

INVARIANT DIFFERENTIAL OPERATORS AND THE GENERALIZED SYMMETRIC GROUP

IBRAHIM NONKANÉ^{1*} AND LATÉVI M. LAWSON²

ABSTRACT. In this paper, we study the decomposition of the direct image of $\pi_+(\mathcal{O}_X)$, the polynomial ring $\mathcal{O}_X$ as a $\mathcal{D}$ -module, under the map $\pi : \text{spec } \mathcal{O}_X \rightarrow \text{spec } \mathcal{O}_X^{G(r,n)}$, where $\mathcal{O}_X^{G(r,n)}$ is the ring of invariant polynomials under the action of the wreath product $G(r, p) := \mathbb{Z}/r\mathbb{Z} \wr \mathcal{S}_n$. We first describe the generators of the simple components of $\pi_+(\mathcal{O}_X)$ and give their multiplicities. Using an equivalence of categories and the higher Specht polynomials, we describe a $\mathcal{D}$ -module decomposition of the polynomial ring localized at the discriminant of π . Furthermore, we study the action of invariants differential operators on the higher Specht polynomials.

1. INTRODUCTION

A fundamental problem in representation theory is the description of all irreducible representations. In this paper, we are interested in the polynomial representation of invariant differential operators with respect to the generalized symmetric group. It is well known by the decomposition theorem that the direct image of a simple module for a proper map is semi-simple [5]. The simplest case is when the map $\pi : X = \text{spec } B \rightarrow Y = \text{spec } A$ is finite, in which case it is easy to give a basic and entirely algebraic proof, using essentially the (generic) correspondence with the differential Galois group, which equals the ordinary Galois group G . The irreducible submodules of the direct image are in one-to-one correspondence with the irreducible representations of G (see [10]). In this paper, we make the differential structure more explicit in the case of the invariants of the complex reflection group $G(r, n)$, $B = \mathbb{C}[x_1, \dots, x_n] \subset A = \mathbb{C}[x_1, \dots, x_n]^{G(r,n)}$.

We explicitly study the simple component of the direct image $\pi_+(\mathcal{O}_X)$ of the polynomial ring $\mathcal{O}_X$ as a $\mathcal{D}$ -module under the map $\pi : \text{spec } \mathcal{O}_X \rightarrow \text{spec } \mathcal{O}_Y$ where $\mathcal{O}_Y = \mathcal{O}_X^{G(r,n)}$; the ring of invariant polynomials under the action of $G(r, n)$. We describe the generators of the simple components of $\pi_+(\mathcal{O}_X)$ and their multiplicities as in [10]. We thus establish the decomposition structure of $\pi_+(\mathcal{O}_X)$ by means of the higher Specht polynomials. This proof uses the fact that the irreducible

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* Corresponding author.

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$\mathcal{D}$ -submodules of $\pi_+(\mathcal{O}_X)$ are in one-to-one correspondence with irreducible representations of $G(r, n)$.

Secondly, we elaborate a $\mathcal{D}$ -module decomposition of the polynomial ring localized at the discriminant of π . Finally, we describe the action of invariant differential operators on higher Specht polynomials. The higher Specht polynomials (introduced combinatorially [1]), are adapted to the $\mathcal{D}$ -module structure.

This paper generalizes the results on modules over the Weyl algebra appeared in [10] and [12]. The case $r = 2$ have been presented at the 10th International Conference on Mathematical Modeling in Physical Sciences to describe the action of the rational Olshanetsky-Perelomov operator Hamiltonian on polynomials [13].

2. PRELIMINARIES

2.1. Direct image. In this section, we briefly recall the definition of the direct image of a $\mathcal{D}$ -module [4]. Let K be a field of characteristic zero, put $X = K^n$. The polynomial ring $K[x_1, \dots, x_n]$ will be denoted by $K[X]$; and the Weyl algebra generated by x_i 's and $\frac{\partial}{\partial x_i}$'s by $\mathcal{D}_X$. The n -tuple $(x_1, \dots, x_n)$ will be denoted by X . Similar conventions will holds for $Y = K^m$, with polynomial ring $K[Y]$ and Weyl algebra $\mathcal{D}_Y$.

Let $\pi : X \rightarrow Y$ be a polynomial map, with $\pi = (\pi_1, \dots, \pi_m)$. Let M be a left $\mathcal{D}_Y$ -module. The inverse image of M under the map π is $\pi^+(M) = K[X] \otimes_{K[Y]} M$. This is a $K[X]$ -module. It becomes a $\mathcal{D}_X$ -module with ∂_{x_i} acting according to the formula

$$\frac{\partial}{\partial x_i}(h \otimes u) = \frac{\partial h}{\partial x_i} \otimes u + \sum_{j=1}^m \frac{\partial \pi_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} u, \quad h \in K[X], u \in M.$$

Since $\mathcal{D}_Y \otimes_{\mathcal{D}_Y} M \cong M$, we have

$$\pi^+(M) \cong K[X] \otimes_{K[Y]} \mathcal{D}_Y \otimes_{\mathcal{D}_Y} M = \pi^+(K[Y]) \otimes_{\mathcal{D}_Y} M.$$

Denoting $\mathcal{D}_{X \rightarrow Y}$ by $\pi^+(K[Y])$, one has that $\pi^+(M) = \mathcal{D}_{X \rightarrow Y} \otimes_{\mathcal{D}_Y} M$. Note that $\mathcal{D}_{X \rightarrow Y}$ is $\mathcal{D}_X$ - $\mathcal{D}_Y$ -bimodule. Let N be a right $\mathcal{D}_X$ -module. The tensor product

$$\pi_+(N) = N \otimes_{\mathcal{D}_X} \mathcal{D}_{X \rightarrow Y}$$

is a right $\mathcal{D}_Y$ -module, which is called the *direct image* of N under the polynomial map π . Let us consider the *standard transposition* $\tau : \mathcal{D}_X \rightarrow \mathcal{D}_X$ defined by $\tau(h\partial^\alpha) = (-1)^{|\alpha|}\partial^\alpha h$, where $h \in K[X]$ and $\alpha \in \mathbb{N}^n$. If N is a right $\mathcal{D}_X$ -module then we define a left $\mathcal{D}_X$ -module N^t as follows. As an abelian group, $N^t = N$. If $a \in \mathcal{D}_X$ and $u \in N^t$ then the left action of a on u is defined by $a \star u = u\tau(a)$. Using the standard transposition for $\mathcal{D}_Y$ and $\mathcal{D}_X$, put $D_{Y \leftarrow X} = (\mathcal{D}_{X \rightarrow Y})^t$, this is a $\mathcal{D}_Y$ - $\mathcal{D}_X$ -bimodule. Let M be a left $\mathcal{D}_X$ -module. The direct image of M under π is defined by the formula

$$\pi_+(M) = D_{Y \leftarrow X} \otimes_{\mathcal{D}_X} M.$$

It is clear that $\pi_+(M)$ is a $\mathcal{D}_Y$ -module. The following is the decomposition theorem

Theorem 2.1. [5] *Let $\pi : X \rightarrow Y$ be a finite polynomial map. If M is a simple (holonomic) module over $\mathcal{D}_X$. Then $\pi_+(M)$ is a semisimple $\mathcal{D}_Y$ -module. we have*

$$\pi_+(M) = \oplus M_i^{\alpha_i},$$

where the M_i are inequivalent irreducible $\mathcal{D}_Y$ -submodules.

2.2. Higher Specht Polynomials for Reflection group. In this subsection we recall some facts about the representation of the generalized symmetric group $G(r, n)$ [1].

Let $\mathcal{S}_n$ be the group of permutations of the set $\{1, \dots, n\}$ and $\mathbb{Z}/r\mathbb{Z}$ be the cyclic group of order r . The generalized symmetric group $G(r, n)$ is the semi-direct product of $(\mathbb{Z}/r\mathbb{Z})^n$ with $\mathcal{S}_n$, written as $(\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathcal{S}_n$, where $(\mathbb{Z}/r\mathbb{Z})^n$ is the direct product of n copies of $\mathbb{Z}/r\mathbb{Z}$. Let ξ be a primitive r -th root of 1. $(\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathcal{S}_n = \{(\xi^{i_1}, \dots, \xi^{i_n}; \sigma) \mid i_k \in \mathbb{N}, \sigma \in \mathcal{S}_n\}$, whose product is given by

$$(\xi^{i_1}, \dots, \xi^{i_n}; \sigma)(\xi^{j_1}, \dots, \xi^{j_n}; \pi) = (\xi^{i_1+j_{\sigma^{-1}(1)}}, \dots, \xi^{i_n+j_{\sigma^{-1}(n)}}; \sigma\pi).$$

Let $\mathcal{O}_X = \mathbb{C}[x_1, \dots, x_n]$ be the ring of polynomials in n indeterminates on which the group $G(r, n)$ acts as follows:

$$(\xi^{i_1}, \dots, \xi^{i_n}; \sigma)f = f(\xi^{i_{\sigma(1)}}x_{\sigma(1)}, \dots, \xi^{i_{\sigma(n)}}x_{\sigma(n)}; \sigma),$$

where $f \in \mathcal{O}_X$ and $(\xi^{i_1}, \dots, \xi^{i_n}; \sigma) \in G(r, n)$. The fundamental invariants under this action are given by the elementary symmetric functions $e_j(x_1^r, \dots, x_n^r)$, $1 \leq j \leq n$. Let J_+ be the ideal of $\mathcal{O}_X$ generated by these fundamental invariants and $\Lambda = \mathcal{O}_X/J_+$ be the quotient ring. It is also known that the $G(r, n)$ -module Λ is isomorphic to the group ring $\mathbb{C}[G(r, n)]$, namely, the left regular representation. A description of all irreducible components of Λ is known in [1], in terms of what is called "higher Specht polynomials". The irreducible representations of $G(r, n)$ are parametrized by the r -tuple of Young diagrams $(\lambda^1, \dots, \lambda^r)$ with $|\lambda^1| + \dots + |\lambda^r| = n$.

Let $\mathcal{P}_{r,n}$ be the set of r -tuples of Young diagrams $\lambda = (\lambda^1, \dots, \lambda^r)$ with $|\lambda^1| + \dots + |\lambda^r| = n$. By filling each cell with a positive integer in such a way that every j ($1 \leq j \leq n$) occurs once, we obtain an r -tableau $T = (T^1, \dots, T^r)$ of shape $\lambda = (\lambda^1, \dots, \lambda^r)$. When the number k occurs in the component T^i , we write $k \in T^i$. The set of r -tableaux of shape λ is denoted by $\text{Tab}(\lambda)$. An r -tableau $T = (T^1, \dots, T^r)$ is said to be standard if the numbers are increasing on each column and each row of T^ν ($1 \leq \nu \leq r$). The set of r -standard tableaux of shape λ is denoted by $\text{STab}(\lambda)$.

Let $S = (S^1, \dots, S^r) \in \text{STab}(\lambda)$. We associate a word $w(S)$ in the following way. First, we read each column of the component S^1 from the bottom to the top starting from the left. We continue this procedure for tableau S^2 and so on. For a word $w(S)$, we define the index $i(w(S))$ inductively as follows. The number 1 in the word $w(S)$ has the index $i(1) = 0$. If the number k has index $i(k) = p$ and the number $k+1$ is sitting on the left (resp. right) of k , then $k+1$ has index $p+1$ (resp. p). Finally, assigning the indices to the corresponding cells, we get a shape $\lambda = (\lambda^1, \dots, \lambda^r)$, each cell filled with a nonnegative integer, which is denoted by $i(S) = (i(S)^1, \dots, i(S)^r)$.

Let $T = (T^1, \dots, T^r)$ be an r -tableau of shape λ . For each component T^ν ($1 \leq \nu \leq r$), the Young symmetrizer $\mathbf{e}_{T^\nu}$ of T^ν is defined by

$$\mathbf{e}_{T^\nu} = \frac{1}{\alpha_{T^\nu}} \sum_{\sigma \in R(T^\nu)} \sum_{\tau \in C(T^\nu)} \text{sgn}(\tau) \tau \sigma,$$

where α_{T^ν} is the product of the hook lengths for the shape λ^ν , $R(T^\nu)$ and $C(T^\nu)$ are the *row-stabilizer* and *column-stabilizer* of T^ν respectively.

We may regard a tableau T on a Young diagram λ as a map

$$T : \{\text{cells of } \lambda\} \rightarrow \mathbb{Z}_{\geq 0},$$

which assigns to a cell ξ of λ the number $T(\xi)$ written in the cell ξ in T .

For $S \in \text{STab}(\lambda)$ and $T \in \text{Tab}(\lambda)$, Ariki, Terasoma, and Yamada defined in [1] the higher Specht polynomial for $G(r, n)$ by

$$F_T^S = \prod_{\nu=1}^r \left(\mathbf{e}_{T^\nu} (x_{T^\nu}^{ri(S)^\nu}) \prod_{k \in T^\nu} x_k^\nu \right),$$

where

$$x_{T^\nu}^{ri(S)^\nu} = \prod_{\xi \in \lambda^\nu} x_{T^\nu(\xi)}^{ri(S)^\nu(\xi)}.$$

The following is the fundamental result in [1] on the higher Specht polynomials for $G(r, n)$.

- Theorem 2.2.** (1) *The space $V_S(\lambda) = \sum_{T \in \text{Tab}(\lambda)} \mathbb{C} F_T^S$ affords an irreducible representation of the reflection group $G(r, n)$.*
 (2) *The set $\{F_T^S \mid T \in \text{STab}(\lambda)\}$ gives a basis over $\mathbb{C}$ for $V_S(\lambda)$.*
 (3) *For $S_1 \in \text{STab}(\lambda)$ and $S_2 \in \text{STab}(\mu)$, the representation $V_{S_1}(\lambda)$ and $V_{S_2}(\mu)$ are isomorphic if and only if S_1 and S_2 has the same shape, i.e. $\lambda = \mu$.*
 (4) *We have the irreducible decomposition*

$$\mathbb{C}[G(r, n)] = \bigoplus_{\lambda \in \mathcal{P}_{r, n}} \bigoplus_{S \in \text{STab}(\lambda)} V_S(\lambda)$$

as a representation of $G(r, n)$.

Theorem 2.3. *The higher Specht polynomials in $\mathcal{F} = \{F_T^S; S, T \in \text{STab}(\lambda), \lambda \vdash n\}$ form a basis of the $\mathbb{C}[x_1, \dots, x_n]^{G(r, n)}$ -module $\mathbb{C}[x_1, \dots, x_n]$*

3. DECOMPOSITION THEOREM

We are interested in studying the decomposition structure of $\pi_+(M)$, where $M = \mathcal{O}_X$, $\pi : X = \text{spec}(\mathcal{O}_X) \rightarrow Y = \text{spec}(\mathcal{O}_X^{G(r, n)})$. Since $\mathcal{O}_X$ is a holonomic $\mathcal{D}_X$ -module [4, Chapter 10], $\pi_+(\mathcal{O}_X)$ is a semisimple $\mathcal{D}_Y$ -module by the decomposition theorem. We construct the simple components of $\pi_+(\mathcal{O}_X)$ and provide their multiplicities. Let us recall some useful facts from [10].

Let $\Delta := \text{Jac}(\pi)$ be the Jacobian of π , Δ^2 the discriminant of π , we denote the complement of the branch locus and the discriminant by U and V , respectively. Assume now that U, V are such that the respective canonical modules are generated by the volume forms dx , and dy , related by $dx = \Delta dy$, where Δ is the Jacobian of π .

Proposition 3.1. (i) *There is an isomorphism of $\mathcal{D}_V$ -modules*

$$T : \pi_+(O_U) \cong O_U, \quad r(dy^{-1} \otimes dx) \mapsto r\Delta^{-1}.$$

(ii) *$T(\pi_+(O_X))$ is isomorphic as a $\mathcal{D}_Y$ -module to $\pi_+(O_X)$.*

Proof. See [10, Lemma 2.3]. □

It is more convenient to study $T(\pi_+(O_X)) \cong \pi_+(O_X)$, as a submodule of O_U , than using the definition of $\pi_+(O_X)$. Therefore, to reach our goal, we will first study the decomposition of O_U into irreducible components as a $\mathcal{D}_V$ -module. The following proposition enables us to reduce the study of the decomposition factors of $\pi_+(O_X)$ to the behavior of the direct image over the complement to the branch locus, or even over the generic point. Let $j : U \hookrightarrow X$ and $i : V \hookrightarrow Y$ be the inclusions.

Proposition 3.2. *Let $\pi : X \rightarrow Y$ be a finite map. Then*

- (i) *$\pi_+(O_X)$ is semi-simple as a $\mathcal{D}_Y$ -module.*
- (ii) *If $\pi_+(O_X) = \oplus M_k$, $k \in I$ is a decomposition into simple (non-zero) $\mathcal{D}_Y$ -modules, then $\pi_+(O_U) = \oplus i^+(M_k)$, $k \in I$, is a decomposition of $\pi_+(O_U)$ into simple (non-zero) $\mathcal{D}_V$ -modules.*

Proof. See [10, Proposition 2.8]. □

3.1. Action description. As we want to study the polynomial representation of a ring of invariant differential operators localized at Δ^2 , it is convenient to precisely describe the action of that ring on the polynomial ring.

Let $\mathcal{D}_X = \mathbb{C}\langle x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle$ be the ring of differential operators associated with the polynomial ring $\mathcal{O}_X = \mathbb{C}[x_1, \dots, x_n]$, and $\mathcal{O}_Y = \mathbb{C}[x_1, \dots, x_n]^{G(r,n)} = \mathbb{C}[y_1, \dots, y_n]$ be the ring of invariant polynomials under the real reflection group $G(r, n)$. We denote by $\mathcal{D}_Y = \mathbb{C}\langle y_1, \dots, y_n, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \rangle$ the ring of differential operators associated with $\mathcal{O}_Y = \mathbb{C}[y_1, \dots, y_n]$. By [9], $\mathcal{D}_Y$ is the ring of invariant differential operators under the action of the reflection group $G(r, n)$.

Notations. We adopt the following notations

$$\mathcal{O}_U := \mathbb{C}[x_1, \dots, x_n, \Delta^{-1}], \quad \mathcal{O}_V := \mathbb{C}[y_1, \dots, y_n, \Delta^{-2}], \quad \mathcal{D}_V := \mathbb{C}\langle y_1, \dots, y_n, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}, \Delta^{-2} \rangle.$$

Lemma 3.3. *$\mathcal{O}_U$ is a $\mathcal{D}_V$ -module.*

Proof. Let us make clear the action of $\mathcal{D}_V$ on $\mathcal{O}_U$.

Let $A = \left(\frac{\partial y_j}{\partial x_i} \right)_{1 \leq i, j \leq n}$ so that $\det(A) = \Delta$. We get the following equation

$$\begin{pmatrix} \frac{\partial}{\partial x_n} \\ \vdots \\ \frac{\partial}{\partial x_1} \end{pmatrix} = A \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix} = A^{-1} \begin{pmatrix} \frac{\partial}{\partial x_n} \\ \vdots \\ \frac{\partial}{\partial x_1} \end{pmatrix}$$

and it is now clear that $\mathcal{O}_U$ is a $\mathcal{D}_V$ -module. $\square$

We know that $\mathcal{O}_U$ a $\mathcal{D}_V$ - semisimple module. What are the simple components of $\mathcal{O}_U$ as $\mathcal{D}_V$ -module and their multiplicities?

3.2. Simple components and their multiplicities. In this section, we state our main result. We use the representation theory of the generalized symmetric group $G(r, n)$ to yield results on modules over the ring of differential operators. It is well-known that

$$\mathcal{O}_X = \mathbb{C}[G(r, n)] \otimes \mathcal{O}_Y \text{ as } \mathcal{O}_Y\text{-modules.}$$

Let us consider the multiplicative closed set $S = \{\Delta^k\}_{k \in \mathbb{N}} \subset \mathcal{O}_X$. It follows that:

$$S^{-1}\mathcal{O}_X = \mathbb{C}[G(r, n)] \otimes S^{-1}\mathcal{O}_Y \text{ as } S^{-1}\mathcal{O}_Y\text{-modules.}$$

where $S^{-1}\mathcal{O}_X$ and $S^{-1}\mathcal{O}_Y$ are the localizations of $\mathcal{O}_X$ and $\mathcal{O}_Y$ at S respectively. But $S^{-1}\mathcal{O}_X = \mathcal{O}_U$ and $S^{-1}\mathcal{O}_Y = \mathcal{O}_V$, whereby we get

$$\mathcal{O}_U = \mathbb{C}[G(r, n)] \otimes \mathcal{O}_V \text{ as } \mathbb{C}[G(r, n)]\text{-modules.}$$

Lemma 3.4. *There exists an injective map*

$$\mathbb{C}[G(r, n)] \hookrightarrow \text{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U).$$

Proof. The $\mathbb{C}[G(r, n)]$ -module $\mathbb{C}[G(r, n)]$ acts faithfully on itself by multiplication, and this multiplication yields an injective map $\mathbb{C}[G(r, n)] \hookrightarrow \text{Hom}_{\mathbb{C}} \left(\mathbb{C}[G(r, n)], \mathbb{C}[G(r, n)] \right)$. Since $\mathcal{O}_V$ is invariant under this action of $\mathbb{C}[G(r, n)]$, we get the expected injective map. $\square$

Proposition 3.5. *There exists an injective map*

$$\mathbb{C}[G(r, n)] \hookrightarrow \text{Hom}_{\mathcal{D}_V}(\mathcal{O}_U, \mathcal{O}_U).$$

Proof. Since $\mathcal{D}_V = \mathbb{C}\langle y_1, \dots, y_n, \partial y_1, \dots, \partial y_n, \Delta^{-2} \rangle$, we only need to show that every element of $\mathbb{C}[G(r, n)]$ commutes with $y_1, \dots, y_n, \partial y_1, \dots, \partial y_n$.

- It is clear that every element of $\mathbb{C}[G(r, n)]$ commutes with y_i , $i = 1, \dots, n$.

- Let us show that every element of $\mathbb{C}[G(r, n)]$ commutes with ∂y_i , $i = 1, \dots, n$. Let $\mathbf{D}$ be a derivation on the field $K = \mathbb{C}(y_1, \dots, y_n)$ of fractions of $\mathcal{O}_V$, then $(K, \mathbf{D})$ is a differential field. Let $L = \mathbb{C}(x_1, \dots, x_n)$ be the field of fractions of $\mathcal{O}_U$. We have that $K = L^{G(r, n)}$ is the fixed field and L is a Galois extension of K , with Galois group $G(r, n)$. Then by [3, Théorème 6.2.6] there exists a unique derivation on L which extends $\mathbf{D}$, then $(L, \mathbf{D})$ is also a differential ring. In this way, $\sigma^{-1}\mathbf{D}\sigma = \mathbf{D}$ for every $\sigma \in G(r, n)$. Therefore, $\sigma\mathbf{D} = \mathbf{D}\sigma$ and σ commute with $\mathbf{D}$.

□

Corollary 3.6.

$$\mathbb{C}[G(r, n)] \cong \text{Hom}_{\mathcal{D}_V}(\mathcal{O}_U, \mathcal{O}_U)$$

Proof. see [10, Corollary 26]

□

Before we state our main result, let us recall some facts.

By Maschke's Theorem [8, Chap XVIII], we know that $\mathbb{C}[G(r, n)]$ is a semi-simple ring, and

$$\mathbb{C}[G(r, n)] = \bigoplus_{\lambda \in \mathcal{P}_{r, n}} R_\lambda,$$

where $\mathcal{P}_{r, n}$ be the set of r -tuples of Young diagrams and R_λ are simple rings. $R_\lambda = \bigoplus_{s \in \text{STab}(\lambda)} V_S(\lambda)$ (see Theorem 2.2). We have the following corresponding decomposition of the identity element of $\mathbb{C}[G(r, n)]$:

$$1 = \sum_{\lambda \in \mathcal{P}_{r, n}} r_\lambda,$$

where r_λ is the identity element of R_λ , with $r_\lambda^2 = 1$ and $r_\lambda r_\mu = 0$ if $\lambda \neq \mu$. $\{r_\lambda\}_{\lambda \in \mathcal{P}_{r, n}}$ is the set of primitive central idempotents of $\mathbb{C}[G(r, n)]$. In fact $R_\lambda = r_\lambda \mathbb{C}[G(r, n)]$, for $\lambda \in \mathcal{P}_{r, n}$. Let $n \in \mathbb{N}^*$, $\lambda \in \mathcal{P}_{r, n}$, we set $\text{Tab}(n) = \cup_{\lambda \in \mathcal{P}_{r, n}} \text{Tab}(\lambda)$ and $\text{STab}(n) = \cup_{\lambda \in \mathcal{P}_{r, n}} \text{STab}(\lambda)$.

Theorem 3.7. *For every primitive idempotent $e \in \mathbb{C}[G(r, n)]$.*

- (1) $e\mathcal{O}_U$ is a nontrivial $\mathcal{D}_V$ -submodule of $\mathcal{O}_U$,
- (2) The $\mathcal{D}_V$ -module $e\mathcal{O}_U$ is simple,
- (3) There exist $\lambda \in \mathcal{P}_{r, n}$ and a higher Specht polynomial F_T^S (with $S, T \in \text{STab}(\lambda)$) such that $e\mathcal{O}_U = \mathcal{D}_V F_T^S$.

Proof.

- (1) Let $e \in \mathbb{C}[G(r, n)]$ be a primitive idempotent, we know that $\mathbb{C}[G(r, n)]e$ is a $G(r, n)$ -irreducible representation. Theorem 2.3 states that there is $\lambda \in \mathcal{P}_{r, n}$ and $S \in \text{STab}(\lambda)$ such that $\mathbb{C}[G(r, n)]e \cong V_S(\lambda)$, with $V_S(\lambda) \subset \mathcal{O}_U$. By [2, Chap III, §4, Theorem 3.9], we have $e\mathbb{C}[G(r, n)]e \cong \mathbb{C}e \neq \{0\}$. $\{0\} \neq eV_S(\lambda) \subset e\mathcal{O}_U$. Since e commutes with every element of $\mathcal{D}_V$ and $\mathcal{O}_U$ is a $\mathcal{D}_V$ -module, it follows that $e\mathcal{O}_U$ is a nontrivial $\mathcal{D}_V$ -module. In fact $V_S(\lambda)$ is a cyclic $\mathbb{C}[G(r, n)]$ -module, i.e., there exist $T, S \in \text{STab}(\lambda)$

and a higher Specht polynomial F_T^S such that $V_S(\lambda) = \mathbb{C}[G(r, n)]F_T^S$, so that $\mathbb{C}[G(r, n)]e \cong \mathbb{C}[G(r, n)]F_T^S$. Then it follows that there is a higher Specht polynomial F_T^S such that eF_T^S is a scalar multiple of F_T^S .

- (2) Assume that $1 = \sum_{i=1}^s e_i$ where the $\{e_i\}_{1 \leq i \leq s}$ is the set of primitive idempotents of $\mathbb{C}[G(r, n)]$, then $\mathcal{O}_U = \sum_{i=1}^s e_i \mathcal{O}_U$. Let $m \in e_i \mathcal{O}_U \cap e_j \mathcal{O}_U$ with $i \neq j$ so that $m = e_i m_i$ and $m = e_j m_j$, but $e_i e_j = 0$ then $e_i m = e_i e_j m = 0$ hence $m = 0$. Therefore $\mathcal{O}_U = \bigoplus_{i=1}^s e_i \mathcal{O}_U$ and we get:

$$\mathrm{Hom}_{\mathcal{D}_V}(\mathcal{O}_U, \mathcal{O}_U) \cong \bigoplus_{i,j=1}^s \mathrm{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_j \mathcal{O}_U),$$

by Corollary 3.6 we know that $\mathbb{C}[G(r, n)] \cong \bigoplus_{i,j=1}^s \mathrm{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_j \mathcal{O}_U)$. For

every $\lambda \in \mathcal{P}_{r,n}$, we pick a unique irreducible representation $V_S(\lambda)$, for a certain $S \in \mathrm{STab}(\lambda)$, which we denote by $V(\lambda) := V_S(\lambda)$. We also have, by [6, Proposition 3.29], that $\mathbb{C}[G(r, n)] \cong \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \mathrm{End}_{\mathbb{C}}(V(\lambda))$. But by the

Wedderburn's decomposition Theorem [2, Chap II, §4, Theorem 4.2] we also know that

$$\mathbb{C}[G(r, n)] = \bigoplus_{\lambda \in \mathcal{P}_{r,n}} r_\lambda \mathbb{C}[G(r, n)] \text{ and } r_\lambda \mathbb{C}[G(r, n)] \cong \mathrm{Mat}_{f^\lambda}(\mathbb{C}) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{f^\lambda})$$

where $f^\lambda = \dim_{\mathbb{C}}(V(\lambda))$. Each r standard tableau T corresponds to a primitive idempotent e_T , so that $r_\lambda = \sum_{T \in \mathrm{STab}(\lambda)} e_T$. In what follows we

denote e_i the primitive idempotent associated with standard tableau T_i , (i.e., $e_i = e_{T_i}$.) Let us show that

$$\mathbb{C}[G(r, n)] \cong \bigoplus_{\lambda \vdash n} \left(\bigoplus_{T_i, T_j \in \mathrm{STab}(\lambda)} \mathrm{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_j \mathcal{O}_U) \right) \text{ where } e_i = e_{T_i}.$$

Let x be an element of $\mathbb{C}[G(r, n)]$ and r_λ the primitive central idempotent associated with $\lambda \in \mathcal{P}_{r,n}$. Then x induces an $\mathcal{D}_V$ -homomorphism $\mathcal{O}_U \rightarrow \mathcal{O}_U, m \mapsto x \cdot m$; the multiplication by x . Since r_λ is in the centre of $\mathbb{C}[G(r, n)]$, $x \cdot (r_\lambda \mathcal{O}_U) = (x \cdot r_\lambda) \mathcal{O}_U \subset r_\lambda \mathcal{O}_U$, which means $x \in \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \mathrm{Hom}_{\mathcal{D}_V}(r_\lambda \mathcal{O}_U, r_\lambda \mathcal{O}_U)$. It follows that

$$\mathrm{Hom}_{\mathcal{D}_V}(\mathcal{O}_U, \mathcal{O}_U) \cong \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \mathrm{Hom}_{\mathcal{D}_V}(r_\lambda \mathcal{O}_U, r_\lambda \mathcal{O}_U).$$

Then $\mathrm{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_j \mathcal{O}_U) = \{0\}$ if $T_i \in \mathrm{STab}(\lambda_i), T_j \in \mathrm{STab}(\lambda_j)$ and $\lambda_i \neq \lambda_j$. We get that

$$\mathrm{Hom}_{\mathcal{D}_V}(\mathcal{O}_U, \mathcal{O}_U) \cong \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \left(\bigoplus_{T_i, T_j \in \mathrm{STab}(\lambda)} \mathrm{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_j \mathcal{O}_U) \right).$$

The number of direct factors in the sum $\bigoplus_{T_i, T_j \in \text{STab}(\lambda)} \text{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_j \mathcal{O}_U)$

is $(f^\lambda)^2$.

Let us show that $\text{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_j \mathcal{O}_U) \cong \mathbb{C}$ if $T_i, T_j \in \text{Tab}(\lambda)$. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}[G(r, n)] & \xrightarrow{\phi} & \text{Hom}_{\mathcal{D}_V}(\mathcal{O}_U, \mathcal{O}_U) \\ \alpha_\lambda \downarrow & & \downarrow \beta_\lambda \\ r_\lambda \mathbb{C}[G(r, n)] & \xrightarrow{\psi} & \text{Hom}_{\mathcal{D}_V}(r_\lambda \mathcal{O}_U, r_\lambda \mathcal{O}_U) \end{array}$$

where $\beta_\lambda : \bigoplus_{\mu \in \mathcal{P}_{r, n}} \text{Hom}_{\mathcal{D}_V}(r_\mu \mathcal{O}_U, r_\mu \mathcal{O}_U) \rightarrow \text{Hom}_{\mathcal{D}_V}(r_\lambda \mathcal{O}_U, r_\lambda \mathcal{O}_U)$ and

$\alpha_\lambda : \bigoplus_{\mu \in \mathcal{P}_{r, n}} r_\mu \mathbb{C}[G(r, n)] \rightarrow r_\lambda \mathbb{C}[G(r, n)]$ are canonical projections and ϕ is

the isomorphism in Corollary 3.6. It follows that ψ is an isomorphism hence $r_\lambda \mathbb{C}[G(r, n)] \cong \text{Hom}_{\mathcal{D}_V}(r_\lambda \mathcal{O}_U, r_\lambda \mathcal{O}_U)$.

Now we identify $r_\lambda \mathbb{C}[G(r, n)]$ with either the set $\text{Mat}_{f^\lambda}(\mathbb{C})$ of square matrices of order f^λ with coefficients in $\mathbb{C}$ either with $\text{End}_{\mathbb{C}}(\mathbb{C}^{f^\lambda})$. Let E_{ij} be the square matrix of order f^λ with 1 at the position (i, j) and 0 elsewhere and $E_i = E_{i, i}$, then we identify the primitive idempotent $e_i \in r_\lambda \mathbb{C}[G(r, n)]$ with E_i in $\text{Mat}_{f^\lambda}(\mathbb{C})$. Let $B = (a_{ij}) \in \text{Mat}_{f^\lambda}(\mathbb{C})$ we get $B = \sum_{i, j} a_{i, j} E_{i, j} = \sum_{i, j} E_i B E_j$, in fact $E_i B E_j$ is the matrix with $a_{i, j}$ in the position (i, j) and 0 elsewhere, if $R = \text{Mat}_{f^\lambda}(\mathbb{C})$, we get that $E_i R E_j \cong \mathbb{C}$.

This isomorphism ψ implies that

$$\bigoplus_{T_i, T_j \in \text{STab}(\lambda)} E_i R E_j \cong \bigoplus_{T_i, T_j \in \text{STab}(\lambda)} \text{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_j \mathcal{O}_U);$$

the restriction of ψ to $E_i R E_j$ yields a map $E_i R E_j \rightarrow \text{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_j \mathcal{O}_U)$ and this map is surjective, moreover we have $E_i R E_j \cong \text{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_j \mathcal{O}_U)$.

Therefore $\text{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_i \mathcal{O}_U) \cong \mathbb{C}$. Let us assume that $e_i \mathcal{O}_U$ is not a simple $\mathcal{D}_V$ -module, then $e_i \mathcal{O}_U$ may be written as $e_i \mathcal{O}_U = \bigoplus_{j \in J} N_j$ where the N_j are simple $\mathcal{D}_V$ -modules and $|J| > 1$. It follows that $\dim_{\mathbb{C}}(\text{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_i \mathcal{O}_U)) \geq |J|$ but $\text{Hom}_{\mathcal{D}_V}(e_i \mathcal{O}_U, e_i \mathcal{O}_U) \cong \mathbb{C}$ so we obtain that $J = 1$, which necessarily implies that $e_i \mathcal{O}_U$ is a simple $\mathcal{D}_V$ -module.

- (3) By the the proof (i), there exists a higher Specht polynomial F_T^S , with $S, T \in \text{STab}(\lambda)$, where $\lambda \vdash n$ such that $e_i \mathcal{O}_U = \mathcal{D}_V F_T^S$.

□

Corollary 3.8. *With the above notations, $e_i \mathcal{O}_U \cong_{\mathcal{D}_V} e_j \mathcal{O}_U$ if and only if T_i and T_j have the same shape i.e. if there is a partition $\lambda \in \mathcal{P}_{r, n}$ such that $T_i, T_j \in \text{STab}(\lambda)$.*

Proof. The $\mathcal{D}_V$ -modules $e_i\mathcal{O}_U$ are simple and $\text{Hom}_{\mathcal{D}_V}(e_i\mathcal{O}_U, e_j\mathcal{O}_U) \cong \mathbb{C}$ whenever there exists a partition $\lambda \in \mathcal{P}_{r,n}$ such that $T_i, T_j \in \text{STab}(\lambda)$. Since $\text{Hom}_{\mathcal{D}_V}(e_i\mathcal{O}_U, e_j\mathcal{O}_U) \neq \{0\}$, we conclude by using the Schur lemma. $\square$

Proposition 3.9. *Let $\lambda \in \mathcal{P}_{r,n}$, $T \in \text{STab}(\lambda)$, and let e be the primitive idempotent associated with T , denote by $F_T := F_T^S$ the corresponding higher Specht polynomial (for some $S \in \text{STab}(\lambda)$), in Theorem 3.7 (iii), such that $e\mathcal{O}_U = \mathcal{D}_V F_T^S$ then we have:*

(1)

$$\mathcal{O}_U = \bigoplus_{T \in \text{STab}(n)} \mathcal{D}_V F_T = \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \left(\bigoplus_{T \in \text{STab}(\lambda)} \mathcal{D}_V F_T \right); \quad (3.1)$$

(2) for each $\lambda \in \mathcal{P}_{r,n}$ fix an r -tableau $T^* \in \text{STab}(\lambda)$, then

$$\mathcal{O}_U = \bigoplus_{\lambda \in \mathcal{P}_{r,n}} f^\lambda \mathcal{D}_V F_{T^*} \quad (3.2)$$

where $f^\lambda = \dim_{\mathbb{C}}(V(\lambda))$

Proof. We have by the proof of Theorem 3.7 that

$$\mathcal{O}_U = \bigoplus_{T_i \in \text{STab}(n)} e_i \mathcal{O}_U$$

and the $e_i\mathcal{O}_U$ are simple $\mathcal{D}_V$ -modules. Since to each primitive idempotent e_i corresponds an r -diagram $\lambda_i \in \mathcal{P}_{r,n}$ and a tableau $T_i \in \text{STab}(\lambda_i)$ such that $e_i\mathcal{O}_U = \mathcal{D}_V F_{T_i}$ then $\mathcal{O}_U = \bigoplus_{T \in \text{STab}(n)} \mathcal{D}_V F_T$. By Corollary 3.6, $\mathcal{D}_V F_{T_i} \cong \mathcal{D}_V F_{T_j}$ if $T_i, T_j \in \text{STab}(\lambda)$, and so we have f^λ isomorphic copies of $\mathcal{D}_V F_{T^*}$ in the direct sum (3.1). $\square$

Using Proposition 3.1 and Proposition 3.2 we get the next theorem.

Theorem 3.10. (i) $N_T = \mathcal{D}_Y F_T$ is an irreducible D_Y -submodule of $\pi_+(\mathcal{O}_X)$.
(ii) There is a direct sum decomposition

$$\pi_+(\mathcal{O}_X) = \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \bigoplus_{T \in \text{STab}(\lambda)} N_T \quad (3.3)$$

(iii)

$$\pi_+(\mathcal{O}_X) \cong \bigoplus_{\lambda \in \mathcal{P}_{r,n}} f^\lambda N_{T^*} \quad (3.4)$$

We get in Theorem 3.10 a decomposition of the $\pi_+(\mathcal{O}_X)$ into irreducible $\mathcal{D}_Y$ modules generated by the higher Specht polynomials.

4. USING CORRESPONDENCE BETWEEN G -REPRESENTATIONS AND D -MODULES

In this section, we use an equivalence of categories between group representations and modules over a differential ring to yield a better version of the decomposition of the polynomial ring as $\mathcal{D}_V$ -modules

Lemma 4.1. *Let $\lambda \in \mathcal{P}_{r,n}$, $S \in \text{STab}(\lambda)$, $V_S(\lambda)$ be the corresponding irreducible representation and e_T the primitive idempotent associated with an r -standard tableau $T \in \text{STab}(\lambda)$. Then $e_T(V_S(\lambda)) = \{e_T(m) | m \in V_S(\lambda)\}$ is a one dimensional $\mathbb{C}$ -vector space.*

Proof. In fact we have

$$\begin{aligned} e_T V_S(\lambda) &\cong e_T \mathbb{C}[G(r, n)] e_T \\ &\cong e_T \mathbb{C}[G(r, n)] e_T \\ &\cong \mathbb{C} e_T \text{ by [2, Chap III, §4, Theorem 3.9] } \end{aligned}$$

□

Recall that if M is a semi-simple module over a ring R , and N is simple R -module, then the isotopic component of M associated to N is the sum $\sum N' \subset M$ of all $N' \subset M$ such that $N' \cong N$.

Lemma 4.2. *Let $\lambda \in \mathcal{P}_{r,n}$, $S \in \text{STab}(\lambda)$ and $V_S(\lambda)$ be the corresponding irreducible module associated. Let $M := \mathcal{O}_U$ and $M(\lambda)$ be the isotopic component of M (as $\mathcal{O}_V$ -module) associated with $V_S(\lambda)$. Then $M(\lambda)$ is $\mathcal{D}_V$ -module.*

Proof. We only have to proof that $\mathcal{D}_V \cdot M(\lambda) \subset M(\lambda)$. Let $D \in \mathcal{D}_V$ and N be a $\mathbb{C}[G(r, n)]$ -module isomorphic to $V_S(\lambda)$, since D commute with the elements of the group algebra $\mathbb{C}[G(r, n)]$, D is an $\mathbb{C}[G(r, n)]$ -homomorphism from N into $D(N)$. Then by virtue of the Schur lemma $D(N) = 0$ or $D(N) \cong N$ as a $\mathbb{C}[G(r, n)]$ -module, and $D(N) \subset M(\lambda)$. Hence $\mathcal{D}_V \cdot M(\lambda) \subset M(\lambda)$. □

Lemma 4.3. *Let $\lambda \in \mathcal{P}_{r,n}$, $S \in \text{STab}(\lambda)_d$, $M(\lambda)$ the isotopic component associated with $V_S(\lambda)$ and e_T the primitive idempotent associative with an r -standard tableau T . Then $e_T(M(\lambda))$ is $\mathcal{D}_V$ -module.*

Proof. Let $D \in \mathcal{D}_V$, we have $D(e_T(M^\lambda)) = e_T(D(M(\lambda))) \subset e_T(M(\lambda))$, so $e_T(M(\lambda))$ is $\mathcal{D}_V$ -module. □

Before we proceed, let us recall the correspondence between G -representations and D -modules [10, Paragraph 2.4]. Let L and K be two extensions fields a field k , denote by $T_{K/k}$ the k -linear derivations of K . We say that a $T_{K/k}$ -module M is L -trivial if $L \otimes_K M \cong L^n$ as $T_{L/k}$ -modules. Denote by $\text{Mod}^L(T_{K/k})$ the full subcategory of finitely generated $T_{K/k}$ -modules that are L -trivial. It is immediate that it is closed under taking submodules and quotient modules. Using a lifting ϕ , L may be thought of as a $T_{K/k}$ -module. If G is a finite group let $\text{Mod}(k[G])$ be the category of finite-dimensional representations of $k[G]$. Let now $k \rightarrow K \rightarrow L$ be a tower of fields such that $K = L^G$. Note that the action of $T_{K/k}$ commutes with the action of G . If V is a $k[G]$ -module, $L \otimes_k V$ is a $T_{K/k}$ -module by $D(l \otimes v) = D(l) \otimes v$, $D \in T_{K/k}$, and $(L \otimes_k V)^G$ is a $T_{K/k}$ -submodule.

Let L and K be two fields, say that a $T_{K/k}$ -module M is L -trivial if $L \otimes_K M \cong L^n$ as $T_{L/k}$ -modules. Denote by $\text{Mod}^L(T_{K/k})$ the full subcategory of finitely generated $T_{K/k}$ -modules that are L -trivial. It is immediate that it is closed under taking submodules and quotient modules. Using the lifting ϕ , L may be thought

of as a $T_{K/k}$ -module. If G is a finite group let $\text{Mod}(k[G])$ be the category of finite-dimensional representations of $k[G]$. Let now $k \rightarrow K \rightarrow L$ be a tower of fields such that $K = L^G$. Note that the action of $T_{K/k}$ commutes with the action of G . If V is a $k[G]$ -module, $L \otimes_k V$ is a $T_{K/k}$ -module by $D(l \otimes v) = D(l) \otimes v$, $D \in T_{K/k}$, and $(L \otimes_k V)^G$ is a $T_{K/k}$ -submodule.

Proposition 4.4. *The functor*

$$\nabla : \text{Mod}(k[G]) \rightarrow \text{Mod}(T_{K/k}), \quad V \mapsto (L \otimes_k V)^G$$

is fully faithful, and defines an equivalence of categories

$$\text{Mod}(k[G]) \rightarrow \text{Mod}^L(T_{K/k}).$$

The quasi-inverse of ∇ is the functor

$$\text{Loc} : \text{Mod}^L(T_{K/k}) \rightarrow \text{Mod}(k[G]), \quad \text{Loc}(M) = (L \otimes_K M)^{\phi(T_{K/k})}.$$

Proof. see [10, Proposition 2.4] □

In the following proposition we take $G = G(r, n)$, $K := \mathbb{C}(y_1, \dots, y_n)$ the field of fractions of $\mathcal{O}_V$ and $L = \mathbb{C}(x_1, \dots, x_n)$ the field of fractions of $\mathcal{O}_U$ so that $K = L^{G(r, n)}$. It is clear that L is a Galois extension of K with Galois $G(r, n)$.

Proposition 4.5. *Let $\lambda \in \mathcal{P}_{r, n}$ $S \in \text{STab}(\lambda)$, $V_S(\lambda)$ be the corresponding irreducible representation and e_T the primitive idempotent associative with an r -standard tableau T .*

Set $M_T := e_T \mathcal{O}_U$. Then we have:

- (1) $M_T = \nabla(V_S(\lambda))$
- (2) $M_T = e_T(M(\lambda))$ is simple $\mathcal{D}_V$ -module;
- (3) $M(\lambda) = \bigoplus_{T \in \text{STab}(\lambda)} e_T(M(\lambda)).$

Proof. (1) Let us consider the right $\mathbb{C}[G(r, n)]$ -module $V = e_T \mathbb{C}[G(r, n)]$ where $T \in \text{STab}(\lambda)$. This is the image of $\mathbb{C}[G(r, n)]$ by right multiplication map $e_T : \mathbb{C}[G(r, n)] \rightarrow \mathbb{C}[G(r, n)]$. By [10, Example 2.5], we may turn this map to a left multiplication $\mathbb{C}[G(r, n)]^r \rightarrow \mathbb{C}[G(r, n)]^r$ and get an image which is isomorphic to $V_S(\lambda)$ for some $S \in \text{STab}(\lambda)$. So we get an induced map

$$\nabla(\mathbb{C}[G(r, n)]^r) \rightarrow \nabla(V_S(\lambda)) \subset \nabla(\mathbb{C}[G(r, n)]^r),$$

which is a multiplication by e_T according to [10, Example 2.5]. Then $\nabla(V_S(\lambda))$ is equal to $e_T \mathcal{O}_U = M_T$.

- (2) Since $V_S(\lambda)$ is a simple $\mathbb{C}[G(r, n)]$ -module, $\nabla(V_S(\lambda))$ is also a simple $\mathcal{D}_V$ -module.
- (3) follows from the fact that $1 = \sum_{T \in \text{STab}(n)} e_T$ and $e_T(M(\lambda)) = 0$ if $T \notin \text{STab}(\lambda)$.

□

Theorem 4.6. *Let T be an r -standard tableau of shape λ where $\lambda \in \mathcal{P}_{r, n}$ and M_T be as in the above proposition. Then*

- (1) $M_T = \bigoplus_{S \in \text{STab}(\lambda)} \mathcal{O}_V F_T^S$ as $\mathcal{D}_V$ -module,
- (2) $\mathcal{O}_U = \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \left(\bigoplus_{S, T \in \text{STab}(\lambda)} \mathcal{O}_V F_T^S \right)$ as a $\mathcal{D}_V$ -module.

Proof. (1) For $S \in \text{STab}(\lambda)$ We know by Theorem 2.2 that the polynomial F_T^S generate a $\mathbb{C}[G(r, n)]$ -module inside $\mathcal{O}_U$ which is isomorphic to $V(\lambda)$. Then $F_T^S \in M^\lambda$ and $M^\lambda = \bigoplus_{S, T \in \text{STab}(\lambda)} \mathbb{C}[G(r, n)] F_T^S \mathcal{O}_V$. Moreover $e_T(F_T^S) = c F_T^S, c \in \mathbb{C}$ and by Lemma 4.1 $e_T(\mathbb{C}[G(r, n)] F_T^S) = \mathbb{C} F_T^S$. Hence $M_T = e_T(M^\lambda) = \bigoplus_{S \in \text{STab}(\lambda)} \mathcal{O}_V F_T^S$.

(2) follows from Proposition 3.9. □

Theorem 4.7. *Let $\lambda \in \mathcal{P}_{r,n}$ and $\mathbf{D} \in \mathcal{D}_V$ such that $\mathbf{D}(F_T^S) \neq 0$ for a higher Specht polynomial F_T^S with $S, T \in \text{STab}(\lambda)$. Then the image of the $\mathbb{C}[G(r, n)]$ -module $V_S(\lambda)$ by $\mathbf{D}$ is an $\mathbb{C}[G(r, n)]$ -module isomorphic to $V_S(\lambda)$. In others words, the actions of the differential operators of $\mathcal{D}_V$ on the higher Specht polynomials generate isomorphic copies of the corresponding irreducible $\mathbb{C}[G(r, n)]$ -module.*

Proof. Let $\lambda \in \mathcal{P}_{r,n}, \mathbf{D} \in \mathcal{D}_V$ such that $\mathbf{D}(F_T^S) \neq 0$ for $S, T \in \text{STab}(\lambda)$ and set $W_{\mathbf{D}}^S(\lambda) = \mathbf{D}(V_S(\lambda))$ the image of the module $V_S(\lambda)$ under the map $\mathbf{D}$. By Theorem 2.2, the $\mathbb{C}$ -vector space $V_S(\lambda)$ is equipped with a basis $\mathcal{F} = \{F_T^S; T \in \text{STab}(\lambda)\}$, then $W_{\mathbf{D}}^S(\lambda)$ is the vector space spanned by the set $\{\mathbf{D}(F_T^S); T \in \text{STab}(\lambda)\}$. The elements of $\{F_T^S; T \in \text{STab}(\lambda)\}$ are linearly independent over $\mathcal{D}_V$, otherwise the direct sum in Theorem 4.7 cannot hold. It follows that the elements $\{\mathbf{D}(F_T^S); T \in \text{STab}(\lambda)\}$ are linear independent over $\mathbb{C}$. Hence $\{\mathbf{D}(F_T^S); T \in \text{STab}(\lambda)\}$ is a basis of $W_{\mathbf{D}}^S(\lambda)$ over $\mathbb{C}$. Since $\mathbf{D}$ commute with elements of $\mathbb{C}[G(r, n)]$, $W_{\mathbf{D}}^S(\lambda)$ is an $\mathbb{C}[G(r, n)]$ -module isomorphic to $V_S(\lambda)$. □

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¹ DÉPARTEMENT D'ÉCONOMIE ET DE MATHÉMATIQUES APPLIQUÉES, IUFIC, UNIVERSITÉ THOMAS SANKARA, BURKINA FASO.

Email address: nonkane.ibrahim@yahoo.fr, ibrahim.nonkane@uts.bf

² AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES (AIMS), SUMMERHILL ESTATES, EAST LEGON HILLS, SANTOE, ACCRA, GHANA P.O. BOX LG DTD 20046, LEGON, ACCRA, GHANA

Email address: latevi@aims.edu.gh