Gulf Journal of Mathematics
Vol 4, Issue 3 (2016) 49-54

SIGN DOMINATION IN ARITHMETIC GRAPHS

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ABSTRACT. Investigation of the graph, whose vertices are labelled with the natural numbers and adjacency among them influenced by concepts of number theory is of prime importance. This marks the collaboration of higher arithmetic with graph theory. In this paper, we compute the sign domination number of one such graph called Arithmetic graph. Also, we characterize the sign domination number with domination and efficient domination number of arithmetic graph.

1. INTRODUCTION AND PRELIMINARIES

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = V$ of order $|V| = n$, edge set $E(G) = E$ of size $|E| = m$ and let v be a vertex of V. The open neighborhood of v is $N(v) = \{u \in V : uv \in E(G)\}$ and closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For any undefined term of graph theory, we refer [6].

Number theory sometimes called higher arithmetic, consist of the study of the properties of whole numbers. Particularly, primes and prime power factorization are very important concepts in number theory. The Fundamental Theorem of Arithmetic says that every integer greater than 1 can be written in the form $p_1^{a_1}p_2^{a_2}, \ldots, p_l^{a_l}$, where $a_i \geq 0$ and the p_i's are distinct primes. The factorization is unique, except possibly for the order of the factors. This interplay between Number theory and Graph theory motivates to define a new class of graph, namely Arithmetic graph $\mathcal{A}_k(G)$ defined as follows: Let k be a positive integer such that $k = p_1^{a_1}p_2^{a_2}, \ldots, p_l^{a_l}$. Then $\mathcal{A}_k(G)$ is the graph whose vertex set consists of the divisors of k and two vertices u, v are adjacent in arithmetic graph $\mathcal{A}_k(G)$ if and only if GCD $(u, v) = p_i$, for any prime divisor p_i of k. As vertex 1 is an isolated vertex, we consider arithmetic graph $\mathcal{A}_k(G)$ without vertex 1. For more details, we follow [1] and [10].

The concept of sign domination was initiated by Dunbar et al. [5] in the following sense: A sign dominating function of a graph G is a function $f : V(G) \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for all $v \in V$. The sign domination number of a graph G is $\gamma_s(G) = min\{w(f) : f$ is sign dominating function$\}$. The classical domination number and efficient domination number of a graph G, $\gamma(G)$ and

Date: Received: Jan 30, 2016; Accepted: Jun 24, 2016.
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2010 *Mathematics Subject Classification.* 11A25, 05C69, 20D60.
Key words and phrases. Graph, arithmetic graph, dominating set,domination function, sign domination.

$\gamma_e(G)$, is similarly defined to be the minimum weight function $f : V(G) \to \{0, 1\}$ such that $f(N[v]) \geq 1$ and $f(N[v]) = 1$ respectively for all $v \in V$. Cockayne et al.[2] introduced the concept of efficient domination and studied by Chaluvaraju et al. [3] and [4]. An existence of a graph with the given domination number was initiated by Vangipuram et al.[12] and studied by Maheshwari et al. [9] and [11]. Complete review on theory of domination and its related parameters can be found in [7] and [8].

In this article, we initiate sign dominating function of an arithmetic graph $\mathcal{A}_k(G)$ as follows: A function $f : V(\mathcal{A}_k(G)) \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for all $v \in V(\mathcal{A}_k(G))$. The sign domination number $\gamma_s(\mathcal{A}_k(G)) = min\{w(f) : f$ is sign dominating function$\}$.

2. Existing results

Theorem 2.1. [11] *If $k = p_1^{a_1} p_2^{a_2} \dots p_l^{a_l}$, where $p_1, p_2, \dots, p_l$ are distinct primes and integers $\{a_1, a_2, \dots, a_l\} \geq 1$, then*

$$\gamma(\mathcal{A}_k(G)) = \begin{cases} l-1 & \text{if } \ a_i = 1 \text{ for more than one } i, \\ l & \text{otherwise}, \end{cases}$$

where l is number of distinct prime divisors of k.

Theorem 2.2. [9] *If k is a product of two distinct primes or p^{a_1} with integer $a_1 > 1$, then $\gamma_e(\mathcal{A}_k(G)) = 1$.*

Theorem 2.3. [9] *If k is a neither product of two distinct primes nor p^{a_1} and $k = p_1^{a_1} p_2$, where p_1, p_2 are two distinct primes and an integer $a_1 > 1$, then $\gamma_e(\mathcal{A}_k(G)) = 2$.*

Theorem 2.4. [9] *If k is a neither product of two distinct primes nor p^{a_1} nor $k = p_1^{a_1} p_2$ and $k = p_1^{a_1} p_2^{a_2}$, where p_1, p_2 are two distinct primes and integers $\{a_1, a_2\} > 1$, then $\gamma_e(\mathcal{A}_k(G)) = 2$.*

Theorem 2.5. [9] *If k is a neither prime nor $p_1 p_2$ nor p^{a_1} nor $k = p_1^{a_1} p_2^{a_2}$ and $k = p_1^{a_1} p_2^{a_2} \dots p_l^{a_l}$, where $p_1, p_2, \dots, p_l$ are distinct primes and integers $\{a_1, a_2, \dots, a_l\} \geq 1$, then the efficient domination number does not exist for arithmetic graph $\mathcal{A}_k(G)$.*

Theorem 2.6. [5] *For any graph G, $\gamma_s(G) = n$ if and only if every non isolated vertex is either an end vertex or adjacent to an end vertex.*

3. Main results

Proposition 3.1. *For any positive integer k,*

$$|V(\mathcal{A}_k(G))| = (a_1 + 1)(a_2 + 1) \dots (a_l + 1) - 1.$$

Theorem 3.2. *For any positive integer k, arithmetic graph $\mathcal{A}_k(G)$ is connected.*

Proof. If k is a prime number, then the arithmetic graph $\mathcal{A}_k(G)$ consist of a single vertex. Thus, it is connected. If k is a product of primes, then by the definition of arithmetic graph $\mathcal{A}_k(G)$, there exist edges between primes, their prime power and also to their prime product vertices. Hence, arithmetic graph $\mathcal{A}_k(G)$ is connected. □

Theorem 3.3. *If $k = p^{a_1}$, where p is a prime number and integer $a_1 \geq 1$, then*
(i) $m(\mathcal{A}_k(G)) = a_1 - 1$.
(ii) $\gamma_s(\mathcal{A}_k(G)) = a_1$.
(iii) $\gamma_s(\mathcal{A}_k(G)) = \gamma(\mathcal{A}_k(G)) + a_1 - 1 = \gamma_e(\mathcal{A}_k(G)) + a_1 - 1$.

Proof. If $a_1 = 1$, then the arithmetic graph $\mathcal{A}_k(G)$ has only one vertex for which number of edges is zero and $\gamma_s(\mathcal{A}_k(G)) = 1$.

Let $k = p^{a_1}$ with $a_1 > 1$. The arithmetic graph $\mathcal{A}_k(G)$ has a_1 vertices namely $p, p^2, \ldots$ and p^{a_1}. For $i = 2$ to a_1, GCD$(p, p^i) = p$, the vertex p is adjacent to all other vertices of arithmetic graph $\mathcal{A}_k(G)$. As GCD $(p^2, p^3) = p^2$, GCD $(p^3, p^4) = p^3, \ldots,$ GCD $(p^{a_1-1}, p^{a_1}) = p^{a_1}$, vertices p^2, $p^3 \ldots$, p^{a_1} are not adjacent to each other. The arithmetic graph $\mathcal{A}_k(G) \cong K_{1,a_1-1}$, for which $m(\mathcal{A}_k(G)) = a_1 - 1$ and $\gamma_s(\mathcal{A}_k(G)) = a_1$. From Theorems 2.1 and 2.2, $\gamma(\mathcal{A}_k(G)) = 1$ and $\gamma_e(\mathcal{A}_k(G)) = 1$. Hence (iii) follows. □

Theorem 3.4. *For any positive integer k and $a_1 \geq 1$, $\gamma_s(\mathcal{A}_k(G)) = a_1$ if and only if $k = p^{a_1}$.*

Proof. If $k = p^{a_1}$, then from (ii) of Theorem 3.3, $\gamma_s(\mathcal{A}_k(G)) = a_1$. Conversely, if $\gamma_s(\mathcal{A}_k(G)) = a_1$, then from Theorem 2.6, every vertex of arithmetic graph $\mathcal{A}_k(G)$ should either be an end vertex or support vertex. Thus $\mathcal{A}_k(G) \cong K_{1,a_1-1}$, when $k = p^{a_1}$. □

Theorem 3.5. *If k is a product of two distinct prime numbers, then*
(i) $m(\mathcal{A}_k(G)) = 2$.
(ii) $\gamma_s(\mathcal{A}_k(G)) = 3$.
(iii) $\gamma_s(\mathcal{A}_k(G))) = \gamma(\mathcal{A}_k(G)) + 2 = \gamma_e(\mathcal{A}_k(G)) + 2$.

Proof. Let $k = p_1p_2$, where p_1 and p_2 are two distinct primes. The arithmetic graph $\mathcal{A}_k(G)$ has three vertices p_1, p_2 and p_1p_2. Since GCD $(p_1, p_2) = 1$, vertices p_1 and p_2 are not adjacent to each other in $\mathcal{A}_k(G)$. As GCD $(p_1, p_1p_2) = p_1$ and GCD $(p_2, p_1p_2) = p_2$, vertex p_1p_2 is adjacent to vertices p_1 and p_2. The arithmetic graph $\mathcal{A}_k(G) \cong K_{1,2}$ for which $m(\mathcal{A}_k(G)) = 2$ and $\gamma_s(\mathcal{A}_k(G)) = 3$. From Theorems 2.1 and 2.2, $\gamma(\mathcal{A}_k(G)) = 1$ and $\gamma_e(\mathcal{A}_k(G)) = 1$. Hence (iii) follows. □

Theorem 3.6. *If $k = p_1^{a_1}p_2$, where p_1, p_2 are distinct primes and a_1 is a positive integer > 1, then*
(i) $m(\mathcal{A}_k(G)) = 4a_1 - 2$.
(ii) $\gamma_s(\mathcal{A}_k(G)) = 3$.
(iii) $\gamma_s(\mathcal{A}_k(G)) = \gamma(\mathcal{A}_k(G)) + 1 = \gamma_e(\mathcal{A}_k(G)) + 1$.

Proof. Let $k = p_1^{a_1}p_2$, where p_1 and p_2 are distinct primes. The arithmetic graph $\mathcal{A}_k(G)$ has $(2a_1+1)$-vertices namely p_1, p_1^2, p_1^3,$\ldots$, $p_1^{a_1}$, p_2, p_1p_2, $p_1^2p_2, \ldots$ and $p_1^{a_1}p_2$. As GCD $(p_1, p_2) = 1$, GCD $(p_1, p_1^i) = p_1$\{for $i = 2$ to a_1\} and GCD $(p_1, p_1^ip_2) = p_1$\{for $i = 1$ to a_1\}, vertex p_1 is adjacent to $(2a_1 - 2)$-vertices of arithmetic graph $\mathcal{A}_k(G)$. Also GCD $(p_2, p_1^ip_2) = p_2$\{for $i = 1$ to a_1\} and GCD $(p_2, p_1^j) = 1$\{for $j = 2$ to a_1\}, implies vertex p_2 is adjacent to a_1 vertices of arithmetic graph $\mathcal{A}_k(G)$. Vertex p_1p_2 is adjacent to $(a_1 + 1)$-vertices as GCD$(p_1p_2, p_1^i) = p_1$\{for

$i = 1$ to $a_1\}$. Except for p_1, p_2 and p_1p_2 all other vertices are of degree 2 in $\mathcal{A}_k(G)$. Let vertices p_1, p_2 and p_1p_2 belongs to V_1. Vertex p_1 is adjacent to $(2a_1 - 1)$-vertices, out of which p_1p_2 is already assigned. Out of remaining $(2a_1 - 2)$-vertices, $(a_1 - 1)$-vertices are assigned 1 and $(a_1 - 1)$-vertices are assigned -1 such that $f(N[p_2]) \geq 1$ and $f(N[p_1p_2]) \geq 1$. Hence $m(\mathcal{A}_k(G)) = 4a_1 - 2$ and $\gamma_s(\mathcal{A}_k(G)) = 3$. From Theorem 2.1 and 2.3, $\gamma(\mathcal{A}_k(G)) = 2$ and $\gamma_e(\mathcal{A}_k(G)) = 2$. Thus, (iii) follows. □

Theorem 3.7. *If $k = p_1p_2p_3$, where p_1, p_2 and p_3 are distinct primes, then*

(i) $m(\mathcal{A}_k(G)) = 12$.
(ii) $\gamma_s(\mathcal{A}_k(G)) = 3$.
(iii) $\gamma_s(\mathcal{A}_k(G)) = \gamma(\mathcal{A}_k(G)) + 1$.

Proof. Let $k = p_1p_2p_3$ be the product of three distinct prime numbers. The arithmetic graph $\mathcal{A}_k(G)$ has seven vertices p_1, p_2, p_3, p_1p_2, p_1p_3, p_2p_3 and $p_1p_2p_3$. As GCD $(p_i, p_j) = 1$ {for $i \neq j$ and $i, j = 1$ to 3}, vertices p_1, p_2 and p_3 are not adjacent to each other. GCD $(p_ip_j, p_jp_q) = p_j$ {for $i \neq j \neq q$ and $i, j, q = 1$ to 3}, vertices p_1p_2, p_2p_3 and p_1p_3 are adjacent to each other. For $i \neq j$, GCD $(p_ip_j, p_i) = p_i$ and GCD$(p_ip_j, p_j) = p_j$ implies vertex p_ip_j is adjacent to both p_i and p_j. GCD $(p_1p_2p_3, p_i) = p_i$ {for $i = 1$ to 3}, vertex $p_1p_2p_3$ is adjacent to vertices p_1, p_2 and p_3. Out of all vertices of arithmetic graph $\mathcal{A}_k(G)$, vertices p_1, p_2, p_3 and $p_1p_2p_3$ are of degree three, whereas vertices p_1p_2, p_2p_3 and p_1p_3 are of degree four. Among p_1, p_2 and p_3 one vertex, say p_1 is assigned -1 and remaining two vertices are assigned 1. Out of p_1p_2, p_2p_3 and p_1p_3 vertices one vertex, say p_2p_3 is assigned -1 such that if p_i is assigned -1 then p_jp_q is assigned -1 for $i \neq j \neq q$. Here $p_1p_2p_3$ cannot be assigned -1 as $f(N[p_i]) < 1$ for any $i = 1$ to 3. Among seven vertices, two vertices are assigned -1 and five vertices are assigned 1. Hence $\gamma_s(\mathcal{A}_k(G)) = 3$. From Theorem 2.1, (iii) follows. □

Proposition 3.8. *If $k = p_1^{a_1}p_2^{a_2}$, where p_1, p_2 are distinct primes and integers $\{a_1, a_2\} > 1$, then*

(i) $m(\mathcal{A}_k(G)) = 4a_1a_2 - 2$.

(ii) $\gamma_s(\mathcal{A}_k(G)) = \begin{cases} 4 & \text{if } n(\mathcal{A}_k(G)) \text{ is even,} \\ 3 & \text{if } n(\mathcal{A}_k(G)) \text{ is odd.} \end{cases}$

(iii) $\gamma_s(\mathcal{A}_k(G)) = \begin{cases} \gamma(\mathcal{A}_k(G)) + 2 & \text{if } n(\mathcal{A}_k(G)) \text{ is even,} \\ \gamma(\mathcal{A}_k(G)) + 1 & \text{if } n(\mathcal{A}_k(G)) \text{ is odd.} \end{cases}$

(iv) $\gamma_s(\mathcal{A}_k(G)) = \begin{cases} \gamma_e(\mathcal{A}_k(G)) + 2 & \text{if } n(\mathcal{A}_k(G)) \text{ is even,} \\ \gamma_e(\mathcal{A}_k(G)) + 1 & \text{if } n(\mathcal{A}_k(G)) \text{ is odd.} \end{cases}$

Proof. Let $k = p_1^{a_1}p_2^{a_2}$, where p_1, p_2 are distinct prime numbers. The graph $\mathcal{A}_k(G)$ has $[(a_1 + 1)(a_2 + 1) - 1]$ number of vertices which are p_1^i, p_2^j, $p_1^ip_2$, $p_1p_2^j$ and $p_1^ip_2^j$ {for $i = 1$ to a_1 and $j = 1$ to a_2}.
(i) Since GCD$(p_1, p_2^j) = 1$ {for j=1 to a_2} and GCD$(p_1, p_1^i) = p_1$ {for i=2 to a_1}, vertex p_1 is adjacent to $a_1a_2 + a_1 - 1$ vertices. GCD$(p_2, p_1^i) = 1$ {for i=1 to a_1} and GCD$(p_2, p_2^j) = p_2$ {for j=2 to a_2}, vertex p_2 is adjacent to $a_1a_2 + a_2 - 1$.

$\mathrm{GCD}(p_1^i, p_1p_2^j) = p_1$ for {i= 2 to a_1 and j=1 to a_2} number of edges added are $(a_1-1)a_2$ and $\mathrm{GCD}(p_2^i, p_2p_1^j) = p_2$ {for i= 2 to a_2 and j= 1 to a_1} number of edges added are $(a_2-1)a_1$. Also vertices $p_1p_2,\ p_1p_2^2, \ldots, p_1p_2^{a_2},\ p_1^2p_2, \ldots p_1^2p_2^{a_2}, \ldots, p_1^{a_1}p_2^{a_2}$ are not adjacent to each other. Total number of edges of arithmetic graph $\mathcal{A}_k(G)$ is $4a_1a_2 - 2$.

(ii) Out of all vertices of $\mathcal{A}_k(G)$, vertices p_1, p_2 and p_1p_2 are assigned 1 as these vertices are of maximum degree.

Case 1. If $[(a_1+1)(a_2+1)-1]$ is even, then $(a_1+1)(a_2+1)$ is odd, a_1 is even and a_2 is also even. Vertex p_1 is adjacent to $a_1a_2 + a_1 - 1$=odd number of vertices out of which p_1p_2 is already assigned. So remaining even number of vertices are assigned 1 and -1 equally. Remaining vertices to be assigned are $p_2^2, p_2^3, \ldots p_2^{a_2}$ which are odd in number, vertices assigned 1 are one more than vertices assigned -1. Here, $|V_1| = |V_{-1}| + 4$. Hence $\gamma_s(\mathcal{A}_k(G)) = 4$.

Case 2. If $[(a_1+1)(a_2+1)-1]$ is odd, then we have following cases:

Subcase 2.1. If both a_1 and a_2 are odd numbers, then vertex p_1 is adjacent to an odd number of vertices. For the assignment of vertices adjacent to p_1, we follow as done in case 1. Remaining vertices to be assigned are $p_2^2, p_2^3, \ldots p_2^{a_2}$ which are even in number. Hence, these vertices are assigned 1 and -1 equally. Here, $|V_1| = |V_{-1}| + 3$. Hence $\gamma_s(\mathcal{A}_k(G)) = 3$.

Subcase 2.2. If a_1 is an even number and a_2 is an odd number, then vertex p_1 is adjacent to an odd number of vertices. For the assignment of vertices adjacent to p_1, we follow as done in case 1. Remaining vertices to be assigned are $p_2^2, p_2^3, \ldots p_2^{a_2}$ which are even in number. Hence, these vertices are assigned 1 and -1 equally. Here, $|V_1| = |V_{-1}| + 3$. Hence $\gamma_s(\mathcal{A}_k(G)) = 3$.

Subcase 2.3. If a_1 is an odd number and a_2 is an even number, then vertex p_1 is adjacent to an even number of vertices out of which p_1p_2 is assigned 1. So an odd number of vertices adjacent to p_1 should be assigned. Number of vertices assigned 1 is one more than the number of vertices assigned -1. Remaining vertices to be assigned are $p_2^2, p_2^3, \ldots p_2^{a_2}$ which are odd in number. Here the assignment is done such that vertices assigned -1 is one more than the number of vertices assigned 1. Here, $|V_1| = |V_{-1}| + 3$. Hence $\gamma_s(\mathcal{A}_k(G)) = 3$.

From all the above three cases, $\gamma_s(\mathcal{A}_k(G)) = 3$. From Theorems 2.1 and 2.4 $\gamma(\mathcal{A}_k(G)) = \gamma_e(\mathcal{A}_k(G)) = 2$, results (iii) and (iv) follow. □

CONCLUSION

The current trend of graph theory to higher arithmetic is assumed to be the new trend. The work carried out in this paper, further gives the scope for study of various dominating functions of arithmetic graph $\mathcal{A}_k(G)$. Many questions are suggested by this research, among them are the following:

If an integer $k = p_1^{a_1}p_2^{a_2}\ldots p_l^{a_l}$, where $p_1, p_2, \ldots, p_l$ are distinct primes and integers $\{a_1, a_2, \ldots, a_l\} \geq 1$, then

(i) In general, find the sign domination number of $\mathcal{A}_k(G)$.
(ii) When is $\gamma_s(\mathcal{A}_k(G)) = \gamma(\mathcal{A}_k(G))$?
(iii) When is $\gamma_s(\mathcal{A}_k(G)) = \gamma_e(\mathcal{A}_k(G))$?

Acknowledgement. Thanks are due to Prof. N. D. Soner for his help and valuable suggestions in the preparation of this paper.

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