subordinate to Gegenbauer polynomials, Journal of Function Spaces, Vol.2021, Art ID 5574673, 7 pages. https://doi.org/10.1155/2021/5574673.
4. I. T. Awolere and A. T. Oladipo, Coefficients of bi-univalent functions involving pseudo-starlikeness associated with Chebyshev polynomials, Khayyam J. Math. 5 (1) (2019), 140-149.
5. K. O. Babalola, Coefficient determinants involving many Fekete-Szego-type parameters, Gulf Journal of Mathematics, 3 (1)(2015), 67-74. https://doi.org/10.1007/s00058-015-1569-2
6. D. A. Brannan and J. G. Clunie, Aspects of contemporary complex analysis, Proceedings of the NATO Advanced study institute held at University of Durhary, Newyork: Academic press, 1979
7. D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, In: S. M. Mazhar, A. Hamoui, N.S. Faour (eds) Mathematical analysis and its applications. Kuwait, pp 53-60, KFAS Proceedings Series, Vol. 3 (1985), Pergamon Press (Elsevier Science Limited), Oxford, 1988; see also Studia Univ. Babeş-Bolyai Math., 31(2) (1986), 70-77.
8. S. M. El-Deeb, T. Bulboacă, and B. M. El-Matary, Maclaurin coefficient estimates of bi-univalent functions connected with the q-derivative, Mathematics, 8 (2020), 418. https://doi.org/10.3390/math8030418
9. P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, (1983).
10. M. Fekete and G. Szegö, Eine Bemerkung Über Ungerade Schlichte Funktionen, J. Lond. Math. Soc., 89 (1933), 85-89. https://doi.org/10.2307/2790166
11. P. Filipponi and Horadam, A. F., Derivative sequences of Fibonacci and Lucas polynomials. In: G. E. Bergum, A. N. Philippou, A. F. Horadam (eds) Applications of Fibonacci Numbers, vol 4 (1990), pp 99-108, Proceedings of the fourth international conference on Fibonacci numbers and their applications, Wake Forest University, Winston-Salem, North

Carolina; Springer (Kluwer Academic Publishers), Dordrecht, Boston and London, 4 (1991), 99-108.

12. P. Filipponi andA. F. Horadam, Second derivative sequence of Fibonacci and Lucas polynomials, Fibonacci Quart., 31 (1993), 194-204.
13. B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. 24 (2011), 1569-1573. https://doi.org/10.1016/j.aml.2011.03.048
14. A. F. Horadam and J. M. Mahon, Pell and Pell-Lucas polynomials, Fibonacci Quart., 23 (1985), 7-20.
15. T. Hörçum and E. Gökçen Koçer, On some properties of Horadam polynomials, Int. Math. Forum., 4 (2009), 1243-1252.
16. B. Khan, H. M. Srivastava, M. Tahir, M. Darus, Q. Z. Ahmed and N. Khan, Applications of a certain q- integral operator to the subclasses of analytic and bi-univalent functions, AIMS Mathematics 6 (1) (2020), 1024-1039.
17. K. Kiepiela, I. Naraniecka and J. Szynal, The Gegenbauer polynomials and typically real functions, J. Compu. Appl. Math.,153(1-2) (2003), 273-282. https://doi.org/10.1016/S0377-0427(02)00642-8
18. A. Lasode and T. Opoola, Fekete-Szego estimates and second Hankel determinant for a generalized subfamily of analytic functions defined by q-differential operator, Gulf Journal of Mathematics, 11 (2) (202), 36-43. https://doi.org/10.56947/gjom.v11i2.583
19. M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc., 18 (1967), 63-68. https://doi.org/10.1090/S0002-9939-1967-0206255-1
20. N. Magesh and S. Bulut, Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functons related to pseudo- starlike functions, Afr. Mat., 29(1-2)(2018), 203-209. https://doi.org/10.1007/s13370-017-0535-3
21. A. O. P*á*ll-Szab*ó* and G. I. Oros, Coefficient related studies for new classes of bi-univalent functions, Mathematics, 8 (2020), 1110, 13 pages. https://doi.org/10.3390/math8071110
22. T. G. Shaba, M. G. Khan and B. Ahmed, Coefficient bounds for certain subclasses of meromorphic bi-univalent functions associated with Al-Oboudi differential operator, Palestine J. Math., 9 (2) (2020), 11 pages.
23. G. Shanmugam, B. A. Stephen and K. O. Babalola, Third Hankel determinant for alpha-starlike functions, Gulf Journal of Mathematics, 2 (2) (2014), 107-113. https://doi.org/10.56947/gjom.v2i2.202
24. H. M. Srivastava, Ş. Altınkaya and S. Yalçın, Certain Subclasses of bi-univalent functions associated with the Horadam polynomials, Iran J. Sci. Technol. Trans. Sci., (2018). https://doi.org/10.1007/s40995-018-0647-0
25. H. M. Srivastava and D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, J. Egyptian Math. Soc., 23(2015), 242-246. https://doi.org/10.1016/j.joems.2014.04.002
26. H. M. Srivastava, S. Bulut, M. Çaglar and N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat 27(2013), 831-842. https://doi.org/10.2298/FIL1305831S
27. H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for some general subclasses of analytic and bi-univalent functions, Afr. Mat., 28(2017), 693-706. https://doi.org/10.1007/s13370-016-0478-0
28. H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23 (2010), 1188-1192.
29. S. R. Swamy, Coefficient bounds for Al-Oboudi type bi-univalent functions based on a modified sigmoid activation function and Horadam polynimials, Earthline J. Math. Sci. 7 (2) (2021,), 251-270.
30. S. R. Swamy, S. Bulut and Y. Sailaja, Some special families of holomorphic and Sălăgean type bi-univalent functions associated with Horadam polynomials involving modified Sigmoid activation function, Hacet. J. Math. Stat., 51(2)(2021), 710-780.

31. S. R. Swamy and P. K. Mamatha, Certain classes of bi-univalent functions associated with q-analogue of Bessel functions, South East Asian J. Math. Math. Sci., 16 (3) (2020), 61-82.
32. S. R. Swamy and Y. Sailaja, Horadam polynomial coefficient estimates for two families of holomorphic and bi-univalent functions, Inter. J. Math. Trends and Tech., 66 (8) (2020), 131-138.
33. S. R. Swamy, A. K. Wanas and Y. Sailaja, Some special families of holomorphic and Sălăgean type bi-univalent functions associated with (m, n)-Lucas polynomials, Communications in Mathematics and Applications, 11 (4) (2020), 563-574. DoI: 10.26713/cma.v11i4.1411. https://doi.org/10.26713/cma
34. S. R. Swamy and S. Yalçın, Coefficient bounds for regular and bi-univalent functions linked with Gegenbauer polynomials, Problemy Analiza-Issues of Analysis, 11(29) No. 1, (2022), 133-144.
35. D. L. Tan, Coefficient estimates for bi-univalent functions, Chin. Ann. Math. Ser. A, 5 (1984), 559-568.
36. A. K. Wanas, New families of bi-univalent functions governed by Gegenbauer polynomials, Earthline Journal of Math. Sci.,7(2)(2021),403-427.
37. T.-T. Wang and W.-P. Zhang, Some identities involving Fibonacci, Lucas polynomials and their applications. Bull. Math. Soc. Sci. Math. Roumanie (New Ser.), 55 (103) (2012), 95-103.

[1]Department of Computer Science and Engineering, RV College of Engineering, Bengaluru - 560 059, Karnataka, India., ORCID:https://orcid.org/0000-0002-8088-4103

Email address: sondekola.swamy@gmail.com

[2] Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyih, Iraq., ORCID:https://orcid.org/0000-0002-7033-8993

Email address: waggas.galib@qu.edu.iq