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# THE COMMON FIXED POINTS IN A BIPOLAR METRIC SPACE

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Abstract. The purpose of this paper is to obtain a new fixed point theorem in the bipolar metric spaces. The contraction used in our theorem is the extension of Boyd-Wong-type contraction in bipolar metric spaces.

## 1. Introduction

The fixed point theorem of Banach (Banach contraction principle), see [3], is a source of inspiration for the past and present researchers of mathematics and different branches of science and technology. Even in the $21^{st}$ century computer scientists, physicists, applied mathematicians, etc. are trying to apply Banach contraction principle to serve the purpose of human beings daily life. In real analysis and functional analysis, metric space theory is a pivoting tool for the applications of many concepts. The metric space is the most general space on which one can think about applications in real life situations of this century. The concept of a topological via a metric space or the concept of a normed linear space to a topological space via a metric space is always an interesting and challenging of proves mathematics among the mathematicians. Metric fixed point theory has a wide range of applications in dynamic programming problems, variational inequalities, solutions of nonlinear differential equations, fractal dynamics, dynamical systems of mathematics as well as the launching of satellites in their appropriate orbits in the space, in medicine the most appropriate diagnosis of patients, in future medical emergencies by using simulation techniques and in mock exercises on the spread of disease, etc.

The study of new space discoveries in mathematics and their basic properties are always favorite topics of interest among the mathematical research community. In this context, the concept of 2-metric spaces, introduced initially by Gähler [8] in his series of papers and given a new thought of new dimensions for ordinary metric spaces. Since the metric for a pair of points is non-negative

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real, (i.e. $[0,+\infty)$) it has wide range of this study. The concept of probabilistic metric spaces in which the probabilistic distance between two points is considered, it given a new height and interest for the study to know more about stars in the universe. In a similar way, the study of fuzzy metric spaces was initially done by Grabiec [9] and Michalek [12] in which the degree of agreement and disagreement were considered. So far the study was done around the real numbers for metric spaces, 2-metric spaces, normed linear spaces, fuzzy metric spaces, modular metric spaces, etc. Let $X$ be a non-empty set and let $d : X\times X \to \mathbb{R}, \|.\| : X \to \mathbb{R}, d : X\times X\times X \to \mathbb{R}$ and $M : X\times X\times[0,1] \to [0,1]$. It was quiet natural to ask "What happens if we replace $\mathbb{R}$ by some other sets which are not completely ordered sets like $\mathbb{R}$?" This was answered by a few of them by introducing the cone metric space, the partially ordered metric space, the modular metric space, the complex valued metric space, fuzzy metric space etc (see [1, 2, 7, 9, 11, 12, 13, 19, 24, 26]).

The purpose of this paper is to obtain a fixed point theorem in the bipolar metric spaces (see [2, 15, 16, 20, 21, 22]). The contraction condition used in our theorem is the extension of Boyd-Wong (see [4]) in bipolar metric spaces.

## 2. The bipolar metric spaces

In this section some basic definitions and examples are given.

**Definition 2.1.** [15] Let $X$ and $Y$ be two non nonempty sets and $d : X \times Y \to [0,+\infty)$ be a function. Then the triplet $(X, Y, d)$ is called bipolar metric space and $d$ is called bipolar metric on $(X, Y)$ if the following conditions hold:
(B1) $d(x, y) = 0$ if and only if $x = y$ where $(x, y) \in X \times Y$,
(B2) If $x, y \in X \cap Y$ then $d(x, y) = d(y, x)$,
(B3) $d(x_1, y_2) \le d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

*Remark* 2.2. In above definition if condition (B1) is replaced by the condition
(B0) $x = y$ implies $d(x, y) = 0$, then $(X, Y, d)$ is called bipolar pseudo-metric space.

**Example 2.3.** Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be two normed linear spaces such that $X \cap Y = \emptyset$. Define $d : X \times Y \to [0,+\infty)$ by

$$d(x, y) = \left|\|x\|_1 - \|y\|_2\right|,$$

then $(X, Y, d)$ is a bipolar pseudo-metric space.

The following definitions are given in [15].

**Definition 2.4.** Let $(X, Y, d)$ be a bipolar metric space. Elements of $X, Y$ and $X \cap Y$ are called left, right and central points respectively. A sequence in $X$ and a sequence in $Y$ are called left and right sequences respectively. By a sequence we mean either a left or right sequence. A sequence $\langle s_n\rangle$ is said to be convergent to a point s if and only if $\langle s_n\rangle$ is a left sequence, $s$ is a right point and $\lim\limits_{n\to+\infty} d(s_n, s) = 0$; or $\langle s_n\rangle$ is a right sequence, $s$ is a left point and $\lim\limits_{n\to+\infty} d(s, s_n) = 0$. A sequence $\langle(x_n, y_n)\rangle$ in $X \times Y$ is called a bisequence on $(X, Y)$. This sequence is simply denoted by $(x_n, y_n)$. If both the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ converge, then the bisequence $(x_n, y_n)$ is said to be convergent. If both the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ converge to a same point $u \in X \cap Y$ then $(x_n, y_n)$ is called biconvergent. If $\lim_{n,m \to +\infty} d(x_n, y_m) = 0$ then the bisequence $(x_n, y_n)$ is called a Cauchy bisequence. In a bipolar metric space, every convergent Cauchy bisequence is biconvergent. A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

**Definition 2.5.** Let $X_1, Y_1, X_2$ and $Y_2$ be four sets. A function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is said to be a covariant map if $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$ and is denoted as $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$. In particular, if $(X_1, Y_1, d_1)$ and $(X_2, Y_2, d_2)$ are two bipolar metric space then we use the notaion $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ for covariant map $f$.

**Definition 2.6.** Let $(X_1, Y_1, d_1)$ and $(X_2, Y_2, d_2)$ be two bipolar metric spaces. A map $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$ is said to be continuous at a point $x_0 \in X_1$, if for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that $y \in Y_1$ and $d_1(x_0, y) < \delta$ implies that $d_2(f(x_0), f(y)) < \varepsilon$. It is continuous at a point $y_0 \in Y_1$ if for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that $x \in X_1$ and $d_1(x, y_0) < \delta$ implies that $d_2(f(x), f(y_0)) < \varepsilon$. If $f$ is continuous at each point $x \in X_1$ and $y \in Y_1$, then it is called continuous.

**Definition 2.7.** Let $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$ be a covariant map from bipolar metric space $(X_1, Y_1, d_1)$ to bipolar metric space $(X_2, Y_2, d_2)$ such that

$$d_2(f(x), f(y)) \leq \alpha d_1(x, y),$$

for all $x \in X_1, y \in Y_1$ where $\alpha > 0$, then $f$ is called Lipschitz continuous. If $\alpha = 1$, then this map is said to be non-expansive and if $\alpha \in [0, 1)$, it is called a contraction. It is obvious that every contraction is non-expansive and every non-expansive map is continuous.

## 3. Main results

Our first new results is the following.

**Theorem 3.1.** *Let $(X, Y, d)$ be a bipolar metric space and let $f, g : (X, Y, d) \rightrightarrows (X, Y, d)$ be two covariant maps and $\phi : [0, +\infty) \to [0, +\infty)$ be an upper semi-continuous function from the right on $[0, +\infty)$ and satisfies $\phi(t) < t$ for all $t > 0$ such that*

$$d(f(x), f(y)) \leq \phi(d(g(x), g(y))),$$

*for all $x \in X$ and $y \in Y$. If $f(X \cup Y) \subseteq g(X \cup Y), g(X \cup Y)$ is complete in $X \cup Y$ and $g$ is injective, then the functions $f, g : X \cup Y \to X \cup Y$ have a unique coincidence point. Further, if $fg = gf$ then $f$ and $g$ have unique common fixed point.*

*Proof.* Let $x_0 \in X$, $y_0 \in Y$ and choose $x_1 \in X$ and $y_1 \in Y$ such that $f(x_0) = g(x_1)$ and $f(y_0) = g(y_1)$. This can be done since $f(X \cup Y) \subseteq g(X \cup Y)$. In general we can choose $(x_n, y_n) \in X \times Y$ such that $f(x_{n-1}) = g(x_n)$ and $f(y_{n-1}) =$

$g(y_n)$ for all $n \in \mathbb{N}$. Let $a_n = d(g(x_n), g(y_n)), b_n = d(g(x_{n-1}), g(y_n))$ and $c_n = d(g(x_n), g(y_{n-1}))$. Then

$$\begin{aligned} a_n &= d(g(x_n), g(y_n)) \\ &= d(f(x_{n-1}), f(y_{n-1})) \\ &\leq \phi(d(g(x_{n-1}), g(y_{n-1}))) \\ &\leq d(g(x_{n-1}), g(y_{n-1})) = a_{n-1}. \end{aligned}$$

Hence $\langle a_n \rangle$ is a monotone non increasing bounded below sequence, therefore converges to some $a \geq 0$. If $a > 0$ then by right upper semi continuity of $\phi$ we obtain

$$\begin{aligned} a &= \limsup_{n\to+\infty} a_{n+1} \\ &= \limsup_{n\to+\infty} \phi(a_n) \\ &\leq \phi(a). \end{aligned}$$

This is contradiction. So, $a = 0$. Similarly, we conclude that $b_n \to 0$ and $c_n \to 0$ as $n \to +\infty$. Now, we show that the bisequence $(g(x_n), g(y_n))$ is a Cauchy bisequence on $(g(X), g(Y), d)$. Let it is not a Cauchy bisequence then we can find $\varepsilon > 0$ such that for any given $k \in \mathbb{N}$ there exist $m_k \geq k, n_k \geq k$ such that

$$d_k = d(g(x_{m_k}), g(y_{n_k})) \geq \varepsilon. \tag{3.1}$$

Further we can choose integer $m_k$ that also satisfies following inequality

$$d(g(x_{m_k-1}), g(y_{n_k})) < \varepsilon. \tag{3.2}$$

Now, using (B3), (3.1) and (3.2), we obtain

$$\begin{aligned} \varepsilon &\leq d_k \\ &\leq d(g(x_{m_k}), g(y_{m_k-1})) + d(g(x_{m_k-1}), g(y_{m_k-1})) + d(g(x_{m_k-1}), g(y_{n_k})) \\ &< c_{m_k} + a_{m_k-1} + \varepsilon. \end{aligned}$$

This implies that $d_k \to \varepsilon$ as $k \to +\infty$
Moreover, we conclude

$$\begin{aligned} d_k &= d(g(x_{m_k}), g(y_{n_k})) \\ &\leq d(g(x_{m_k}), g(y_{m_k})) + d(g(x_{m_{k+1}}), g(y_{m_k})) + d(g(x_{m_{k+1}}), g(y_{n_k}) \\ &\leq d(g(x_{m_k}), g(y_{m_k})) + d(g(x_{m_{k+1}}), g(y_{m_k})) + d(g(x_{m_{k+1}}), g(y_{n_{k+1}})) \\ &+ d(g(x_{n_k}), g(y_{n_{k+1}})) + d(g(x_{n_k}), g(y_{n_k})) \\ &\leq a_{m_k} + c_{m_k+1} + \phi(d(x_{m_k}, y_{n_k})) + b_{n_k+1} + a_{n_k} \\ &\leq 2a_k + c_k + b_k + \phi(d_k). \end{aligned}$$

So, taking the limit as $k \to +\infty$ we get $\varepsilon \leq \phi(\varepsilon)$. This is a contradiction since $\varepsilon > 0$. Thus $(g(x_n), g(y_m))$ is a Cauchy bisequence. Since $(g(X), g(Y), d)$ is complete in bipolar metric space $(X, Y, d)$ we obtain that $(g(x_n), g(y_m))$ biconverges to a point $u \in g(X \cap Y) \subseteq g(X) \cap g(Y) \subseteq X \cap Y$. Now, since $u \in g(X \cap Y)$ then there exists $v \in X \cap Y$ such that $u = g(v)$. Since,

$$d(f(x_n), f(v)) \leq \phi(d(g(x_n), g(v)) \leq d(g(x_n), g(v)),$$

we obtain that $f(x_n)$ converges to $f(v)$. This implies that $g(x_n)$ converges to $f(v) \in X \cap Y$. But we already know that $g(x_n)$ converges to $g(v)$, therefore $f(v) = g(v) = u$, by uniqueness of limit, and hence $v$ is a coincidence point of $f$ and $g$.
For uniqueness of coincidence point let us assume that $v$ and $w$ are two distinct coincidence point of $f$ and $g$. So, $f(v) = g(v)$ and $f(w) = g(w)$. Now,

$$d(f(v), f(w)) \leq \phi(d(g(v), g(w))).$$

By assumption we get

$$d(g(v), g(w)) \leq \phi(d(g(v), g(w))) < (d(g(v), g(w))).$$

This is a contradiction. So, there is unique coincidence point.
Now from equation $f(v) = g(v) = u$ we get $g(f(v)) = g(u)$ and $f(g(v)) = f(u)$ This implies that $g(u) = f(u)$ if $f$ and $g$ are commuting mappings. Hence $u$ is also a coincidence point of $f$ and $g$. Thus from uniqueness of coincidence point we have $u = v$ and hence $f(v) = g(v) = v$ that is $v$ is the unique common fixed point of $f$ and $g$. □

From Theorem 3.1 we obtain the following result (Czerwik type contraction in bipolar metric space, see [5]).

**Theorem 3.2.** *Let $(X, Y, d)$ be a complete bipolar metric space and let $f : (X, Y, d) \rightrightarrows (X, Y, d)$ be a covariant map and $\phi : [0, +\infty) \to [0, +\infty)$ be a upper semicontinuous function from the right on $[0, +\infty)$ and satisfies $\phi(t) < t$ for all $t > 0$ such that*

$$d(f(x), f(y)) \leq \phi(d(x, y)),$$

*for all $x \in X, y \in Y$. Then the function $f : X \cup Y \to X \cup Y$ has a unique fixed point.*

*Proof.* The proof is clear if we take $g = I$ in Theorem 3.1 where $I$ is identity mapping. □

**Corollary 3.3.** [15] *Let $(X, Y, d)$ be a complete bipolar metric space and let $f : (X, Y, d) \rightrightarrows (X, Y, d)$ be a contraction map. Then the function $f : X \cup Y \to X \cup Y$ has a unique fixed point.*

*Proof.* There exists $\alpha \in (0, 1)$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$, for all $(x, y) \in X \times Y$. If we take $\phi(t) = \alpha \cdot t$. Then by Theorem 3.2 we obtain that $f$ has unique fixed point. □

**Example 3.4.** Let $X = \mathbb{R}^2$ and $Y = \mathbb{R} \times \{0\}$ and a function $d : X \times Y \to [0, +\infty)$ defined by

$$d((x, y), (z, 0)) = \sqrt{(x - z)^2 + y^2},$$

for all $x, y, z \in \mathbb{R}$. Then $(X, Y, d)$ is a complete bipolar metric space. Consider two covariant maps $f, g : (X, Y, d) \rightrightarrows (X, Y, d)$ defined by

$$f(x, y) = \left(8 + \frac{x}{3}, \frac{y}{3}\right) \text{ and } g(x, y) = \left(3 + \frac{3x}{4}, \frac{3y}{4}\right),$$

for all $(x, y) \in X$ and function $\phi : [0, +\infty) \to [0, +\infty)$ is defined by

$$\phi(t) = \frac{t}{2}.$$

Then $\phi$ is continuous function satisfying the condition $\phi(t) < t$ for all $t > 0$. For any element $(x, y) \in X$ and any element $(u, 0) \in Y$, we have

$$d(f(x, y), f(u, 0)) = d((8 + \frac{x}{3}, \frac{y}{3}), (8 + \frac{u}{3}, 0)) = \frac{1}{3}\sqrt{(x-u)^2 + y^2}.$$

Similarly,

$$d(g(x, y), g(u, 0)) = \frac{3}{4}\sqrt{(x-u)^2 + y^2}$$

and hence

$$\phi(d(g(x, y), g(u, 0))) = \frac{3}{8}\sqrt{(x-u)^2 + y^2}.$$

Both the functions $f$ and $g$ are bijective and commuting. Hence all the conditions of the Theorem 3.1 are satisfied. So, $f$ and $g$ must have unique common fixed point. In fact $(12, 0)$ is the unique common fixed point of $f$ and $g$.

**Example 3.5.** Let $X = \mathbb{R}^2$ and $Y = \mathbb{R} \times \{0\}$. A function $d : X \times Y \to \mathbb{R}^+$ defined by

$$d((x, y), (z, 0)) = \sqrt{(x-z)^2 + y^2},$$

for all $(x, y) \in X$ and$(z, 0) \in Y$. Then $(X, Y, d)$ is a complete bipolar metric space. If we define covariant map $f : (X, Y, d) \rightrightarrows (X, Y, d)$ by

$$f(x, y) = \left(\frac{x}{3}, \frac{y}{3}\right),$$

then $f$ is a contraction map with $\alpha = \frac{1}{3}$. Hence, by the Corollary 3.3, we obtain that $f$ has unique fixed point. In fact $(0, 0)$ is a unique fixed point of $f$.

*Remark* 3.6. In this paper, as we have seen, the famous Boyd-Wong contraction in bipolar metric spaces is considered. It is natural to consider the following as a questions in the setting of bipolar metric spaces:
1) Ćirić type of quasi-contraction, [6];
2) Sehgal-Guseman type of contraction, [10, 25];
3) The completion of bipolar metric space;
4) Meir-Keeler type contraction, [14];
5) Nadler type of contraction, [18].

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