

COMPUTER EXTENDED SERIES AND HOMOTOPY ANALYSIS METHOD FOR THE SOLUTION OF MHD FLOW OF VISCOUS FLUID BETWEEN TWO PARALLEL POROUS PLATES

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ABSTRACT. The study of steady two-dimensional incompressible magnetohydrodynamic (MHD) viscous fluid between two parallel porous plates is considered. The governing equations of the flow problem are reduced into nonlinear ordinary differential equation by using similarity transformations. The resulting equation of the problem is solved by using computer extended series (CES) and homotopy analysis method (HAM) with polynomial coefficients. The universal coefficients in the low Reynolds number perturbation expansion are generated. The analytic continuation of series solution unravel the flow structure which could not be fully revealed in earlier perturbation as well as numerical studies. The present analysis of the flow problem enables us to extend the study for higher Reynolds number $R = 10$ to 30 . Both the solutions are found to be in an excellent agreement and the results are demonstrated graphically.

1. INTRODUCTION

The steady laminar flow of an electrically conducting viscous incompressible fluid between two parallel porous plates of a channel in the presence of magnetic field is considered. The flow situations encountered in many industrial applications such as MHD power generators, accelerators, aerodynamic heating, polymer technology, electrostatic precipitations, purification of crude oil, fluid droplets and sprays etc. Hartmann [12], analysed the flow of a conducting fluid between two stationary, insulated and infinite parallel plates which are influenced by a transverse magnetic field. Bermann [6], studied the low Reynolds number for the problem of steady flow of an incompressible viscous fluid through a porous channel with rectangular cross section and obtained a solution by using perturbation method, assumed that the normal wall velocities to be equal. Perlmutter and Seigal [16], discussed the transverse magnetic field of forced convection heat transfer for an electrically conducting liquid flow in a channel. Tani [20], considered the steady motion of electrically conducting viscous fluids in channels and obtained the Hall effects. Sondalgekar et al. [18] and Soundalgekar and Uplekar [19], examined

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the steady MHD couette flow with heat transfer and the effect of the Hall currents, also they considered the temperature of the two plates were assumed either to be constant or to vary linearly along the plates in the direction of the flow. Abo-El-Dhat [1], considered the steady Hartmann flow subjected to a uniform suction/injection at the boundary plates and discussed the effect of Hall currents. In the hydromagnetic case the effect of temperature dependent viscosity on the flow in a channel has been analysed by Attia and Kotb [2, 3]. Bujurke et al. [7, 8, 4, 9], studied the flow in a narrow channel of varying gap using computer extended series solution and his associates have used this technique successfully.

In the present paper, we study the problem for small and moderately high Reynolds number and present some results based on new type of series analysis. In numerical methods boundary value problems (BVP) involve more than one integration process, however the use of a series solution represents an effective alternate approach provided convergence characteristics of the series are guaranteed. Van Dyke [21] and his associates have shown the probable applications of these methods in computational fluid dynamics. For simple models the semi-analytical and semi-numerical methods proposed here is to provide accurate results and have advantages over pure numerical schemes. In numerical methods a separate algorithm scheme is to be developed for determining the derived quantities and if the computation of differential derivatives are required, the numerical algorithm used which will be very sensitive to the step/grid size, this itself will be an comprehensive scheme. However this difficulty does not arise in the case of a series methods. In addition, these methods reveals the analytical structure of the solution, which disappears in case of numerical solutions. The few manually calculated approximations in the low Reynolds number perturbation solution of the boundary value problem, enables us to propose series expansion to generate the universal polynomial coefficients by using recurrence relation and Mathematica. The resulting series which will be limited in convergence by nonphysical singularities, is extended to moderately high Reynolds numbers using an analytic continuation of the series solution. The location and nature of the singularity restricts the convergence of the series as predicted by using Domb-Sykes plot [10]. The sign pattern of the coefficients decides the nature of the singularity, then recast the series into continued fraction representation and use Pade' approximants of various orders for summing it.

We also investigate the same problem by using HAM. Liao [13] proposed HAM for the solution of non-linear ordinary and partial differential equations of the physical problems. The HAM is based on basic concept in topology and it is widely used in numerical techniques. All perturbation techniques are based on small/large parameters so that at least one unknown must be expressed in a series of small parameters. Unlike perturbation techniques, the HAM is independent of any small/large physical parameters. By using HAM one can transfer a nonlinear problem into an infinite number of linear subproblems. The HAM provides us a convenient way to guarantee the convergence of series solution, so that it is valid even if nonlinearity becomes rather strong as compared to all other analytic techniques. In general, the HAM logically contains some traditional methods such as Lyapunovs small artificial method, Adomian decomposition method and

δ - expansion method, so that it has the great flexibility and generality over all other analytical methods.

This paper is organised as follows. Section 1 describes introduction. In section 2 the mathematical formulation of the proposed problem with relevant boundary conditions is given. The solution of the proposed problem is obtained using CES and HAM in section 3 and 4 respectively. Section 5 presents results and discussion and section 6 is about the conclusion.

2. MATHEMATICAL FORMULATION

Consider a steady two-dimensional laminar flow of an incompressible viscous fluid between two parallel porous plates in the presence of a transverse magnetic field of strength H_0 applied perpendicular to the walls. The origin is considered as the centre of the channel, x and y are the co-ordinate axes parallel and perpendicular to the walls of the channel [11].

Let L be the length of the channel and $2h$ is the distance between the two parallel porous plates. Let u and v are the velocity components in the x and y directions respectively. The governing equations for the present flow problem becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\sigma B^2}{\rho} u \quad (2.2)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (2.3)$$

The appropriate boundary conditions of the flow problems are

$$u(x, y) = 0, \quad v(x, y) = +V \quad \text{at} \quad y = +h \quad (2.4)$$

$$u(x, y) = 0, \quad v(x, y) = -V \quad \text{at} \quad y = -h \quad (2.5)$$

where σ is an electrical conductivity of the fluid, ρ be the density of the fluid, ν is the kinematic viscosity of the fluid, p be the pressure of the fluid, B is the magnetic field and V is the suction velocity at the walls of the channel.

Let ψ be the stream function and $\eta = \frac{y}{h}$ be the dimensionless variable such that

$$u = \frac{1}{h} \frac{\partial \psi}{\partial \eta} \quad (2.6)$$

$$v = -\frac{\partial \psi}{\partial x} \quad (2.7)$$

The conservation of the mass Equation (2.1) satisfied by a stream function of the form

$$\psi = [hU(0) - Vx]f(\eta) \quad (2.8)$$

where $U(0)$ is the average entrance velocity at $x=0$ and the velocity components are given by

$$u = \frac{1}{h} \frac{\partial \psi}{\partial \eta} = \frac{1}{h} [hU(0) - Vx]f'(\eta), \quad ' = \frac{d}{d\eta} \quad (2.9)$$

$$v = -\frac{\partial\psi}{\partial x} = Vf(\eta). \quad (2.10)$$

Using Equations (2.9) and (2.10) the conservation of the mass Equation (2.1) satisfied automatically and the momentum Equations (2.2) and (2.3) reduces to the nonlinear ordinary differential equation

$$f''' + R(f'^2 - ff'') - aRf' = K \quad (2.11)$$

where $R = \frac{hV}{\nu}$ is the Suction Reynolds number, $M = Bh(\frac{\sigma}{\nu\rho})^{1/2}$ is the Hartmann number, $a = \frac{H_0^2\mu_e^2\sigma h}{\rho V}$ and K are arbitrary constants. Differentiating the above Equation (2.11) w.r.t η once, we get

$$f'''' + R(f'f'' - ff''') - aRf'' = 0, \quad ' = \frac{d}{d\eta}. \quad (2.12)$$

and the appropriate boundary conditions of the flow problem becomes

$$f(1) = 1, \quad f(-1) = -1, \quad f'(1) = 0, \quad f'(-1) = 0 \quad (2.13)$$

3. SERIES SOLUTION

We seek a regular perturbation series solution of Equation (2.12) in the form of

$$f(\eta) = \sum_{n=0}^{\infty} R^n f_n(\eta) \quad (3.1)$$

where R is the Reynolds number. Substituting Equation (3.1) into Equation (2.12) and equating like powers of R on both sides, we get

$$f_0'''' = 0 \quad (3.2)$$

$$f_{n+1}'''' = f_0 f_n'''' + f_n f_0'''' - f_0 f_n'' - f_n f_0'' + a f_n'' + \sum_{L=1}^{n-1} [f_L f_m'''' - f_L' f_m''], \quad n = 1, 2, \dots \quad (3.3)$$

where $m = n - L$. The appropriate boundary conditions of the flow problem are

$$f_0(-1) = -1, \quad f_0(1) = 1, \quad f_0'(-1) = 0, \quad f_0'(1) = 0 \quad (3.4)$$

$$f_n(-1) = 0, \quad f_n(1) = 0, \quad f_n'(-1) = 0, \quad f_n'(1) = 0, \quad n = 1, 2, \dots \quad (3.5)$$

The required solutions of the above equations up to the $O(R^2)$ are given by

$$\begin{aligned} f_0 &= \frac{3}{2}\eta - \frac{1}{2}\eta^3, \\ f_1 &= -\frac{1}{280}(2 + 7a)\eta + \frac{1}{280}(3 + 14a)\eta^3 - \frac{a}{40}\eta^5 - \frac{1}{280}\eta^7, \\ f_2 &= (-1406 - 5313a + 3388a^2)\eta - \left(\frac{-438 - 1771a + 2079a^2}{646800}\right)\eta^3 \\ &\quad + \left(\frac{3a + 14a^2}{5600}\right)\eta^5 - \left(\frac{-9 + 63a + 35a^2}{58800}\right)\eta^7 - \left(\frac{2 + a}{6720}\right)\eta^9 + \left(\frac{1}{92400}\right)\eta^{11} \end{aligned} \quad (3.6)$$

It is not sufficient to analyze the problem by considering only first few terms of the series, whether the series is slowly convergent or divergent. So we require

the large number of universal polynomial coefficients to analyze the behaviour of the function represented by the series. As we proceed to higher order of approximations the algebra becomes complicated and it is very difficult to calculate the higher order terms manually. The behaviour of the solution Equation (3.6) enables us to propose a systematic series expansion with the universal polynomial coefficients which is quite useful and efficient in the calculation of higher order approximations of the series. Towards this goal, we found $f_n(\eta)$ to be of the form

$$f_n(\eta) = (1 - \eta^2)^2 \sum_{k=1}^{2n} A_{(n,2k-1)} \eta^{(2k-1)}, \quad n \geq 1. \quad (3.7)$$

The above expression derives exactly the earlier calculated approximations $f_i(i = 1, 2)$ using Mathematica. Substituting Equation (3.7) into Equation (3.3) and equate various powers of η on both sides and obtained a recurrence relation for generating the unknowns coefficients $A_{n,(2k-1)}$ in the form

$$\begin{aligned} A_{(n+1,2N_2-(2J+1))} &= A_{(n+1,2N_2-(2J-1))} - A_{(n+1,2N_2-(2J-3))} \\ &\quad + \frac{1}{(2N_2 - (2J - 3))(2N_2 - (2J - 2))(2N_2 - (2J - 1))(2N_2 - (2J))} \\ &\quad \times \left[\sum_{i=1}^4 A_{(n,2N_2-2i-2J+3)} S_i(N_2 - i - J + 2) + \sum_{L=1}^{n-1} \left[\sum_{r=-2}^2 \left(\sum_{k_1=2L-J+r}^{2L} A_{L,(2k_1-1)} \right. \right. \right. \\ &\quad \left. \left. \left. \times A_{(m,2N_2-2k-(2J+(1-2r)))} S_{7-r}(k_1, N_2 - k_1 - (J - r)) \right) \right] \right] \quad (3.8) \end{aligned}$$

where $N_2 = 2n$, $m = n - L$ and J varies from $-2, -1, 0, 1, \dots, (2n - 1)$.

$$\begin{aligned} S_1 &= \frac{3}{2}(2k - 1)(2k - 2)(2k - 3) - \frac{3}{2}(2k - 1)(2k - 2) + a(2k - 1)(2k - 2) \\ S_2 &= -3(2k + 1)2k(2k - 1) - \frac{1}{2}(2k - 1)(2k - 2)(2k - 3) - 3 + 3(2k + 1)2k \\ &\quad + \frac{3}{2}(2k - 1)(2k - 2) \\ S_3 &= (2k + 1)2k(2k - 1) + 6 - \frac{3}{2}(2k + 3)(2k + 2) - 3(2k + 1)2k - 6(2k + 1) \\ &\quad + a(2k + 3)(2k + 2) \\ S_4 &= -\frac{1}{2}(2k + 3)(2k + 2)(2k + 1) - 3 + \frac{3}{2}(2k + 3)(2k + 2) + 3(2k + 3) \\ S_5 &= (2k_2 - 1)(2k_2 - 2)(2k_2 - 3) - (2k_1 - 1)(2k_2 - 1)(2k_2 - 2) \\ S_6 &= -2(2k_2 - 1)(2k_2 - 2)(2k_2 - 3) - 2(2k_2 + 1)2k_2(2k_2 - 1) \\ &\quad + 2(2k_1 + 1)(2k_2 - 1)(2k_2 - 2) + 2(2k_1 - 1)(2k_2 + 1)2k_2 \\ S_7 &= 4(2k_2 + 1)2k_2 + (2k_2 + 3)(2k_2 + 2)(2k_2 + 1) - (2k_1 + 3)(2k_2 - 1) \\ &\quad \times (2k_2 - 2) - 4(2k_1 + 1)(2k_2 + 1)2k_2 - (2k_1 - 1)(2k_2 + 3)(2k_2 + 2) \end{aligned}$$

$$\begin{aligned}
S_8 &= -2(2k_2 + 1)2k_2(2k_2 - 1) - 2(2k_2 + 3)(2k_2 + 2)(2k_2 + 1) \\
&\quad + 2(2k_1 + 3)(2k_2 + 1)2k_2 + 2(2k_1 + 1)(2k_2 + 3)(2k_2 + 2) \\
S_9 &= (2k_2 + 3)(2k_2 + 2)(2k_2 + 1) - (2k_1 + 3)(2k_2 + 3)(2k_2 + 2)
\end{aligned}$$

and $A_{(1,1)} = -\frac{1}{280}$, $A_{(1,3)} = -\frac{a}{40} - \frac{1}{140}$, for obtaining other $A_{(i,j)}$'s, we use the above recurrence relation.

The expression for the radial velocity is given by

$$f'(\eta) = \frac{3}{2} - \frac{3}{2}\eta^2 + \sum_{n=1}^{\infty} R^n \sum_{k=1}^{2n} A_{(n,2k-1)} [(2k+3)(\eta^{(2k+2)} - 2(2k+1)\eta^{2k} + (2k-1)\eta^{(2k-2)})]. \quad (3.9)$$

The expression for shear stress at the walls of the channel is given by

$$f''(-1) = 3 - 8 \sum_{n=1}^{\infty} R^n \sum_{k=1}^{2n} A_{(n,2k-1)}. \quad (3.10)$$

The analytic continuation of the region and validity of series can be achieved by taking various Pade' approximants. The coefficients of the series Equations (3.9) and (3.10) representing the radial velocity $f'(\eta)$ and shear stress $f''(-1)$ are decreasing in magnitude but having irregular sign pattern. The results are further extrapolated using rational approximation [17] for determining the radius of convergence. Figs. 8 and 9 shows the Domb-Sykes plots, after extrapolation confirms the radius of convergence of the series to be $R_0 = 12.91489$ and 12.96176 for the series $f'(0)$ and $f''(-1)$ respectively. The direct sum of the series for radial and axial velocities are valid only up to the radius of convergence. We use pade' approximants for summing the series which give a converging sum for sufficiently large value of R .

4. HOMOTOPY ANALYSIS METHOD

4.1. Zeroth-order deformation problem. We seek a solution of Equation (2.12) by using HAM [14, 15] and due to the boundary conditions (2.13) we choose the base function $(\eta^{2k+1} | k \geq 0)$ to express $f(\eta)$. The initial guess is chosen as

$$f_0(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \quad (4.1)$$

and also auxiliary linear operator defined as

$$L[f] = f'''' \quad (4.2)$$

and the above linear operator satisfying the following property

$$L[C_1 \frac{\eta^3}{6} + C_2 \frac{\eta^2}{2} + C_3 \eta + C_4] = 0 \quad (4.3)$$

where C_1, C_2, C_3 and C_4 are constants to be determined later. If $q \in [0, 1]$ then the zeroth order deformation problem can be constructed

$$(1 - q)L[f(\eta, q) - f_0(\eta)] = qhH(\eta)\aleph[f(\eta, q)] \quad (4.4)$$

subjected to the boundary conditions are

$$f(-1, q) = -1, \quad f(1, q) = 1, \quad f'(-1, q) = 0, \quad f'(1, q) = 0 \quad (4.5)$$

where $q \in [0, 1]$ is an embedding parameter, h and H are the non-zero auxiliary parameter and auxiliary function respectively. Further $\aleph$ is the non-linear differential operator defined as

$$\aleph[f(\eta, q)] = \frac{\partial^4 f(\eta, q)}{\partial \eta^4} + R \frac{\partial f(\eta, q)}{\partial \eta} \frac{\partial^2 f(\eta, q)}{\partial \eta^2} - R f(\eta, q) \frac{\partial^3 f(\eta, q)}{\partial \eta^3} - a R \frac{\partial^2 f(\eta, q)}{\partial \eta^2} \quad (4.6)$$

For $q = 0$ and $q = 1$, Equation (4.4) have the solutions

$$f(\eta, 0) = f_0(\eta), \quad f(\eta, 1) = f(\eta) \quad (4.7)$$

As q varies from 0 to 1, $f(\eta, q)$ also varies from the initial guess $f_0(\eta)$ to the exact (final) solution $f(\eta)$. By Taylor's theorem, Equation (4.7) can be expressed as

$$f(\eta, q) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) q^m \quad (4.8)$$

where $f_m(\eta) = \frac{1}{m!} \frac{\partial^m f(\eta, q)}{\partial q^m} |_{q=0}$. The convergence of the series (4.8) strictly depends upon the convergence control parameter h and assume that h is selected in such a way that the series (4.8) is convergent at $q = 1$, then we have

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) \quad (4.9)$$

4.2. mth-order deformation problem. Differentiating the zeroth order deformation problem Equation (4.4) ' m ' times with respect to q and lastly setting $q = 0$. The resulting mth order deformation problem becomes

$$L[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h H(\eta) \aleph_m(\eta) \quad (4.10)$$

and the homogeneous boundary conditions are

$$f_m(-1) = 0, \quad f_m(1) = 0, \quad f'_m(-1) = 0, \quad f'_m(1) = 0 \quad (4.11)$$

where

$$\aleph_m(\eta) = f_{m-1}'''' + R \sum_{n=0}^{m-1} [f'_n f_{m-1-n} - f_n f_{m-1-n}'''] - a R f_{m-1}'' \quad (4.12)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1; \\ 1, & m > 1. \end{cases} \quad (4.13)$$

We use the Mathematica software to solve the system of linear Equations (4.10) with the appropriate homogeneous boundary conditions (4.11) and obtain the solutions as follows

$$f_1(\eta) = \left(\frac{3}{2} + \frac{1}{40} h M^2 + \frac{1}{140} h R\right) \eta + \left(-\frac{1}{2} - \frac{1}{20} h M^2 - \frac{3}{280} h R\right) \eta^3 + \frac{1}{40} h M^2 \eta^5 + \frac{1}{280} h R \eta^7$$

$$\begin{aligned}
f_2(\eta) = & \left(\frac{3}{2} + \frac{1}{20}hM^2 + \frac{1}{40}h^2M^2 + \frac{11}{8400}h^2M^4 + \frac{1}{70}hR + \frac{1}{140}h^2R \right. \\
& - \frac{23}{11200}h^2M^2R - \frac{703}{1293600}h^2R^2 \Big) \eta + \left(-\frac{1}{2} - \frac{1}{10}hM^2 - \frac{1}{20}h^2M^2 \right. \\
& - \frac{9}{2800}h^2M^4 - \frac{3}{140}hR - \frac{3}{280}h^2R + \frac{23}{8400}h^2M^2R + \frac{73}{107800}h^2R^2 \Big) \eta^3 \\
& + \left(\frac{1}{20}hM^2 + \frac{1}{40}h^2M^2 + \frac{1}{400}h^2M^4 + \frac{3}{5600}h^2M^2R \right) \eta^5 + \left(-\frac{1}{1680}h^2M^4 \right. \\
& + \frac{1}{140}hR + \frac{1}{280}h^2R - \frac{3}{2800}h^2M^2R + \frac{3}{19600}h^2R^2 \Big) \eta^7 + \left(-\frac{1}{6720}h^2M^2R \right. \\
& - \frac{1}{3360}h^2R^2 \Big) \eta^9 + \frac{1}{92400}h^2R^2\eta^{11}
\end{aligned} \tag{4.14}$$

where $aR = M^2$.

4.3. Convergence of HAM. The series (4.9) contains the auxiliary parameter h which is known as convergence control parameter and which influences the convergent rate and region of the series. To ensure that this series converges, we first focus on how to choose the proper value of h . To obtain the permissible ranges of the parameter h , by drawing the line segment of the h curves parallel to η -axis (one example of h curve is plotted in Fig. 10 and Fig. 11 for 10^{th} order of approximations). From Fig. 10 and Fig. 11 it is observed that admissible ranges for $f'(0)$ and $f'''(-1)$ are $-2.3 \leq h \leq -0.7$ and $-1.9 \leq h \leq -0.5$ respectively. Our computation shows that the series converges in the whole region of η when $h = -1$.

5. RESULTS AND DISCUSSION

The laminar flow of an incompressible MHD viscous fluid between two parallel porous plates is analysed by the more suggestive ways using Fortran programming and Mathematica. The governing fourth order nonlinear ordinary differential Equation (2.12) subject to boundary conditions (2.13) has been solved semi-analytically and semi-numerically by using CES and HAM. By using these methods, the graphs of axial and radial velocity profiles have been drawn for different values of M and R in the range $0 \leq R \leq 30$. Fig. 1 and Fig. 2 shows the axial velocity profiles for different values of R and M , these curves (profiles) decreases in the central region and increase near the walls with increase of Reynolds number R . Fig. 3, 4 and 7 shows that as M increases the axial velocity profiles become flat in the central region and steep (increase) near the walls of the channels. For large values of M the fluid moves like a block which shows some sort of rigidity and flow becomes turbulent. In conducting fluids this confirms the idea that magnetic field brings rigidity in the fluid. It is observed that increase in the values of R , $f'(\eta)$ decreases and the profile is parabolic. In Fig. 5 and Fig. 6 the radial velocity profiles have drawn for various values of M and R . It is observed that for different values of M and $R > 0$ in the region $-1 \leq \eta \leq 0$, radial velocity (f) decreases with increase of R while in the region $0 \leq \eta \leq 1$, f increases with

increase of Reynolds number R .

Fig. 10 and 11 shows the h -curves to know the convergence range and the rate of approximation for the series $f'(0)$ and $f''(-1)$ when $M = 0$ and $R = 5$ respectively. It is also observed that the 10th order approximation is enough for the solution of velocity profile and shear stress profile for different values of M and R . The results of HAM are further accelerated by using Pade' approximants. Fig 12 shows the comparison of shear stress profiles by using CES, HAM and numerical results(NDSolve function in Mathematica) for different values of M and R , and the results are comparable.

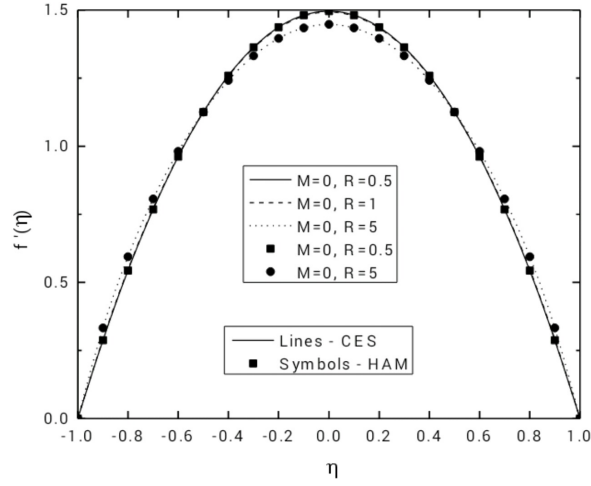


Fig. 1 Axial velocity profiles when $M = 0$ for various values of R .

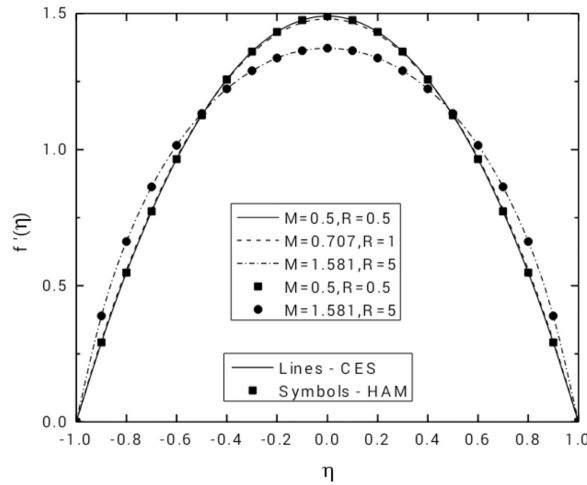


Fig. 2 Axial velocity profiles when $M = 0.5, 0.707, 1.581$ for various values of R .

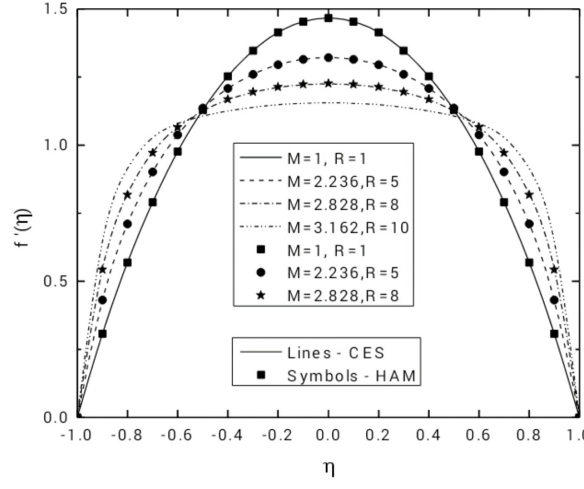


Fig. 3 Axial velocity profiles for different values of R and $M = 1.0, 2.236, 2.828, 3.162$.

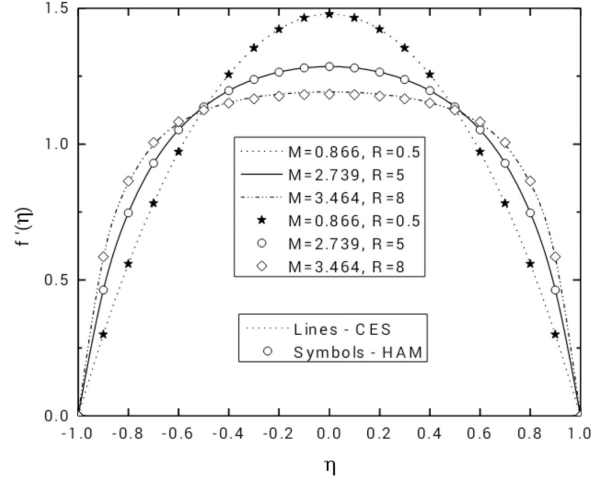


Fig. 4 Axial velocity profiles for different values of R and $M = 0.866, 2.739, 3.464$.

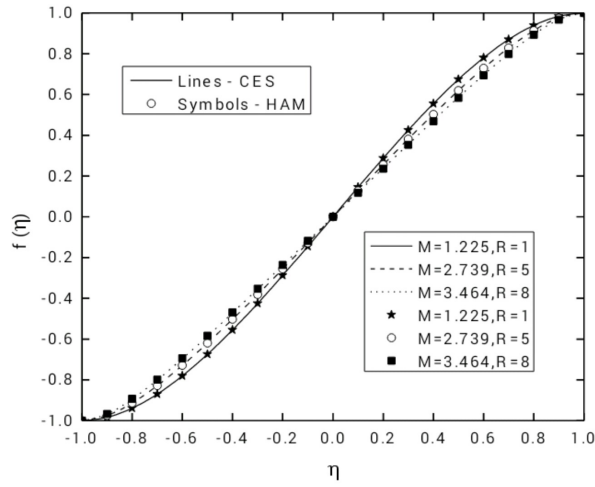


Fig. 5 Radial velocity profiles for different values of R and $M = 1.225, 2.739, 3.464$.

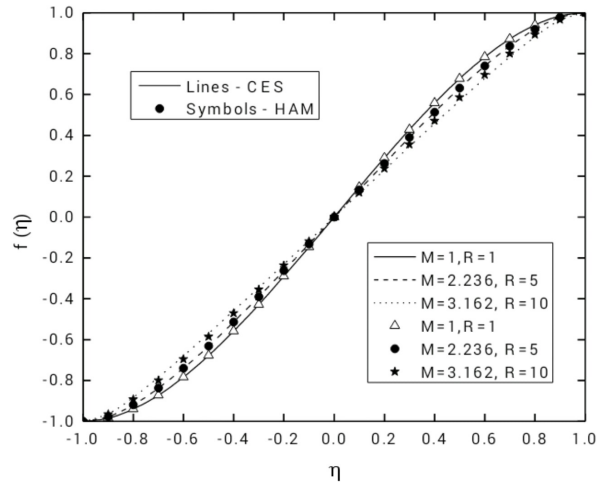


Fig. 6 Radial velocity profiles when $M = 1.0, 2.236, 3.162$ for various values of R .

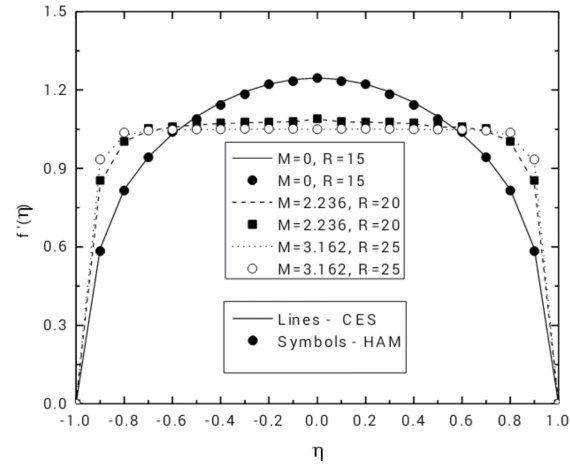


Fig. 7 Axial velocity profiles for different higher values of R and $M = 0, 2.236, 3.162$.

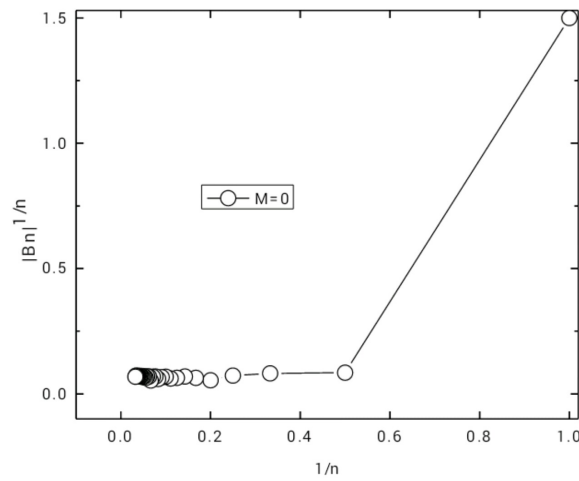


Fig. 8 Domb-Sykes plot for the series $f'(0)$ when $M = 0$.

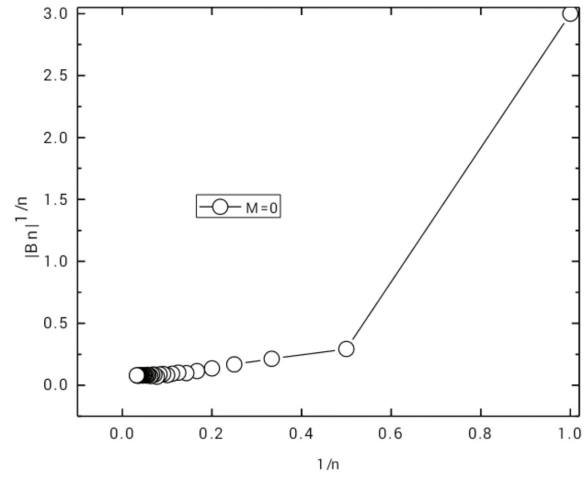


Fig. 9 Domb-Sykes plot for the series $f''(-1)$ when $M = 0$.

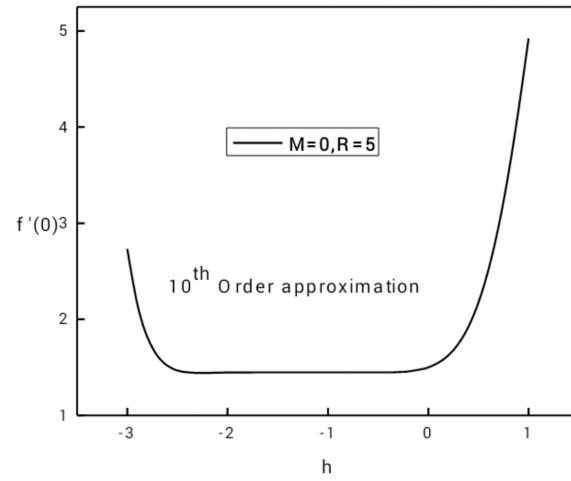


Fig. 10 h curve for the series $f'(0)$ when $M = 0$ and $R = 5$.

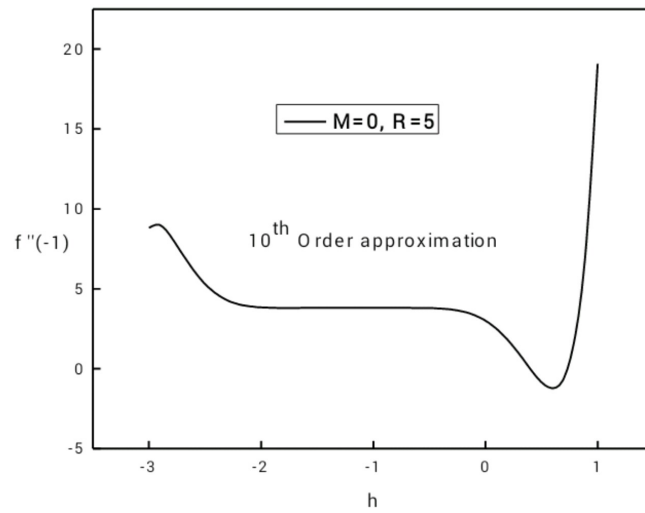


Fig. 11 h curve for the series $f''(-1)$ when $M = 0$ and $R = 5$.

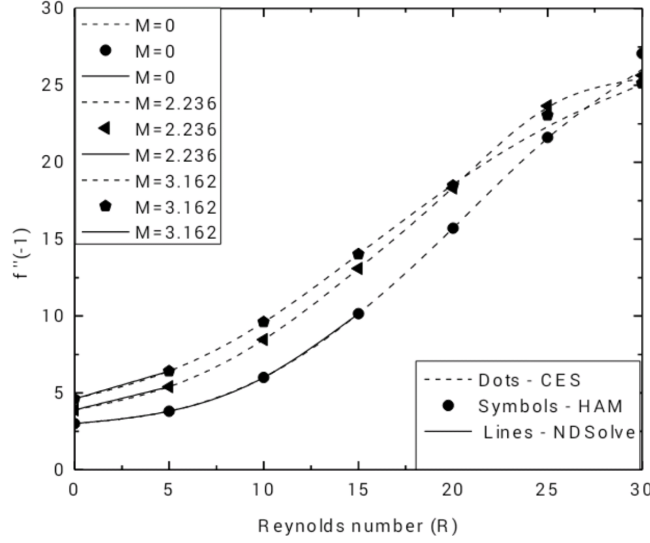


Fig. 12 Comparison of values of shear stress $f''(-1)$.

6. CONCLUSION

In this article, we use the CES and HAM to obtain solution for the fourth order nonlinear ordinary differential equation which arises in laminar flow of an incompressible MHD viscous fluid between two parallel porous plates. We extended the region of validity for higher Reynolds number. The analytic continuation of series solution unravel the flow structure of the problem. The convergence of the HAM solution is properly discussed and undoubtedly the HAM provides us with a convenient way to control the convergence rate and region of the series. The results are presented graphically and the effect of the magnetic parameter M is discussed. Further the results of shear stress $f''(-1)$ are presented in Fig. 12 by using CES, HAM and numerical results for different values of M and R .

Appendix

Pade' Approximants

The basic idea of Pade' summation is to replace a power series $\sum C_n R^n$ by a sequence of rational functions of the form

$$P_M^N(R) = \left(\sum_{n=0}^N A_n R^n \right) \left(1 + \sum_{n=1}^M B_n R^n \right)^{-1} \quad (i)$$

To determine the remaining $(M+N+1)$ coefficients $A_0, A_1, A_2, \dots, A_N; B_1, B_2, \dots, B_M$ so that the first $(M+N+1)$ terms in the Taylor's series expansion of $P_M^N(R)$ match with first $(M+N+1)$ terms of the power $\sum C_n R^n$. The resulting function $P_M^N(R)$ is called a Pade' approximant. The Pade' approximant often gives better approximation of the function than truncating its Taylor series, and it may still work where the Taylor series does not converge. For these reasons Pade' approximants are used extensively in computer calculations. If $\sum C_n R^n$ is a power series representation of the function $f(R)$ than in favourable cases $P_M^N(R) \rightarrow f(R)$, pointwise as $N, M \rightarrow \infty$. There are many methods for the construction of Pade'

approximants. One of the most efficient methods for constructing Pade's approximants is recasting of the series into continued fractions form. A continued fraction is an infinite sequence of fractions whose $(N + 1)$ th member has the form

$$F_N(R) = \frac{D_0}{1 + \frac{D_1 R}{1 + \frac{D_2 R}{\ddots \frac{D_{N-1} R}{1 + D_N R}}}} \quad (\text{ii})$$

The coefficients D_n are determined by expanding the terminated continued fraction $F_N(R)$ in a Taylor series and comparing with those of the power series to be summed. An efficient procedure for calculating coefficients D_n 's of the continued fraction (E) derived from the algebraic identities of Bendor and Orszag [5](8.4.2a) – (8.4.2c). In contrast, to representations by power series, continued fractions representations may converge in regions that contain isolated singularities of the function to be represented, and in many cases convergence is improved. Based on these D_n 's we get terminated continued fractions of various order from other algorithms [5] [(8.4.7), (8.4.8a) and (8.4.8b)]. The truncation of the continued fraction (ii) yields the successive members of the Pade's sequence $P_0^1, P_1^0, P_1^1, P_2^1, P_2^2, \dots$.

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